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# A LOCAL HOPF BIFURCATION THEOREM FOR A CERTAIN CLASS OF IMPLICIT DIFFERENTIAL EQUATIONS

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ABSTRACT. The local Hopf Bifurcation theorem is extended to implicit differential equations in $R^n$, of the form $\dot{x} = f(x, \dot{x}, \alpha)$, which are not solvable for the variable $\dot{x}$. The proof uses the $S^1$-degree of convex-valued mappings. An example of an implicit differential equation in $R^3$ to which the presented theorem applies is provided.

**Introduction.** There is a considerable amount of research that has been devoted to the existence of continua of periodic solutions from an equilibrium of a parametrised dynamical system, *i.e.* the so called *Hopf Bifurcation Problem.* In his original work, Hopf (*cf.* [7]) proved the result on bifurcating periodic orbits under very strong and restrictive assumptions such as: the analyticity of the function, simple characteristic root transversally crossing the imaginary axis, and no other imaginary characteristic roots. Even though his result allowed to deal with stability properties and other characteristics of bifurcating orbits, there was need for a more general type of bifurcation theorems for a diversity of problems arising from applications. We refer to the book by Marsden and McCracken [12] for a more detailed discussion of the history and background of the problem.

Many techniques were developed and used for the study of Hopf bifurcation problems in more general settings. We could mention such methods as Lyapounov-Schmidt, center manifold, Fuller index, regular approximation, decomposition theory, *etc.* We refer the reader to [2] for details and an extensive bibliography.

The concept of $S^1$-degree recently developed in [3] and [6] has been used to provide purely topological proofs of local and global Hopf bifurcation theorems for certain classes of differential equations. In particular, [5] proved the existence of a bifurcation of nonconstant periodic solutions from a trivial solution of the functional differential equation

$$\dot{x} = F(x_t, \alpha). \tag{0.1}$$

The aim of this paper is to show how one can extend such results to implicit differential equations, *i.e.*, those where the derivative $\dot{x}$ is involved in the nonlinear part and the equation cannot be reduced to a quasi-linear equation by solving it for $\dot{x}$. The proof

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of our local bifurcation theorem is based on the $S^1$-degree of convex-valued mappings developed in [15] and on the techniques used in [5] and [11].

For simplicity of the presentation, we restrict the study to an ordinary differential equation (without delay or advanced arguments) of the form

$$\dot{x}(t) = f\big(x(t), \dot{x}(t), \alpha\big). \tag{0.2}$$

Analogous results for functional differential equations of the form

$$F(x_t, \dot{x}_t, \dot{x}, \alpha) = 0 \tag{0.3}$$

will be presented in another paper.

The existence of solutions of boundary value problems for various classes of implicit differential equations has been proved by using multi-valued mappings in [1], [4], [8], [9], and [10]. Nontrivial periodic solutions of a second order implicit differential equations were studied in [14] with the use of $A$-proper mappings. In that paper, the equation was actually solvable for the second order derivative but the solution function did not have nice properties which would permit studying the solved quasilinear problem. It seems, however, that the bifurcation problem for implicit equations unsolvable for the variable $\dot{x}(t)$ is treated here for the first time.

In Section 1 we formulate the problem and in Section 2 we prove the local bifurcation theorem. Section 3 contains two simple examples illustrating our theorem and its applications.

1. **Formulation of the problem.** We shall study periodic solutions $x: R \longrightarrow R^n$, with an unknown period $p$, of the following implicit differential equation

$$\dot{x}(t) = f\big(x(t), \dot{x}(t), \alpha\big), \tag{1.1}$$

where $\alpha \in R$ is a parameter and $f: R^n \times R^n \times R \longrightarrow R^n$ is a $C^1$ function satisfying the following conditions (*cf.* Remark at the end of this section):

(i) $\|D_y f(x, y, \alpha)\| \leq 1$ for all $(x, y, \alpha) \in R^n \times R^n \times R$;

(ii) For any constant $K > 0$, there exist constants $M > 0$ and $0 < c < 1$ such that $\|f(x, y, \alpha)\| \leq M + c\|y\|$ provided $\|(x, \alpha)\| \leq M$, for all $y \in R^n$.

