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New characterizations of the unit vector basis of c_0 or ℓ_p

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Abstract. Motivated by Altshuler's famous characterization of the unit vector basis of c_0 or ℓ_p among symmetric bases (Altshuler [1976, *Israel Journal of Mathematics*, 24, 39–44]), we obtain similar characterizations among democratic bases and among bidemocratic bases. We also prove a separate characterization of the unit vector basis of ℓ_1.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space with dual space $(X^*, \|\cdot\|_*)$, let (e_i) be a semi-normalized basic sequence in X, and let $\alpha = \sum_{i=1}^n a_i e_i$ and $\beta = \sum_{i=1}^m b_i e_i$, where $b_m \neq 0$, belong to $\text{span}((e_i))$. Define

$$\alpha \otimes \beta = \sum_{i=1}^n a_i \left(\sum_{j=1}^m b_j e_{(i-1)m+j} \right) \tag{1.1}$$

and $\alpha \otimes 0 = 0$. Note that $(\text{span}((e_i)), \otimes)$ is a noncommutative semigroup with identity element e_1. However, the semigroup multiplication is not continuous.

If (e_i) is equivalent to the unit vector basis (u.v.b.) of c_0 or ℓ_p $(1 \leq p < \infty)$, then there exists $K > 0$ such that for all $\alpha, \beta \in \text{span}((e_i))$,

$$\frac{1}{K}\|\alpha\|\|\beta\| \leq \|\alpha \otimes \beta\| \leq K\|\alpha\|\|\beta\|. \tag{1.2}$$

The following is the main open question related to this note.

Question 1.1 Let (e_i) be a semi-normalized basis for X satisfying (1.2). Is (e_i) equivalent to the u.v.b. of c_0 or ℓ_p for some $1 \leq p < \infty$?

We prove below that Question 1.1 has a positive answer when (e_i) is either bidemocratic, almost greedy, or invariant under spreading. However, we do not know the answer in general or for other natural classes of bases such as the class of unconditional bases.

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Suppose that (e_i) is a symmetric basis for X and that both X and $[(e_i^*)]$ have a unique symmetric basic sequence up to equivalence. Altshuler [3] proved that (e_i) is equivalent to the u.v.b. of c_0 or ℓ_p. This theorem was recently extended to subsymmetric bases [6]. In Section 3, we adapt Altshuler's proof to give an answer to Question 1.1 for bidemocratic bases (see Section 2 for the definition of democratic and bidemocratic bases) by imposing on (e_i) a condition that is formally weaker than (1.2). In Section 4, we provide a new characterization of ℓ_1 which implies a solution to Question 1.1 under additional assumptions. The proof uses Szemerédi's theorem on arithmetic progressions. In Section 5, we use the latter characterization to provide an answer to Question 1.1 for the class of democratic bases.

The final section contains results about subsequences. We prove that Question 1.1 has a positive solution if (e_i) is invariant under spreading or, more generally, if every subsequence of (e_i) satisfies (1.2). Using infinite Ramsey theory, we also prove a related characterization of basic sequences which are saturated by subsequences equivalent to the u.v.b. of c_0 or ℓ_p.

2 Notation and terminology

We use standard Banach space theory notation and terminology as in [13]. We also require some terminology and results from the theory of greedy bases which we briefly review here. For further information on this topic, we refer the reader to [2, 15].

Let (e_i) be a semi-normalized basic sequence in X. For finite $A \subset \mathbb{N}$, let $\lambda(A) = \|\sum_{i\in A} e_i\|$. The *fundamental function* $(\Phi(n))$ of (e_i) is defined by

$$\Phi(n) = \sup\{\lambda(A) : |A| \le n\}.$$

We say that (e_i) is *democratic* with constant Δ if $\Phi(|A|) \le \Delta\lambda(A)$ for all finite $A \subset \mathbb{N}$. Democratic bases were introduced in [12] in order to prove the celebrated characterization of greedy bases as unconditional and democratic.

