Can. J. Math., Vol. XXXV, No. 1, 1983, pp. 33–42

THE RADII OF POLYHEDRONS

T. C. ENNS

1. Introduction. Let P be a polyhedron (i.e., a 3-dimensional polytope). A *path* in P is defined as a sequence of edges $(x_1, x_2), \ldots, (x_{i-1}, x_i)$, $(x_i, x_{i-1}), \ldots, (x_{n-1}, x_n)$ where (x_i, x_{i+1}) denotes the edge with endpoints x_i and x_{i+1}. Define the *length* $|A|$ of a path A to be the number of edges of said path. The *distance* between any two vertices x and y of P is defined to be the least length of all paths of P between x and y. For the purposes of this paper, if x and y lie on a particular path A, the *distance between x and y along A* will be defined to be the length of the segment of A between x and y. The *radius* of P is defined to be the smallest integer r for which there exists a vertex v of P such that the distance from v to any other vertex of P is at most r. It has been conjectured by Jucovic and Moon (see [**1**]) that the maximum radius among all polyhedrons with n vertices $(n \geqq 6)$ is $[n/4 + 1]$ where the brackets about the value indicate the greatest integer less than or equal to said value. (Note that for $n = 4$ or 5, the maximum radius is one.) This conjecture is resolved by the following theorem.

THEOREM. *Given any integer $n \geqq 6$, the greatest value of the radii of all convex polyhedrons with n vertices is $[n/4 + 1]$.*

It is easily shown that, given any integer $n \geqq 6$, there exists a polyhedron with radius $[n/4 + 1]$. (This is done at the end of the introduction.) The difficult part of the proof is in showing that no polyhedron with $n \geqq 6$ vertices has radius greater than $[n/4 + 1]$.

The basic idea of the proof of this latter part is to show that if a polyhedron P has radius r, it must have at least $4r - 4$ vertices. This is done using the following two lemmas.

LEMMA 1. *Let P be a polyhedron and x and y be vertices of P. Then there exist three disjoint (except at x and y) paths A, B, and C connecting x and y with the following property: Given any vertex v of $(A \cup B \cup C)\backslash\{x, y\}$, v is connected to some vertex $w \notin \{x, y\}$ of one of the paths A, B, or C to which v does not belong by a path of P disjoint from $A \cup B \cup C$ except at v and w.*

LEMMA 2. *Let P be a polyhedron of radius r and let x and y be vertices of P of distance at least r apart. Let paths A, B, and C connect x and y as in Lemma 1. Then there must be enough vertices of P in addition to those of A, B, and C to total at least $4r - 4$.*

Received January 21, 1981 and in revised form October 20, 1981 and June 4, 1982.

Section 2 will prove Lemma 1 and Section 3 will prove Lemma 2. The following definitions and theorems will be used in these proofs.

The image G of the projection of a polyhedron P onto the plane of one of its faces is called a *Schlegel diagram* of the polyhedron. A graph G is called 3-*connected* if it cannot be disconnected by the removal of any two of its vertices.

THEOREM (Whitney-Menger). *A graph G is 3-connected if and only if any two vertices of G are connected by three paths in G which are disjoint except at their endpoints.*

THEOREM (Steinitz). *A planar graph G is isomorphic to the Schlegel diagram of some convex polyhedron P if and only if G is 3-connected.*

Using Steinitz's theorem, it is simple to prove that, given any integer $n \geqq 6$, there exists a polytope P with radius $[n/4 + 1]$. Note that the radius of P is the same value as the radius of any of its Schlegel diagrams. It then suffices to exhibit a 3-connected graph G with exactly n vertices and radius $[n/4 + 1]$. If n is even, a graph as illustrated in figure 1 may be used. (This is the Schlegel diagram of a prism with two $n/2$-gonal faces and is taken after [**1**].) If n is odd, a graph as indicated by figure 2 may be used.