A point $x_0 \in R^n$ is called a *stationary point* of (1.1) for $\alpha_0 \in R$ if $f(x_0, 0, \alpha_0) = 0$; thus $x(t) \equiv x_0$ is a solution of (1.1). A stationary point $(x_0, \alpha_0)$ (*i.e.* a stationary point $x_0$ for $\alpha_0$) is called *nonsingular* if the matrix $D_x f(x_0, 0, \alpha_0)$ is nonsingular. It follows from the implicit function theorem that if $(x_0, \alpha_0) \in R^n \times R$ is a nonsingular stationary point then there is a unique $C^1$ curve $(x_\alpha, \alpha)$ of nonsingular stationary points passing through $(x_0, \alpha_0)$.

We may imbed (1.1) into the family of equations parametrized by a constant $0 < k \leq 1$,

$$\dot{x}(t) = kf\big(x(t), \dot{x}(t), \alpha\big). \tag{1.2}$$

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Let us note that (1.2) has the same set of nonsingular stationary points and the same curves $(x_\alpha, \alpha)$ as (1.1), for all $k \in (0, 1]$ The linearization of (1.2) at its stationary point $(x_0, \alpha_0)$ leads to the following *characteristic equation*:

$$\det \Delta_{x_0,\alpha_0,k}(\lambda) = 0 \tag{1.3}$$

where $\Delta_{x_0,\alpha_0,k}(\lambda) = \lambda[I - kD_yf(x_0, 0, \alpha_0)] - kD_xf(x_0, 0, \alpha_0)$ is a complex $n \times n$ matrix-valued function of $\lambda \in C$. Since $\det \Delta_{x_0,\alpha_0,k}\colon C \longrightarrow C$ is a polynomial in $\lambda$, the equation (1.3) has finitely many solutions $\lambda$ for each given $(x_0, \alpha_0, k)$. Any such solution is called a *characteristic value* of $(x_0, \alpha_0, k)$. The assumption that $(x_0, \alpha_0)$ is nonsingular implies that $\lambda = 0$ is not a characteristic value of $(x_0, \alpha_0, k), 0 < k \leq 1$.

The conditions (i) and (ii) imply that $f$, as a function of $y$ only, is lipschitzian with the constant 1 and that it maps a ball of a certain radius about the origin in $R^n$ into itself. By the fixed point theorem for nonexpansive mappings, the set

$$F(x, \alpha, k) = \{y \in R^n : y = kf(x, y, \alpha)\} \tag{1.4}$$

is nonempty, compact and convex. The Banach contraction principle implies that if $k < 1$ then $F(x, \alpha, k)$ is a singleton which we identify with its unique element, so that (1.2) is equivalent to $\dot{x}(t) \in F\big(x(t), \alpha, 1\big)$ if $k = 1$ and to $\dot{x}(t) = F\big(x(t), \alpha, k\big)$ if $k < 1$. In the last case, $I - kD_yf$ is nonsingular and $D_xF = k[I - D_yf]^{-1}D_xf$. As a consequence, we obtain the following

PROPOSITION 1. *If $k < 1$, then, for any $(x, \alpha) \in R^n \times R$,*

$$\det \Delta_{x,\alpha,k}(\lambda) = 0 \Leftrightarrow \det[\lambda I - D_xF(x, \alpha, k)] = 0.$$

A nonsingular stationary point $(x_0, \alpha_0)$ of (1.1) is called a *center* if $(x_0, \alpha_0, 1)$ has a purely imaginary characteristic value. It is called an *isolated center* if there is a neighbourhood of $(x_0, \alpha_0)$ where (1.1) has no other centers than $(x_0, \alpha_0)$. In what follows we make the following assumption:

(iii) There exists an isolated center $(x_0, \alpha_0)$ for (1.1).