We say that (e_i) is *unconditional for constant coefficients* if there exists $C > 0$ such that, for all finite $A \subset \mathbb{N}$ and all choices of signs $\pm$,

$$\frac{1}{C}\lambda(A) \le \left\|\sum_{i\in A} \pm e_i\right\| \le C\,\lambda(A).$$

Now suppose that (e_i) is a Schauder basis for X with biorthogonal functionals $(e_i^*) \subset X^*$. Let $(\Phi^*(n))$ be the fundamental function of (e_i^*). We say that (e_i) is *bidemocratic* if

$$\Phi^*(n) \asymp \frac{n}{\Phi(n)}.$$

(We write $a_n \asymp b_n$ if there exists $C > 0$ such that $a_n/C \le b_n \le Ca_n$ for all $n \in \mathbb{N}$.) It is known that if (e_i) is bidemocratic, then both (e_i) and (e_i^*) are democratic and unconditional for constant coefficients [8, Proposition 4.2]. Moreover, every subsymmetric basis is bidemocratic [13, Proposition 3.a.6].

Remark 2.1 As is well known, if (e_i) is bidemocratic and $(\Phi(n))$ is bounded, then there exists $C > 0$ such that $\|\sum_{i=1}^{n} \pm e_i\| \le C$ for all $n \ge 1$ and for all choices of signs.

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Hence (e_i) is equivalent to the u.v.b. of c_0. On the other hand, if (e_i) is bidemocratic and $\Phi(n) \asymp n$, then $(\Phi^*(n))$ is bounded. Hence, (e_i^*) is equivalent to the u.v.b. of c_0. By duality, (e_i) is equivalent to the u.v.b. of ℓ_1.

Example 2.2 (1) For $1 < p < \infty$, let (e_i) be the u.v.b. of p-convexified Tsirelson space T_p [10]. Then (e_i) is democratic and unconditional and $\Phi(n) \asymp n^{1/p}$. Since $(n^{1/p})$ has the *upper regularity property* (see [8, p. 586]), it follows from [8, Proposition 5.4] that (e_i) is bidemocratic. Moreover, from the special properties of block bases in T_p [7, Corollary II.5], it follows that there exists $K > 0$ such that for all $\alpha, \beta \in \mathrm{span}((e_i))$,

$$\|\alpha\|\|\beta\| \le K\|\alpha \otimes \beta\|. \tag{2.1}$$

(2) Let (e_i) be a subsymmetric basis and suppose that all subsymmetric block bases of (e_i) are equivalent to (e_i). Altshuler [4] constructed the first symmetric example of this type (other than the u.v.b. of c_0 or ℓ_p). A second symmetric example was constructed in [7]. Recently, the first example of a subsymmetric basis of this type, which in addition is *not* symmetric, was constructed in [6].

It was proved in [6, Lemma 17] that if (e_i) is any subsymmetric basis of this type, then there exists $K > 0$ such that for all $\alpha^*, \beta^* \in \mathrm{span}((e_i^*))$,

$$\|\alpha^*\|_*\|\beta^*\|_* \le K\|\alpha^* \otimes \beta^*\|_*.$$

Remark 2.3 Suppose that (e_i) is a basis that satisfies (1.2) with constant K. Then, there exists a constant $\tilde{K}$ such that for every $n \in \mathbb{N}$ and $\alpha \in \mathrm{span}((e_i))$, $\alpha^* \in \mathrm{span}((e_i^*))$,

$$\|\alpha\|^n \le \tilde{K}^n\|\alpha^n\| \quad \text{and} \quad \|\alpha^*\|_*^n \le \tilde{K}^n\|\alpha^{*n}\|_*. \tag{2.2}$$

Note that the left inequality follows simply by iterating (1.2). For the right one, assume without loss of generality that (e_i) is normalized and monotone. First, make the following easy observation. If $a = \sum_{i=1}^m a_i e_i$ and $a^* = \sum_{i=1}^m b_i e_i^*$ with $a_m \neq 0$ and $b_m \neq 0$, then, for all $n \in \mathbb{N}$, $a^{*n}(a^n) = (a^*(a))^n$.

Next, take an $a^* = \sum_{i=1}^m b_i e_i^*$ and $n \in \mathbb{N}$. By monotonicity, find a norm-one $a = \sum_{i=1}^\ell a_i e_i$ with $\ell \le m$ such that $a^*(a) = \|a^*\|_*$. Assume first that $\ell < m$. For $\varepsilon > 0$, let $a_\varepsilon = a + \varepsilon e_m$ and note that $|a^*(a_\varepsilon)| \ge (1-\varepsilon)\|a^*\|_*$, while $\|a_\varepsilon\| \le 1 + \varepsilon$. Therefore,

$$(1-\varepsilon)^n\|a^*\|_*^n \le |a^*(a_\varepsilon)|^n = |a^{*n}(a_\varepsilon^n)| \le \|a^{*n}\|_*\|a_\varepsilon^n\|_* \le K^n(1+\varepsilon)^n\|a^{*n}\|,$$

and let $\varepsilon \to 0$. If $\ell = m$, the argument is slightly simpler and does not require the perturbation of a.