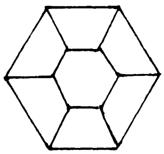

FIG. 1

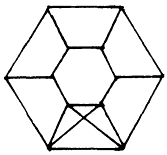

FIG. 2

2. Proof of lemma 1. This is done by noting that between any two vertices of P there are three disjoint paths and then constructing the paths mentioned in Lemma 1 from these paths. Let G be a Schlegel diagram of a convex polyhedron P and x and y be distinct vertices of G. Then by the 3-connectedness of G and the Whitney-Menger theorem there exist three paths A', B', and C' which connect x and y and which are pairwise disjoint except where they all meet x and y. Pick one of these paths, say B'. If each vertex of B' is connected to $A' \cup C'$ by a path not intersecting B' at other than an endpoint, then let $B = B'$. Otherwise, let z be the first vertex, going from x to y on B', which is not connected to A' or C' by a path not intersecting B' except at an endpoint. Let A'' be the region enclosed by B' and A' but not including B' and A' and let C'' be the region enclosed by B' and C' but not including B' and C'. Then let K_1 be the set of all vertices and edges of G connected to z by paths in A'' which intersect B' at an endpoint and no vertices other than

https://doi.org/10.4153/CJM-1983-002-3 Published online by Cambridge University Press

their endpoints. Define K_2 similarly with respect to B', C', and C''. Let w_1 and w_2 be the vertices of K_1 and K_2 respectively lying on B' and closest to x along B'. Let w_3 and w_4 be defined similarly with respect to K_1 and K_2 respectively and y.

Note that any of w_1, w_2, w_3, or w_4 may be z, but at least one, say w_1 or w_3, must be distinct from z. Let P_1' be a path in K_1 connecting w_1 and w_3 (at least one such path exists because paths in K_1 exist connecting z to w_1 and w_3, and the union of a pair of such would do). If some path K of K_1 goes outside the region bounded by P_1' and the segment of B' between w_1 and w_3 then let e and f be the vertices where K or some extension of K in K_1 meets P'. By replacing ef on P_1' by k or its extension one can denote a new path P_1'' as indicated by the heavy line in figure 3. Note that the

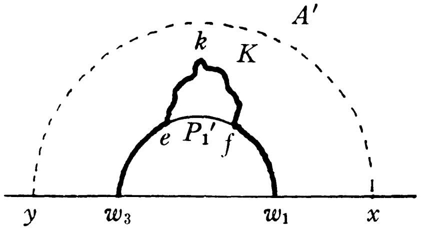

FIG. 3

region then enclosed by and including P_1'' and the segment w_1w_3 of B' contains the region so determined by the previous path P_1'. One can then so construct a path P_1 which is maximal in the sense that it includes or encloses (along with w_1w_3) all the paths in K_1. Let P_2 be similar relative to the vertices of K_2.

Assume first that w_1 or w_2, say w_1, is not the vertex z nor the vertex adjacent to z on $zx \subset B'$. If z is on P_1, replace the segment of B' between z and w_1 by the segment of P_1 between z and w_1, otherwise replace the segment of B' between w_1 and w_3 by P_1. Then P_1 has the property that any vertex t of it is connected to C' by a path not intersecting B'. To see this, first note that the segment of B' which has been replaced has, by the selection of z, at least one vertex z' which is connected to C' by a path not intersecting B'. If $t \in P_1$ is not connected to z by a path not intersecting P_1, let x_1 and x_3 be the vertices of P_1 closest to w_1 and w_3 respectively (along P_1) to which t is connected by some paths D_1 or D_2 intersecting P_1 only at their endpoints (see figure 4). Then any path S where $\{x_1, x_3\} \not\subset S$, which connects t to B' must intersect D_1 or D_2 as indicated in figure 4. Thus some path between t and $w_1w_2 \subset B'$ not intersecting P_1 between x_1 and x_3 exists (as indicated by the dark line in figure 4) a contradiction to the assumption that none such exists. Therefore t is connected to z only by paths containing x_3 or x_1. But then the removal of x_1 and x_3 disconnects