Let now $(x_0, \alpha_0)$ be an isolated center for (1.1) and $i\beta_0$ its purely imaginary characteristic value. We should emphasize that we allow $i\beta_0$ to be a multiple root of the characteristic equation and we do not exclude the possibility of the existence of other purely imaginary roots. Since the roots of (1.3) are conjugate, we may assume that $\beta_0 > 0$. Let $(x_\alpha, \alpha)$ be the previously discussed curve of nonsingular stationary points through $(x_0, \alpha_0)$, with $x_{\alpha_0} = x_0$. For simplicity, we write $\Delta_{\alpha,k}$ for $\Delta_{x_\alpha,\alpha,k}$ and $\Delta_\alpha$ for $\Delta_{x_\alpha,\alpha,1}$. We choose $a > 0$ and $b > 0$ such that the closure of $\Omega = (0, a) \times (\beta_0 - b, \beta_0 + b) \subseteq R^2 = C$ contains no root of $\det \Delta_{\alpha_0}(\lambda)$ other then $i\beta_0$. By (iii), for all sufficiently small $\delta > 0$ if $0 < |\alpha - \alpha_0| \leq \delta$ then $iR \cap \{\lambda \in C : \det \Delta_\alpha(\lambda) = 0\} = \emptyset$. The continuity of $(\alpha, k, \lambda) \longrightarrow \det \Delta_{\alpha,k}(\lambda)$ and the compactness of $\partial\Omega$ imply the existence of $\delta > 0$ and $\tau \in (0, 1]$ such that $\det \Delta_{\alpha_0 \pm \delta,k}(\lambda)$ has no zero on $\partial\Omega$ for all $k \in [\tau, 1]$.

We define the *crossing number* for (1.2) at $(x_0, \alpha_0)$ by

$$\gamma(k) = \deg\big(\det_{\alpha_0-\delta,k}(\cdot), \Omega\big) - \deg\big(\det_{\alpha_0+\delta,k}(\cdot), \Omega\big) \tag{1.5}$$

where deg is the usual Brouwer degree. Explicitly, $\gamma(k)$ is the difference between the number of zeros of $\Delta_{\alpha_0-\delta,k}$ in $\Omega$ and that of $\Delta_{\alpha_0+\delta,k}$, counting the multiplicities. As an immediate consequence of the homotopy invariance, we get the following

PROPOSITION 2. *$\gamma(k)$ is independent of $k \in [\tau, 1]$.*

We may therefore write $\gamma = \gamma(k), k \in [\tau, 1]$. Our main result is the following

THEOREM 1. *Suppose that the hypotheses (i), (ii), and (iii) are satisfied. If $\gamma \neq 0$ then there exists a bifurcation of nonconstant periodic solutions of (1.1) from $(x_0, \alpha_0)$ with derivatives defined a.e. and square integrable. More precisely, there exists a sequence $\big\{\big(x_n(t), \alpha_n, \beta_n\big)\big\}$ such that $x_n(t) \longrightarrow x_0$ for all $t \in R^n$, $\alpha_n \longrightarrow \alpha_0$, $\beta_n \longrightarrow \beta_0$ as $n \longrightarrow \infty$, $x_n(t)$ is a nonconstant periodic solution of (1.1) with $\alpha = \alpha_n$, period $p_n = \frac{2\pi}{\beta_n}$, and $\dot{x}_n \in L^2_{loc}(R, R^n)$.*

REMARK. The conditions (i) and (ii) are imposed on the global behaviour of $f$, for the simplicity of arguments. Since our result has a local character, we only need to assume (i) and (ii) for $(x, \alpha)$ in some neighbourhood of our discussed stationary point $(x_0, \alpha_0)$.

2. **Proof of the theorem.** The first step is to normalize the period of the problem. By the change of variable $z(t) = x(\frac{p}{2\pi}t)$, we bring (1.2) to an equivalent form

$$\dot{z}(t) = \frac{k}{\beta} f\big(z(t), \beta\dot{z}(t), \alpha\big) \tag{2.1}$$

where $\beta = \frac{2\pi}{p}$ is the circular frequency. Clearly $x(t)$ is p-periodic if $z$ is $2\pi$ periodic. The equation (2.1) is next equivalent to the differential inclusion

$$\dot{z}(t) \in F_k(z(t), \alpha, \beta) \tag{2.2}$$

where $F_k(z, \alpha, \beta) = \{y \in R^n : y = \frac{k}{\beta} f(z, \beta, y, \alpha\}$. By the same arguments as in (1.3), $F_k(z, \alpha, \beta)$ is nonempty, compact, convex, and by the closed graph property, the map $(z, \alpha, \beta, k) \longrightarrow F_k(z, \alpha, \beta)$ is upper semicontinuous.

