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THE ACTION OF A MAXIMAL PARABOLIC SUBGROUP ON THE TRANSPOSITIONS OF THE BABY MONSTER

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We prove a technical result required by Ivanov and Shpectorov in their construction of a non-split extension of 3^{4371} by the Baby Monster simple group.

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1. Introduction

The construction by Ivanov and Shpectorov [1] of a non-split extension $3^{4371} \cdot B$, where B denotes Fischer's $\{3,4\}$-transposition group, often called the Baby Monster, depends on a number of factors. First, Shapiro's Lemma is applied to the inertial group $2 \cdot {}^2E_6(2)$ to show that there is a non-split group $3^{13571955000} \cdot B$, where the action is given by inducing up the alternating character of $2 \cdot {}^2E_6(2){:}2$. Then the 2-local geometry of B is lifted to this group, and is eventually shown to generate a subgroup $3^{4371} \cdot B$.

To prove this, many detailed facts are required about the action on the module $3^{13571955000}$ of the parabolic subgroups of this geometry. At the time when this problem was drawn to my attention, in a talk given by Shpectorov at the Oberwolfach meeting on Groups and Geometries in August 1991, there was one major problem remaining. It was suggested that my new explicit construction of the Baby Monster [3] could be used to solve this. What follows is the story of this solution.

2. The problem

The 2-local geometry of B has the diagram shown in Fig. 1, in which the node for a particular type in the geometry is labelled with the subgroup of B which stabilizes an element of that type. Let $P \cong 2^{5+5+10+10} \cdot L_5(2)$ be a maximal parabolic of the end type, and let $Q = O_2(P) \cong 2^{5+5+10+10}$. Let $S = C(t) \cong 2 \cdot {}^2E_6(2){:}2$ be the centralizer of an involution t of class $2A$. The module M for B, of dimension 13571955000 over $GF(3)$, is the monomial induced up from the non-trivial linear character of S. The result required by Ivanov and Shpectorov can be restated in group-theoretic terms as follows:

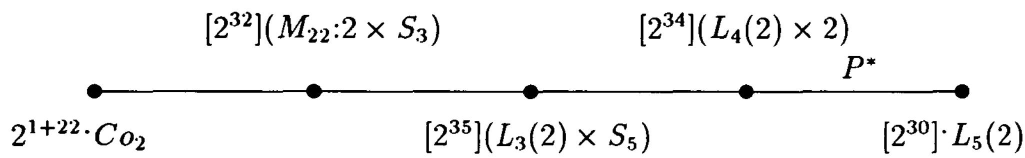

FIGURE 1. The 2-local geometry of B.

Conjecture 1. *For each $g \in B$, $Q^g \cap S$ is not contained in S'.*

Alternatively, we can say that for all $g \in B$, $Q \cap S^g$ is not contained in S'^g. We will adopt whichever point of view is computationally more convenient.

To prove this conjecture, we need to find an element in $(Q^g \cap S) - S'$ for a representative Q^g of each orbit of S on conjugates of Q. First, therefore, we need to determine the orbits of S on conjugates of Q, or equivalently, the orbits of S on conjugates of P. This is in turn equivalent to finding the orbits of P on conjugates of S, or the orbits of P on the conjugates of t, where $S = C_B(t)$.

Once we have found representatives t^g for these orbits we need to find in each case an element in the corresponding set $(Q \cap S^g) - S'^g = Q \cap (S - S')^g$.

3. Computerising the problem

In [3] we showed how to construct 4370×4370 matrices over $GF(2)$, generating the Baby Monster. These can be manipulated using the programs of the "Meat-axe" [2]. A matrix multiplication takes typically 2 minutes CPU time on an IBM3090, or 15 minutes on a SUN ELC.

To begin with we computed the complete module structure for $V|_S$, where $V = V_{4370}$ is the 4370-dimensional module for B over $GF(2)$. This structure is illustrated in Fig. 2.

The first crucial observation is that S fixes a unique non-zero vector in V. Thus there is a one-to-one correspondence between the transpositions in B and the 13571955000 images of this vector. We call these vectors *sacred vectors*. Thus to find the orbits of P on transpositions, it suffices to find the orbits of P on sacred vectors.

The second observation is that there is a 2-dimensional indecomposable subquotient, which represents S/S'. Restricting to S', this becomes a direct sum of two trivial modules. Thus there is a vector fixed by S' but not by S. Lifting back to V, we obtain a 1782-dimensional subspace which is invariant under S' but not under S. This can now be used to determine fairly quickly whether a particular element of S is in S'.

The next task is to find explicit subgroups S and P in the matrix group isomorphic to B. It is quite straightforward to find (a conjugate of) S, since it is an involution centralizer. It is much harder to find P. In fact we had a subgroup $2^5 \cdot L_5(2) < Th < B$ from an earlier calculation (see [4]). We then took an element a of order 10 in $2^5 \cdot L_5(2)$ and found its centralizer in B, by first finding $C(a^5)$ and then finding the centralizer of

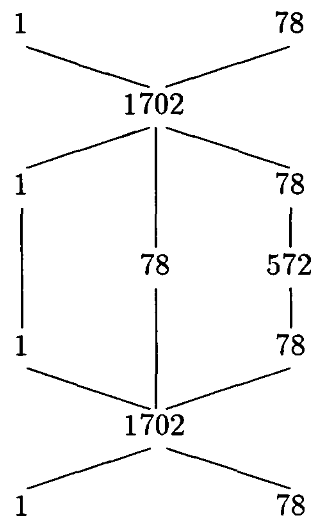

FIGURE 2. Module structure for $V|_S$.