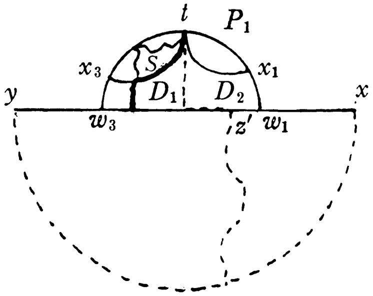

FIG. 4

t from B' and so contradicts the 3-connectedness of the graph of P. Therefore such a t is connected to B' by a path not intersecting P_1. This path can be connected to a segment of w_1w_3 with z' (given above) as an endpoint. These two paths combined, along with the previously mentioned path connecting z' to C' give the desired path connecting t to C' but not intersecting $(B'/w_1w_3) \cup P_1$ except at its endpoints as indicated by the dotted line in figure 4. From here on, a path "not intersecting" another shall mean not meeting it except at the first path's endpoint.

If neither w_1 nor w_2 are as assumed above, then either w_3 or w_4, say w_3, is distinct from z, and at least as close as the other (along B') to y. If some path not intersecting B' connects w_3w_1 to C' a situation such as that in the previous paragraph follows. If not, then some path R disjoint from B' from a vertex s on w_3z to a vertex q' on w_3y or w_1x on B' exists. (Otherwise, G would not be 3-connected, i.e., removal of w_3 and one of w_1 or w_2 would disconnect G.) Assume $q' \in w_3y \subset B'$ with no such on $w_1x \in B'$. Pick the vertex s just mentioned to be the closest such to w_1 along B' and then $q' \in B'$ the closest such along B' to y which is connected to s by a path R disjoint from B'. Make R maximal in the way that P_1 and P_2 were made maximal. If some vertex v on $w_3q' \subset B'$ is connected to A' by a path not intersecting B', then replace sq' on B' by R (see figure 5). Note that every vertex of $R \cup sw_1$ must be connected to $P_1 \cup q's$ by a path not intersecting $R \cup sw_1$ by an argument similar to the one con-

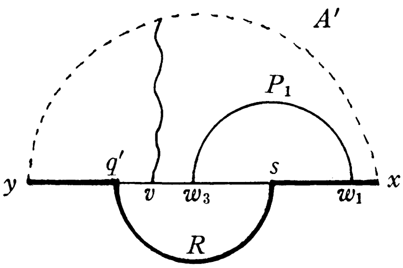

FIG. 5

cerning the vertices of P_1 previously. Replacing $q's$ by R on B', note that any vertex of $R \cup sw_1$ can then be connected to A' by a path not intersecting the new B'. If a vertex such as v above does not exist, continue constructing paths as above (making them maximal and alternating on the sides of B') until a vertex q' of one of them on B' comes between y and such a vertex v on B'. Note that such a v must exist or else removing w_1 and y would disconnect G, which is 3-connected. Replace $q'w_1 \cup B'$ by the paths constructed as P_1 or P_2 which lie on the opposite side of B' to the path from v to A' or C', along with the segments of B' which connect the consecutive paths constructed as P_1 or P_2 (see figures 6 and 7). By extensions of previous arguments, any vertex t in this newly constructed segment of B' has the desired property that any vertex of it is connected to A' or C' by a path not intersecting the new B'. If $q' \in w_1x$ on B', do the same type of construction as above.

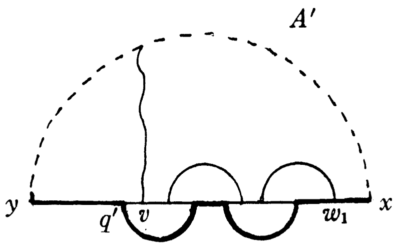

FIG. 6

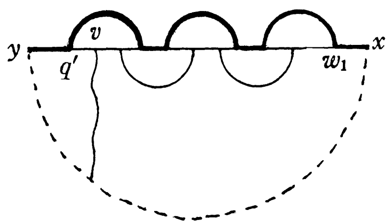

FIG. 7

After each of the above constructions has been completed, consider the vertex adjacent to q' on $q'y \subset B'$ and continue as before. When y is finally reached, designate the resulting path B. Then, by the constructions each vertex of B is connected to A' or C' by a path not intersecting B. This construction can then be done for A' and C' as well. Noting that the resulting paths are disjoint, the statement of Lemma 1 is proven.