3 Bidemocratic bases

The main theorem of this section is the following.

Theorem 3.1 *Suppose that (e_i) is a bidemocratic basis for X that satisfies (1.2). Then (e_i) is equivalent to the u.v.b. of c_0 or ℓ_p.*

The above is an immediate consequence of the next proposition and Remark 2.3.

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Proposition 3.2 *Suppose that (e_i) is a bidemocratic basis for X that satisfies (2.2). Then (e_i) is equivalent to the u.v.b. of c_0 or ℓ_p.*

Proof For $n \geq 1$, let $\lambda(n) = \| \sum_{i=1}^{n} e_i \|$ and $\lambda^*(n) = \| \sum_{i=1}^{n} e_i^* \|_*$. Note that if $\alpha = \sum_{i=1}^{n} e_i$ and $k \geq 1$, then $\alpha^k = \sum_{i=1}^{n^k} e_i$. So (2.2) applied to α yields

$$[\lambda(n)]^k \leq K^k \lambda(n^k). \tag{3.1}$$

Similarly (2.2) applied to $\alpha^* = \sum_{i=1}^{n} e_i^*$ gives

$$[\lambda^*(n)]^k \leq K^k \lambda^*(n^k).$$

Since (e_i) is bidemocratic, there exists $C > 0$ such that

$$\frac{n}{\lambda(n)} \leq \lambda^*(n) \leq C \frac{n}{\lambda(n)}.$$

Hence,

$$\left[\frac{n}{\lambda(n)}\right]^k \leq CK^k \frac{n^k}{\lambda(n^k)},$$

i.e.,

$$\lambda(n^k) \leq CK^k [\lambda(n)]^k. \tag{3.2}$$

By the proof (see page 60) of [13, Theorem 2.a.9], (3.1) and (3.2) imply that $\lambda(n) \asymp n^{1/p}$ for some $1 \leq p \leq \infty$. Since (e_i) is bidemocratic, it follows that $\Phi(n) \asymp n^{1/p}$ and $\Phi^*(n) \asymp n^{1/q}$, where $q = p/(p-1)$. By Remark 2.1, if $p = 1$ or $p = \infty$, then (e_i) is equivalent to the u.v.b. of ℓ_1 or c_0, respectively.

So suppose that $1 < p < \infty$. Consider $\alpha = \sum_{i=1}^{m} a_i e_i \in \operatorname{span}((e_i))$. By (2.2), $\|\alpha\|^n \leq K^n \|\alpha^n\|$ for each $n \geq 1$. Note that

$$\alpha^n = \sum_{i_1+\cdots+i_m=n} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \left(\sum_{j \in A(i_1,\ldots,i_m)} e_j \right),$$

where

$$|A(i_1,\ldots,i_m)| = \binom{n}{i_1 \ldots i_m}.$$

Hence,

$$\begin{aligned} \|\alpha^n\| &\leq \sum_{i_1+\cdots+i_m=n} |a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}| \lambda(|A(i_1,\ldots,i_m)|) \\ &\leq \sum_{i_1+\cdots+i_m=n} |a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}| \Phi(|A(i_1,\ldots,i_m)|) \\ &\leq C \sum_{i_1+\cdots+i_m=n} |a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}| \binom{n}{i_1 \ldots i_m}^{1/p} \end{aligned}$$

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(for some $C > 0$)

$$\leq C(n+1)^{m/q}\left(\sum_{i_1+\cdots+i_m=n} |a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}|^p \binom{n}{i_1 \ldots i_m}\right)^{1/p}$$

by Hölder's inequality. Hence,

$$\|\alpha\|^n \leq K^n C(n+1)^{m/q}\left(\sum_{i=1}^{m} |a_i|^p\right)^{n/p}.$$

Taking the nth root and then the limit as $n \to \infty$ gives

$$\left\|\sum_{i=1}^{m} a_i e_i\right\| \leq K\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p}. \tag{3.3}$$

Since (e_i^*) also satisfies (2.2) and $\Phi^*(n) \asymp n^{1/q}$, the same argument gives

$$\left\|\sum_{i=1}^{m} a_i e_i^*\right\|_* \leq K\left(\sum_{i=1}^{m} |a_i|^q\right)^{1/q}. \tag{3.4}$$

Hence, by duality, (e_i) is equivalent to the u.v.b. of ℓ_p. ■

The following corollary replaces (2.2) by a weaker condition but adds an assumption about the fundamental function.

