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CERTAIN nth ORDER DIFFERENTIAL INEQUALITIES IN THE COMPLEX PLANE

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ABSTRACT. Let $w(z)$ be regular in the unit disc $U:|z|<1$, with $w(0)=0$ and let $h(r, s, t)$ be a complex function defined in a domain D of C^3. The author determines conditions on h such that if

$$|h(D^n w(z), D^{n+1}w(z), (n+2)(D^{n+2}w(z)-D^{n+1}w(z)))|<1$$

$z \in U$, then $|w(z)|<1$ for $z \in U$ and $n = 0, 1, 2, \ldots$. Here $D^n w(z) = (z/(1-z)^{n+1} * w(z)$, where $*$ stands for the Hadamard product (convolution). Some applications of the results to certain differential equations are given.

1. **Introduction.** Let A be the family of functions $w(z)$ regular in the unit disc $U:|z|<1$, with $w(0)=0$. For $w(z) \in A$, let

(1) $$D^n w(z) = z(z^{n-1}w(z))^{(n)}/n!$$

where $n \in N_0 = 0, 1, 2, \ldots$ and $z \in U$.

Ruscheweyh [4] observed that (1) can also be written in the form

(2) $$D^n w(z) = (z/(1-z)^{n+1}) * w(z),$$

where $(*)$ stands for the Hadamard product, that is, if

$$f(z) = \sum_1^\infty a_n z^n, \qquad g(z) = \sum_1^\infty b_n z^n, \quad \text{then} \quad f(z) * g(z) = \sum_1^\infty a_n b_n z^n.$$

Recently, Miller [2], and Miller and Mocanu [3] have determined the complex valued functions $h(r, s, t)$ defined in a domain of C^3 such that if $w(z) \in A$ satisfies the inequalities

$$|h(w(z), zw'(z), z^2 w''(z))|<1 \quad \text{for} \quad z \in U, \quad \text{then} \quad |w(z)|<1 \quad \text{for} \quad z \in U.$$

The purpose of this note is to extend and to generalize the above results. To be specific we shall determine the class of complex valued functions H_n defined in some domain D of C^3 such that for $w(z) \in A$ and $h \in H_n$ satisfying

$$|h(D^n w(z), D^{n+1}w(z), (n+2)(D^{n+2}w(z)-D^{n+1}w(z)))|<1$$

for $z \in U$ and $n \in N_0$, then $|w(z)|<1$ for $z \in U$.

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Applications of this result in the theory of differential equations are also given.

2. **Results and applications.** We need the following lemma of Miller and Mocanu [3].

LEMMA. *Let* $w(z)\in A$ *with* $w(z)\not\equiv 0$ *in* U. *If* $z_0=r_0e^{i\theta_0}$, $0<r_0<1$ *and* $|w(z_0)|=\max_{|z|\le r_0}|w(z)|$, *then*

(i) $z_0w'(z_0)/w(z_0)=m$, *and*

(ii) $\operatorname{Re}\{z_0w''(z_0)/w'(z_0)+1\}\ge m$,

where $m\ge 1$.

Part (i) of the *Lemma* can also be found in [1].

DEFINITION. For $n\in N_0$, let H_n be the set of functions $h(r,s,t):C^3\to C$ such that

(i) $h(r,s,t)$ is continuous in a domain D of C^3,

(ii) $(0,0,0)\in D$ and $|h(0,0,0)|<1$,

(iii)

$$\left|h\left(e^{i\theta},\frac{n+m}{n+1}e^{i\theta},\frac{n(m-1)e^{i\theta}+L}{n+1}\right)\right|>1$$

when

$$\left(e^{i\theta},\frac{n+m}{n+1}e^{i\theta},\frac{n(m-1)e^{i\theta}+L}{n+1}\right)\in D,$$

where $\operatorname{Re} e^{-i\theta}L\ge m(m-1)$ for θ and real $m\ge 1$.

We now state and prove our main result.

THEOREM 1. *Let* $w\in A$ *and* $n\in N_0$ *satisfying*:

(i) $(D^nw(z),D^{n+1}w(z),(n+2)(D^{n+2}w(z)-D^{n+1}w(z)))\in D$,

(ii) $|h(D^nw(z),D^{n+1}w(z),(n+2)(D^{n+2}w(z)-D^{n+1}w(z)))|<1$,

when $z\in U$. *Then* $|D^nw(z)|<1$ *for* $z\in U$.

Proof. Using (2) we have

(3) $$z(D^kw(z))'=(k+1)D^{k+1}w(z)-kD^kw(z),\qquad k\in N_0.$$

Let

(4) $$\ell(z)=D^nw(z).$$

From (3) and (4) with $k=n$ we obtain

(5) $$(n+1)D^{n+1}w(z)=n\ell(z)+z\ell'(z).$$

Differentiation of (5) and using (3) with $k=n+1$ lead to

(6) $$(n+2)(D^{n+2}w(z)-D^{n+1}w(z))=(-n\ell(z)+nz\ell'(z)+z^2\ell''(z))/(n+1).$$