We next put $S^1 = R/2\pi Z$, $C^0 = C(S^1, R^n)$, $H^1 = H^1(S^1, R^n)$, and $L^2 = L^2(S^1, R^n)$. The spaces $H^1$ and $L^2$ are isometric Hilbert representations of the group $G = S^1$ acting by shifting the argument, and the equivariant operator $\mathcal{L}x = \dot{x}$ is regarded as an unbounded Fredholm operator of index zero from $\mathrm{Dom}(\mathcal{L}) = H^1 \subset L^2$ into $L^2$. Note that $\mathrm{Ker}(\mathcal{L}) \simeq R^n$ is the set of all constant functions and the operator $K\colon L^2 \longrightarrow L^2$ defined by $Kx = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$ is an equivariant resolvent of $\mathcal{L}$. We next define the Nemytskii operator $F^*\colon C^0 \times R^2 \times [\tau, 1] \longrightarrow \mathrm{co}\, L^2$ (convex subsets of $L^2$) by

$$F^*_k(z, \alpha, \beta) = \big\{u \in L^2 : u(t) \in F_k\big(z(t), \alpha, \beta\big) \text{ a.e.t.}\big\}$$

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It is known that $F^*$ has nonempty weakly compact convex values, that it is $S^1$-equivariant ($S^1$ acts trivially on the component $R^2$), and that its composition with any linear compact operator is upper semicontinuous (*cf.* [13] and [15]). We have the following diagram

$$\begin{tikzcd} C^0 \arrow[rr, "F_k^*(\cdot,\alpha,\beta)"] & & L^2 \\ & H^1 \arrow[ul, "j"] \arrow[ur, "\mathcal{L}"'] & \end{tikzcd}$$

where $j: H^1 \longrightarrow C^0$ is the completely continuous inclusion. The equation (1.2) subject to periodic condition $x(t+p) = x(t)$ is now equivalent to

$$z \in G_k(z, \alpha, \beta) \tag{2.3}$$

where $G: C^0 \times R^2 \times [\tau, 1] \longrightarrow \text{co}\, C^0$ the homotopy defined by

$$G_k(z, \alpha, \beta) = j[\mathcal{L} + K]^{-1}[F_k^*(z, \alpha, \beta) + Kz]$$

Let now $\delta$ and $b$ be as in the discussion following the assumption (iii), Section 1, and let $N \subset \text{Ker}(\mathcal{L}) \times R^2 \simeq R^n \times R^2$ be a 2-dimensional submanifold defined by

$$N = \{(x_\alpha, \alpha, \beta) : \alpha_0 - \delta < \alpha < \alpha_0 + \delta \text{ and } \beta_0 - b < \beta < \beta_0 + b\}.$$

One may regard the space of constant functions $\text{Ker}(\mathcal{L})$ as a closed (complemented) subspace of $C^0$ and thus $N$ is regarded as a submanifold of $C^0 \times R^2$.

We will now argue by contradiction. Suppose that $(x_0, \alpha_0)$ is not a bifurcation point. Then, following [5], one can construct an $S^1$ invariant tubular neighbourhood $U$ of $N$ with the property that $(z, \alpha, \beta) \in \bar{U}$ is a trivial solution of (2.3) (*i.e.* $z$ is a constant solution corresponding to $\alpha$ and $\beta$) if and only if $(z, \alpha, \beta) \in \bar{N} \cap \bar{U}$. By again following [5], one can construct an $S^1$-invariant complementing function $\varphi: \bar{U} \longrightarrow R^2$ with the property $\varphi(z, \alpha, \beta) \neq 0$ if $(z, \alpha, \beta) \in \bar{N} \cap \bar{U}$. It follows that the convex-valued mapping $\Gamma: \bar{U} \times [\tau, 1] \longrightarrow \text{co}(C^0 \times R^2)$ defined by

$$\Gamma_k(z, \alpha, \beta) = [z - G_k(z, \alpha, \beta)] \times \{\varphi(z, \alpha, \beta)\}, \quad \tau \leq k \leq 1,$$