3. Proof of lemma 2. Let r denote the radius of the polyhedron P, x and y vertices of its Schlegel diagram G a distance r apart, and A, B, and C paths between x and y as described in Lemma 1. Let a, b, and c be vertices of A, B, and C respectively which are of distances $[|A|/2]$ along A, $[|B|/2]$ along B, and $[|C|/2]$ along C respectively from x.

The strategy of the proof will be to show that if there exists some vertex k of G a distance at least r from b, then at least $4r - 4$ vertices must be constructed, that is $4r - 3 - |A| - |B| - |C|$ vertices in addition to those of A, B, and C.

Call a path D in the following constructions *proper* if it connects some vertex v ($\neq x$ or y) of A, B, or C to some other vertex v' ($\neq x$ or y) of another of the paths A, B, or C and intersects A, B, and C only at the endpoints of D. Then, by Lemma 1, there exist proper paths D, E, and F

with endpoints at a, b, and c (as previously described) respectively. The following restrictions may be made without loss of generality.

a. If one of A, B, or C is met by D, E, and F, it is B.

b. A vertex k which is a farthest vertex of $A \cup B \cup C$ from b lies on C.

Then one of the following hold.

I. D, F meet B, E meets C
II. D, F meet B, E meets A
III. D meets C, E meets A, F meets B
IV. D meets B, E meets C, F meets A

Let t be a vertex of G of distance r (the radius of G) from b, k as above, and h the distance of k from b. Then, if $h \neq r$, there exist, by the 3-connectedness of G, three disjoint paths from $A \cup B \cup C$ to t. If k is the only vertex of $A \cup B \cup C$ a distance h from b, then the total number of vertices added by paths connecting t to $A \cup B \cup C$ must be at least $3r - 3h$ (t is included). If two or three vertices of $A \cup B \cup C$ are of distance h from b, then the number of added vertices of paths connecting $A \cup B \cup C$ to t must total at least $3r - 3h - 1$ or $3r - 3h - 2$ respectively. It will then suffice to show that, if there are one, two, or three vertices of $A \cup B \cup C$ a distance h from b, at least $3h + r - |A| - |B| - |C| - 3$, $3h + r - |A| - |B| - |C| - 2$, or $3h + r - |A| - |B| - |C| - 1$ vertices respectively must be added to those which connect $A \cup B \cup C$ to t. These vertices will come from those of D, E, F and other necessary paths.

Cases I and IV, which have the common properties that D meets B and E meets C, will be looked at first. Define

$$a' = D \cap (B \cup C), b' = E \cap (A \cup C), \quad \text{and} \quad c' = F \cap (A \cup B).$$

Let k be as described above. By Lemma 1, k is connected by a proper path J to either E, $bx \subset B$, or A (see figure 8, 9a, or 10a for example). If k is so connected to E, the path indicated by the heavy line in figure 8 is of length at least h and so at least $h - 1$ vertices are added by it to those of $A \cup B \cup C$. In this and the following cases it will be assumed that the paths added do not intersect those which connect the vertex t (as described

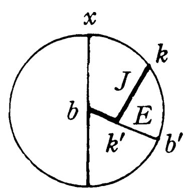

FIG. 8

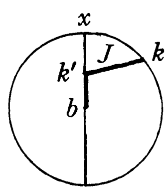

FIG. 9a

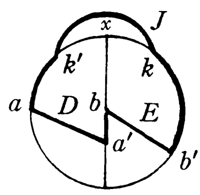

FIG. 10a

above) to $A \cup B \cup C$, as such a situation can be handled in the same manner as the cases given here. Finally note that when $h - 1$ vertices are added, at least $3h + r - |A| - |B| - |C| - 1$ are added and so by the preceding paragraph the theorem holds in this case.