Corollary 3.3 *Let $1 < p < \infty$. Suppose that (e_i) is a bidemocratic basis for X such that $\Phi(n) \asymp n^{1/p}$. Suppose also that there exists $K > 0$ such that for all $\alpha \in \operatorname{span}((e_i))$ and $\alpha^* \in \operatorname{span}((e_i^*))$,*

$$\|\alpha\|^2 \leq K\|\alpha^2\| \quad \text{and} \quad \|\alpha^*\|_*^2 \leq K\|\alpha^{*2}\|_*. \tag{3.5}$$

Then (e_i) is equivalent to the u.v.b. of ℓ_p.

Proof Iteration of (3.5) yields $\|\alpha\|^n \leq K^n\|\alpha^n\|$ and $\|\alpha^*\|_*^n \leq K^n\|\alpha^{*n}\|_*$ when n is a power of 2. This is enough to prove (3.3) and (3.4) and conclude the proof as above. ■

4 A characterization of ℓ_1

Theorem 4.1 *Let (e_i) be a semi-normalized basis for X. Suppose that there exists $K > 0$ such that, for all $\alpha, \beta \in \operatorname{span}((e_i))$,*

$$\|\alpha \otimes \beta\| \leq K\|\alpha\|\|\beta\|, \tag{4.1}$$

and also that

$$\limsup \frac{1}{n} \operatorname*{Ave}_{\pm} \left\|\sum_{i=1}^{n} \pm e_i\right\| > 0. \tag{4.2}$$

Then (e_i) is equivalent to the u.v.b. of ℓ_1.

Remark 4.2 Note that (1.2) implies that $\lambda(n) \asymp n^{1/p}$ for some $1 \le p \le \infty$ (see (5.1)). If $p = 1$ and (e_i) is unconditional for constant coefficients, then (4.2) is satisfied. So Theorem 4.1 gives a positive answer to Question 1.1 in this case.

Proof We may assume that $\|e_i\| \le 1$ for all $i \ge 1$. By assumption, there exist an infinite $M \subseteq \mathbb{N}$ and $\delta > 0$ such that

$$\operatorname*{Ave}_{\pm} \left\| \sum_{i=1}^{m} \pm e_i \right\| > \delta m$$

for all $m \in M$. By Elton's "ℓ_1^n theorem" [9], there exist $\delta_1 > 0$ and $c > 0$ such that for each $m \in M$ there exists $A_m \subset \{1, 2, \ldots, m\}$, with $|A_m| \ge \delta_1 m$, such that for all scalars $(a_i)_{i \in A_m}$,

$$c \sum_{i \in A_m} |a_i| \le \left\| \sum_{i \in A_m} a_i e_i \right\| \le \sum_{i \in A_m} |a_i|.$$

Let $k \in \mathbb{N}$. By Szemerédi's theorem [14], when m is sufficiently large A_m contains an arithmetic progression of length k, $\{n_1, n_1 + d, n_1 + 2d, \ldots, n_1 + (k-1)d\}$. Let $n_1 = bd + r$, where $1 \le r \le d$. Fix scalars $(a_i)_{i=1}^k$ and $\varepsilon > 0$, and set

$$\alpha = \sum_{i=1}^{k} a_i e_{b+i} \quad \text{and} \quad \beta_\varepsilon = e_r + \varepsilon \sum_{i=r+1}^{d} e_i.$$

Note that

$$\alpha \otimes \beta_\varepsilon = \sum_{i=1}^{k} a_i e_{n_1 + (i-1)d} + \varepsilon y$$

for some $y \in \operatorname{span}((e_i))$. Thus, applying (4.1),

$$c \sum_{i=1}^{k} |a_i| - \varepsilon \|y\| \le \|\alpha \otimes \beta_\varepsilon\| \le K \|\alpha\| \|\beta_\varepsilon\| \le K(1 + (d-r)\varepsilon) \|\alpha\|.$$