Table 1. Sacred vectors.

Orbit no.	First 0 image	Orbit length	$\dim \langle v \rangle^P$
1	V_0	10401873920	4370
2	V_{10}	1950351360	4360
3	V_{34}	1137704960	4336
4	V_{74}	71106560	3982
5	V_{269}	8888320	3382
6	V_{879}	1904640	2717
7	V_{1894}	119040	1348
8	?	4960	363
9	?	1240	154

a^2 inside $C(a^5)$. We obtained a group $C(a) \cong 5 \times 2_+^{1+4} \cdot A_5$. A random search in the latter group soon produced an element to extend $2^5 \cdot L_5(2)$ to the whole group P.

4. Finding the orbits of P on sacred vectors

It turns out that there are just nine orbits of P on sacred vectors, as listed in Table 1. In this section we indicate how we found the orbits, and in the next section we indicate how we proved that this list is correct.

The two smallest orbits correspond to transpositions inside Q, which can easily be found directly, for example as the 21st powers of random elements of order 42. The corresponding sacred vectors can then be found by chopping the representation with the "Meat-axe", or by some short-cut method. Finally the orbit of the vector under B can be found with the "Meat-axe" program VP ("vector permute").

The remaining seven orbits were found by random methods. A large number (around 100,000) of sacred vectors was produced by multiplying a fixed sacred repeatedly by elements of B. Invariants of the various orbits were then obtained by finding the images of the vectors in a few P-invariant quotients of $V = V_{4370}$. We took a chain of quotients

$$V_{4370} \to V_{1894} \to V_{879} \to V_{269} \to V_{74} \to V_{34} \to V_{10} \to V_0,$$

where the subscript denotes the dimension. We found 7 sacred vectors which map to zero at different points in this chain. It follows that they are in different P-orbits. For completeness, as well as simplifying some later calculations, we also determined the P-invariant subspaces generated by each of these vectors. The results are given in the appropriate columns of Table 1.

By this stage we had enough statistical evidence to be able to estimate the lengths of most of the orbits accurately. Moreover, we used the program VP on certain quotients to obtain divisors of the orbit lengths. This information, together with the fact that all the orbit lengths are divisible by 155 and divide $|P| = 2^{30}.3^2.5.7.31$, was sufficient to conjecture the orbit lengths as given in Table 1, with a high degree of confidence. Since these numbers add up to the total number of sacred vectors, namely 13571955000, it is now sufficient to prove that they are lower bounds for the actual orbit sizes.

5. Proving that the orbit lengths are correct

For each of our nine sacred vectors, we show that it belongs to an orbit at least as big as stated in Table 1. In fact, we have already done this for the last two cases. The strategy in the remaining cases is best illustrated by an example. Let us take the seventh vector, which is conjectured to belong to an orbit of length 119040. First, we worked inside the 1348-dimensional P-invariant subspace $W = \langle v_7 \rangle^P$ generated by our sacred vector v_7. Then we found a quotient $\bar{W}$ of W in which the image $\overline{v_7}$ of v_7 has just 465 images under P. Let $\bar{P}$ denote P modulo the kernel of its action on these 465 vectors. We found words in the generators for $\bar{P}$ giving elements stabilizing $\overline{v_7}$. Then we noted that v_7 has 256 images under the same words in the corresponding generators of P. It follows that the full orbit of v_7 under P has length at least $465 \times 256 = 119040$.

The same principle applies to the other orbits, although the larger orbits were more difficult to deal with and required more than two steps.

6. Finding elements in $(Q^g \cap S) - S'$

For each of the nine orbits of S on conjugates of Q, we need to find an element of Q^g that is in S but not in S'. We adopted different strategies for the large and small orbits.

For the small orbits, where $Q^g \cap S$ is large, a random element of Q^g has a reasonable probability of being in S. Moreover, this is a very easy property to check, since it is equivalent to fixing our given sacred vector. We can then apply our check (see Section 3) to see if the element is in S', and continue until we find one which is not. By using

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some further refinements we were able to apply this method for all but the three largest orbits.

In the three large orbits, however, $Q^g \cap S$ is too small for a random method to be effective, and we more or less had to find the group $Q^g \cap S$ explicitly. In fact, it was more convenient to conjugate by g^{-1}, and look for elements of Q fixing a particular sacred vector. We worked down a series of quotient spaces similar to that used in Section 4. At each stage we had an element of Q which fixed the image of the sacred vector in the given quotient. We then moved to the next quotient and "refined" our element, so that we obtained a new element fixing the image in the new quotient. Eventually we obtained an element of Q which actually fixed the given sacred vector, and tested to see if it was in the appropriate conjugate of S'. It did not take long to find an element of the required type in each case.

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