is an $S^1$-equivariant homotopy between convex-valued compact vector fields $\Gamma_\tau$ and $\Gamma_1$. The inclusion $0 \in \Gamma_k(z, \alpha, \beta)$ is equivalent to the pair of equations

$$\beta \dot{z}(t) = kf\big(z(t), \beta \dot{z}(t), \alpha\big); \tag{2.4}$$

$$\varphi\big(z(t), \alpha, \beta\big) = 0; \tag{2.5}$$

for $(z, \alpha, \beta) \in \bar{U}$. It follows that $\Gamma_1$ has no zero in $\bar{U}$, so $S^1\text{-Deg}(\Gamma_1, U) = 0$ (*cf.* [15]). Consequently, there exists $\tau_0 \in [\tau, 1]$ such that $S^1\text{-Deg}(\Gamma_k, U) = 0$ for all $k \in [\tau_0, 1]$. However, if $k \in [\tau_0, 1)$, the equation (2.4) is equivalent to

$$\dot{z}(t) = \frac{1}{\beta} F\big(z(t), \alpha, k\big),$$

where $F$ is the continuous function defined by (1.4). It follows from Proposition 1, [5], and Proposition 3.2 in [11] that $0 = S^1\text{-Deg}(\Gamma_k, U) = \gamma(k)$ which in the view of Proposition 2, contradicts the assumption $\gamma \neq 0$.

3. **Examples.** Let $a, b, \alpha \in R$ and let $u$ and $v$ be real functions defined by

$$u(t) = \begin{cases} t & \text{if } |t| \leq 1 \\ -1 - \ell n(-t) & \text{if } t < -1 \\ 1 + \ell n(t) & \text{if } t > 1 \end{cases}$$

$$v(t) = \frac{t^2}{1+t^2}$$

We have chosen $u$ and $v$ so to have $C^1$ functions of sublinear growth with $|\dot{u}| \leq 1$, $u(0) = 0$, $u(t) = t$ for small $|t|$, $v(0) = \dot{v}(0) = 0$.

Let us consider the following system of equations

$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 + v(\|\dot{x}\|) \\ \dot{x}_2 = x_1 + \alpha x_2 + v(\|\dot{x}\|) \\ \dot{x}_3 = a x_1 + b x_2 + x_3 + u(\dot{x}_3) \end{cases} \tag{3.1}$$

Due to the third equation, the system cannot be continuously solved for the variable $\dot{x}$ in any neighbourhood of the origin in $R^3 \times R^3 \times R$. Our theorem, however, can be applied to show the existence of a bifurcation: Indeed, $x_0 = 0$ is a unique stationary point for all $\alpha \in R$ with characteristic values $\lambda = \alpha \pm i$ of multiplicity 1, and $(x_0, \alpha_0) = (0, 0)$ is the unique center. The assumptions (i), (ii) and (iii) are clearly satisfied. For the computation of the crossing number, we may take $\Omega = (0, \alpha) \times (0, 2)$, where $\alpha > 0$ is arbitrary, and $\delta > 0$ any number less than $\alpha$. Then $\deg(\Delta_\delta(\cdot), \Omega) = 1, \deg(\Delta_{-\delta}(\cdot), \Omega) = 0$, so $\gamma = -1$.

For the second example, we simplify our system by taking $a = b = 0$ and $v \equiv 0$. The new system

$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 \\ \dot{x}_2 = x_1 + \alpha x_2 \\ \dot{x}_3 = x_3 + u(\dot{x}_3) \end{cases} \tag{3.2}$$

also is unsolvable for the variable $\dot{x}$. In this case, however, we may substitute $\dot{x}_3 = 0$. Consequently, (3.2) has the bifurcation of periodic solutions whose trajectories are circles $x_1^2 + x_2^2 = c^2, x_3 = 0$. Evidently, any such solution also is a solution of the linear system

$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 \\ \dot{x}_2 = x_1 + \alpha x_2 \\ \dot{x}_3 = 0 \end{cases} \tag{3.3}$$

which shows that our theorem may even bring improvements to explicit (quasi-linear) equations such as those considered in [5], [6], or [11]: It may allow to weaken the non-singularity conditions imposed on the differential of the right-hand side.

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