Taking the limit as $\varepsilon \to 0$, we get

$$\|\alpha\| \ge \frac{c}{K} \sum_{i=1}^{k} |a_i|.$$

Let $n = \lfloor k/3 \rfloor$. There exists $s \in \mathbb{N}$ such that $[(s-1)n + 1, sn] \subset [b+1, b+k]$. We may assume that $a_n \ne 0$. Then, applying (4.1) again,

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| \ge \frac{1}{K} \left\| e_s \otimes \sum_{i=1}^{n} a_i e_i \right\| = \frac{1}{K} \left\| \sum_{i=1}^{n} a_i e_{(s-1)n+i} \right\| \ge \frac{c}{K^2} \sum_{i=1}^{n} |a_i|.$$

Since k (and hence n) are arbitrary, it follows that (e_i) is equivalent to the u.v.b. of ℓ_1. ∎

Corollary 4.3 *Suppose that (e_i) satisfies (4.1) and is unconditional for constant coefficients and that $\lambda(n) \asymp n$. Then (e_i) is equivalent to the u.v.b. of ℓ_1.*

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Remark 4.4 Corollary 4.3 does not admit an ℓ_p version for $p > 1$. To see this, let $q = p/(p-1)$. As observed in Example 2.2, the u.v.b. (e_i) of T_q^* is unconditional and bidemocratic with $\phi(n) \asymp n^{1/p}$. Moreover, (2.1) dualizes due to the special property of block bases of T_q [7, Corollary II.5]. In particular, (e_i) satisfies (4.1). Thus, Corollary 4.3 does not admit an ℓ_p version for $p > 1$.

5 Democratic bases

Democratic bases are more general than bidemocratic bases. In this section, we prove a characterization of the u.v.b. of ℓ_p or c_0 among democratic bases.

The characterization involves the notion of quasi-greedy basis which we now recall. Let (e_i) be a basis for X. For $x \in X$ and $\delta > 0$, let

$$G(x,\delta) = \{i \in \mathbb{N}: |e_i^*(x)| \geq \delta\} \quad \text{and} \quad \mathcal{G}_\delta(x) = \sum_{i \in G(x,\delta)} e_i^*(x) e_i.$$

Then (e_i) is *quasi-greedy* if there exists $A > 0$ such that for all $x \in X$ and for all $\delta > 0$, $\|\mathcal{G}_\delta(x)\| \leq A\|x\|$. Quasi-greedy bases were introduced in [12] in connection with the Thresholding Greedy Algorithm. While unconditional bases are quasi-greedy, there are also important examples of conditional quasi-greedy bases [1, 12, 16]. However, quasi-greedy bases are always unconditional for constant coefficients [16]. We refer to [8] for the definition of an *almost greedy* basis. Almost greediness of a basis is proved in [8] to be equivalent to being quasi-greedy and democratic.

Theorem 5.1 *Let (e_i) be a quasi-greedy democratic (i.e., almost greedy) basis for X which satisfies (1.2). Then (e_i) is equivalent to the u.v.b. of c_0 or ℓ_p.*

Proof By considering $\alpha = \sum_{i=1}^m e_i$ and $\beta = \sum_{i=1}^n e_i$, (1.2) implies that, for all $m, n \in \mathbb{N}$,

$$\frac{1}{K}\lambda(m)\lambda(n) \leq \lambda(mn) \leq K\lambda(m)\lambda(n). \tag{5.1}$$

Hence, by [13], $\lambda(n) \asymp n^{1/p}$ for some $1 \leq p \leq \infty$. Since (e_i) is democratic, it follows that $\Phi(n) \asymp n^{1/p}$.

If $(\Phi(n))$ is bounded, then, since (e_i) is unconditional for constant coefficients, there exists $C > 0$ such that $\|\sum_{i=1}^n \pm e_i\| \leq C$ for all $n \geq 1$ and all choices of signs. Hence, (e_i) is equivalent to the u.v.b. of c_0.

If $\phi(n) \asymp n$, then by the democratic assumption and unconditionality for constant coefficients, it follows that

$$\operatorname*{Ave}_{\pm} \left\| \sum_{i=1}^n \pm e_i \right\| \asymp n.$$

Hence, by Theorem 4.1, (e_i) is equivalent to the u.v.b. of ℓ_1.