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Now suppose there exists $z_0 = r_0 e^{i\theta_0}$, $0 < r_0 < 1$ such that

$$|D^n w(z_0)| = \max_{|z| \le r_0} |D^n w(z)|. \tag{7}$$

Since $D^n w(0) = 0$, the proof of Theorem 1 would be completed if (7) is shown to be impossible for any $z_0 \in U$. Applying the *Lemma* to (4), (5) and (7) yield

$$D^n w(z_0) = \ell(z_0) = e^{i\theta}, \tag{8}$$

$$D^{n+1} w(z_0) = \frac{m+n}{n+1} e^{i\theta}, \tag{9}$$

and

$$\operatorname{Re}\{z_0^2 \ell''(z_0)/z_0 \ell'(z_0) + 1\} = \operatorname{Re} z_0^2 \ell''(z_0) \cdot e^{-i\theta} \ge m(m-1), \tag{10}$$

Thus substitution of (8), (9) and (10) in (6) give

$$(n+2)(D^{n+2} w(z_0) - D^{n+1} w(z_0)) = (n(m-1)e^{i\theta} + L)/(n+1), \tag{11}$$

where $L = z_0^2 \ell''(z_0)$ and therefore $\operatorname{Re} \mathrm{L}e^{-i\theta} \ge m(m-1)$, $m \ge 1$.

Finally from (8), (9), (11) and part (iii) of the *Definition* assuming part (i) of the hypothesis we get

$$\begin{aligned} &|h(D^n w(z_0), D^{n+1} w(z_0), (n+2)(D^{n+2} w(z_0) - D^{n+1} w(z_0)))| \\ &= \left| h\left(e^{i\theta}, \left(\frac{m+n}{n+1}\right) e^{i\theta}, \frac{n(m-1)e^{i\theta} + L}{n+1}\right) \right| > 1, \end{aligned}$$

where $\operatorname{Re} \mathrm{L}e^{-i\theta} \ge m(m-1)$. This contradicts part (ii) of the hypothesis. Thus the proof is completed.

REMARKS 1. The condition on L may be replaced with $\operatorname{Re} \mathrm{L}e^{-i\theta} \ge 0$.

2. For $n = 0$, Theorem 1 reduces to Theorem 2 of [3] for the class, A, and also includes Theorem 2 of [2].

Let $h_0(r, s, t) = s$. It is obvious that $h_0 \in H_n$. By iteration of Theorem 1 applied to h_0 we get the following corollary.

COROLLARY 1. $\|D^{n+1} w(z)| < 1 \Rightarrow |w(z)| < 1$ *for* $z \in U$ *and* $n \in N_0$. *Theorem* 1 *and Corollary* 1 *give the following.*

COROLLARY 2. *If* $h(r, s, t)$ *and* $w(z)$ *are as in Theorem* 1, *then* $|w(z)| < 1$ *for* $z \in U$.

Corollary 2 can be used to show that certain nth order differential equations have bounded solutions.

THEOREM 2. *Let $h(r, s, t)$ and $w(z)$ be as in Theorem 1 and let $b(z)$ be a regular function in U with $|b(z)|<1$, $z \in U$. If the differential equation*

$$h(D^n w(z), D^{n+1} w(z), (n+2)(D^{n+2} w(z) - D^{n+1} w(z))) = b(z)$$

has a solution $w(z)$ regular in the unit disc U then $|w(z)|<1$.

We close this brief note with some examples and applications, see [2] and [3] for further details.

EXAMPLES. It is easy to check that each of the following functions are in H_n.

$h_1(r, s, t) = \alpha r + s,$	$\operatorname{Re} \alpha \geq 0, D = C^3, n \in N_0.$
$h_2(r, s, t) = 2s/(1-r),$	$D = (C - \{1\}) \times C^2, n \in N_0.$
$h_3(r, s, t) = r^i s^j (t^k + r^\ell),$	$D = C^3$, $i, j, k \in N_0$ and $\ell \in N.$
$h_4(r, s, t) = r + s + t,$	$D = C^3, n \in N_0.$
$h_5(r, s, t) = -r + 2s + 2t,$	$D = C^3, n \in N.$

Applying Corollary 2 to h_4 with $n = 0$ we obtain the Euler equation

$$z^2 w''(z) + z w'(z) + w(z) = b(z).$$

If $|b(z)|<1$ then we must have $|w(z)|<1$.

Similarly applying the same Corollary to h_5 with $n = 1$ we get

$$z^3 w'''(z) + 4z^2 w''(z) + z w'(z) = b(z).$$

And if $|b(z)|<1$ and the above differential equation has a solution $w(z)$ regular in U then $|w(z)|<1$ for $z \in U$.

If we apply same corollary to h_3, $r = j = k = \ell = 1$, $n = 2$, then the resulting nonlinear differential equation

$$zw'(2zw' + z^2 w'')(2zw' + 3z^2 w'' + z^3 w''') = 4b(z)$$

would have a bounded solution $|w(z)|<1$ for $z \in U$ provided that there exists a solution $w(z)$ regular in the unit disc U when $|b(z)|<1$.

Problem. A natural problem arises from the results and examples of this note; namely does there exist a function $h(r_1, r_2, \ldots, r_n)$ defined in some suitable domain D of C^n such that if

$$|h(w(z), zw'(z), \ldots, z^n w^{(n)}(z))| < 1 \quad \text{for} \quad z \in U,$$

then $|w(z)|<1$ for $z \in U$.

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