So suppose that $1 < p < \infty$. The sequence $(n^{1/p})$ has the "upper regularity property" for $1 < p < \infty$ (see [8, p. 586]). So (e_i) is quasi-greedy and democratic and $(\Phi(n))$ has the upper regularity property. Hence, by [8, Proposition 4.4], (e_i) is bidemocratic. In particular, $\Phi^*(n) \asymp n^{1/q}$, where $q = p/(p-1)$.

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The proof is now concluded as in Proposition 3.2. More precisely, the left-hand inequality of (1.2) gives an upper p-estimate for (e_i) and (2.3) gives an upper q-estimate for (e_i^*). ■

6 Subsequences

In this section, we use results from the theory of spreading models initiated by Brunel and Sucheston [5]. We start with a brief review. A basic sequence (e_i) is *invariant under spreading* (IS) if there exists $C > 0$ such that

$$\frac{1}{C}\left\|\sum_{i=1}^{n} a_i e_i\right\| \le \left\|\sum_{j=1}^{n} a_j e_{i_j}\right\| \le C\left\|\sum_{i=1}^{n} a_i e_i\right\|$$

for all $n \ge 1$, $i_1 < \cdots < i_n$, and scalars $(a_i)_{i=1}^n$.

We say (e_i) *generates a spreading model* if there exists a Banach space $(Y, \|\cdot\|_Y)$ with basis (s_i) such that

$$\left\|\sum_{i=1}^{n} a_i s_i\right\|_Y = \lim_{\substack{i_1\to\infty \\ i_1<\cdots<i_n}} \left\|\sum_{j=1}^{n} a_j e_{i_j}\right\|$$

for all $n \ge 1$ and scalars $(a_i)_{i=1}^n$. It was proved in [5] that every basic sequence has a subsequence (x_i) which generates a spreading model. Note that (s_i) is IS with $C = 1$. Moreover, $(s_{2i} - s_{2i-1})$ is suppression 1-unconditional [5]. Note that an IS sequence (e_i) is equivalent to the spreading model (s_i) generated by a subsequence of (e_i). Hence $(e_{2i} - e_{2i-1})$ is IS and unconditional.

Theorem 6.1 *Suppose that (e_i) is IS and satisfies (1.2). Then (e_i) is equivalent to the u.v.b. of c_0 or ℓ_p.*

Proof As remarked above, $(e_{2i} - e_{2i-1})$ is IS and unconditional. Because, for every choice of scalars $a_1, \dots, a_n$,

$$\left(\sum_{i=1}^{n} a_i e_i\right) \otimes (e_2 - e_1) = \sum_{i=1}^{n} a_i (e_{2i} - e_{2i-1}),$$

(1.2) yields that $(e_i)_i$ is equivalent to $(e_{2i} - e_{2i-1})$. Therefore, $(e_i)_i$ is almost greedy and, thus, by Theorem 5.1, the result follows. ■

To state the next result, we first make some clarifying remarks to avoid a possible source of confusion. For any given semi-normalized basic sequence (e_i), we defined a multiplication $\otimes$ on span$((e_i))$ by (1.1). We must emphasize, however, that for a different choice of basic sequence, (f_i) say, the corresponding multiplication will also be different. But to avoid cumbersome notation, we use the same symbol $\otimes$ for both. Likewise, when we say in the next result that all subsequences of (e_i) satisfy (1.2), it is to be understood that the constant K in (1.2) will depend on the subsequence, i.e., we are not assuming *a priori* that there is a uniform K for all subsequences. (Of course, as the result shows, a uniform K does in fact exist.)

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Theorem 6.2 *Let (e_i) be a semi-normalized basic sequence. Suppose that every subsequence of (e_i) satisfies (1.2). Then (e_i) is equivalent to the u.v.b. of c_0 or ℓ_p.*

The proof requires the following lemma.

Lemma 6.3 *Let (e_i) be a semi-normalized basic sequence which satisfies (1.2). Then all sequences of the form $(e_{mn+i})_{i=1}^n$ $(m, n \in \mathbb{N})$ are uniformly equivalent to $(e_i)_{i=1}^n$, i.e., there exists $C > 0$ such that for all $m, n \in \mathbb{N}$ and all scalars $(a_i)_{i=1}^n$,*

$$\frac{1}{C}\left\|\sum_{i=1}^n a_i e_i\right\| \le \left\|\sum_{i=1}^n a_i e_{mn+i}\right\| \le C\left\|\sum_{i=1}^n a_i e_i\right\|.$$

Proof This follows at once from (1.2) along with the observation that

$$e_{m+1} \otimes \left(\sum_{i=1}^n a_i e_i\right) = \sum_{i=1}^n a_i e_{mn+i}.$$

■

Proof of Theorem 6.2 By Theorem 6.1, it suffices to prove that (e_i) is IS. Let (y_i) be a subsequence of (e_i) which generates a spreading model $(Y, \|\cdot\|_Y)$ with basis (s_i). We define a subsequence (x_i) of (e_i) inductively. For $1 \le i \le 3$, let $x_i = e_i$. For the inductive step, suppose that $n \ge 1$ and that $(x_i)_{i=1}^{3^n}$ have been defined with $x_i = e_{N(i)}$, where $(N(i))_{i=1}^{3^n}$ is strictly increasing. Choose $m \in \mathbb{N}$ with $m3^n > N(3^n)$ and define $x_{3^n+i} = e_{m3^n+i}$ for $1 \le i \le 3^n$. Thus, x_i has now been defined for $1 \le i \le 2 \cdot 3^n$.

Now choose $p > (m+1)3^n$ such that

$$\frac{1}{2}\left\|\sum_{i=1}^{3^n} a_i s_i\right\|_Y \le \left\|\sum_{i=1}^{3^n} a_i y_{p+i}\right\| \le 2\left\|\sum_{i=1}^{3^n} a_i s_i\right\|_Y \tag{6.1}$$

for all scalars $(a_i)_{i=1}^{3^n}$. This is possible because (y_i) generates the spreading model with basis (s_i). Define $x_{2\cdot 3^n+i} = y_{p+i}$ for $1 \le i \le 3^n$. This completes the inductive definition of $x_i = e_{N(i)}$ for $1 \le i \le 3^{n+1}$. Note that

$$N(3^n + 1) = m3^n + 1 > N(3^n)$$

and

$$N(2 \cdot 3^n + 1) \ge p + 1) > (m+1) \cdot 3^n = N(2 \cdot 3^n).$$

Hence, $(N(i))_{i=1}^{3^{n+1}}$ is strictly increasing as desired.

By assumption, (x_i) satisfies (1.2) for some $K > 0$. Hence, by Lemma 6.3, $(x_{3^n+i})_{i=1}^{3^n} = (e_{m\cdot 3^n+i})_{i=1}^{3^n}$ is uniformly equivalent to $(e_i)_{i=1}^{3^n}$. Again, by Lemma 6.3, $(x_{3^n+i})_{i=1}^{3^n}$ is uniformly equivalent to $(x_{2\cdot 3^n+i})_{i=1}^{3^n}$, which in turn is uniformly equivalent to $(s_i)_{i=1}^{3^n}$ by (6.1). So $(e_i)_{i=1}^{3^n}$ is uniformly equivalent to $(s_i)_{i=1}^{3^n}$, i.e., (e_i) is equivalent to (s_i) as desired. ■

We conclude with a characterization of basic sequences that are saturated by subsequences equivalent to the u.v.b. of c_0 or ℓ_p.

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Let $[\mathbb{N}]^\omega$ denote the collection of increasing sequences $(n_k)_{k=1}^\infty$ of natural numbers endowed with the product topology.

Theorem 6.4 *Let (e_i) be a semi-normalized basic sequence. The following are equivalent:*

(a) *Every subsequence of (e_i) contains a further subsequence equivalent to the u.v.b. of c_0 or ℓ_p.*

(b) *Every subsequence of (e_i) contains a further subsequence satisfying (1.2) for some $K > 0$.*

Proof (a) ⇒ (b) is obvious. Suppose (b) holds. Let (f_i) be any subsequence of (e_i). Let

$$B = \{(n_k)_{k=1}^\infty \in [\mathbb{N}]^\omega : (f_{n_k})_{k=1}^\infty \text{ satisfies (1.2) for some } K > 0\}.$$

Then B is easily seen to be a Borel set. Hence, by the infinite Ramsey theorem of Galvin and Prikry [11], there exists $(n_k)_{k=1}^\infty \in [\mathbb{N}]^\omega$ such that either every subsequence of $(n_k)_{k=1}^\infty$ belongs to B or every subsequence belongs to the complement of B. The latter contradicts (b). By Theorem 6.2, the former implies that (f_{n_k}) is equivalent to the u.v.b. of c_0 or ℓ_p. ∎

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