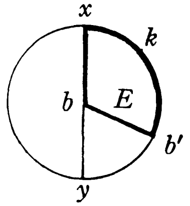

FIG. 9b

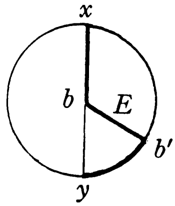

FIG. 9c

Next assume that J meets $bx \subset B$ and $k \in b'x \subset C$. The paths indicated by figures 9a, 9b, and 9c are of length at least h, $2h$, and r respectively and the corresponding inequalities are

$$|J| + |k'b| \geqq h,\ |E| + |b'x| + |xb| \geqq 2h, \text{ and}$$
$$|xb| + |E| + |b'y| \geqq r.$$

Adding two times the first inequality to the second and third inequalities, one may conclude that

$$|J| + |E| \geqq 2h + r/2 - 2[|B|/2] - |C|/2 + 1.$$

Thus at least $2h + r/2 - |B| - |C|/2 - 1 \geqq 3h + r - |A| - |B| - |C| - 1$ vertices are added and the theorem holds in this case. When J meets $ax \subset A$ and $a' \in by \subset B$, the paths indicated by the figures 9c, 10a, 10b,

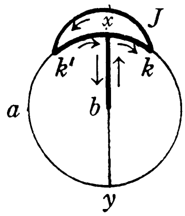

FIG. 10b

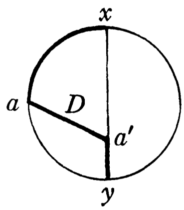

FIG. 10c

and 10c lead to the conclusion that at least $2h + r - [|A|/2] - |B|/2 - [|B|/2] - |C|/2 - 3$ vertices are added. From this one may conclude that at least $3h + r - |A| - |B| - |C| - 1$ vertices are added when $h \leqq r - 1$ and $4r - 3 - |A| - |B| - |C|$ when $r = h$, proving the theorem in this case. Finally, if J meets $ay \subset A$ and $a' \in by \subset B$, then paths as indicated by figures 9c, 10c, 11a, and 11b lead to inequalities which show that D, E, and J add at least $3h + r - |A| - |B| - |C| - 3$ vertices. Similar paths lead to the same conclusion when $a' \in bx \subset B$ as well as in the cases when $k \in b'y \subset C$.

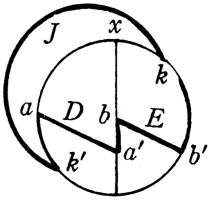

FIG. 11a

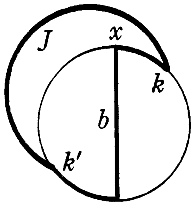

FIG. 11b

Note that cases II and III have the common property that E meets A and F meets B. Paths and calculations analogous to those preceding then apply except when J meets F. As before, assume that $k \in xc \subset C$. Then assuming J meets F, there is a vertex $w \in xc$ which is connected by a proper path ww' to F and which is the closest such to x along C. Let u be the next closest vertex along C to x.

Assuming the conditions and definitions of the above, assume further that $c' \in by \subset B$. Then u is connected by a proper path to either $xc' \subset B$, $xb' \subset A$, or $b'y \subset A$. In the first case, the circuit and paths indicated by figures 12a, 12b, and 12c yield the inequality

$$|uu'| + |ww'| + |F| \geqq h + r - |B|/2 - |C|/2 - 1.$$

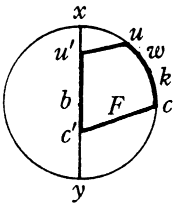

FIG. 12a

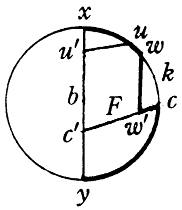

FIG. 12b

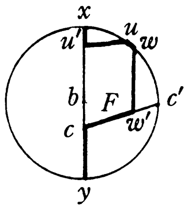

FIG. 12c

In the second case, the paths and circuits indicated by figures 13a, 13b, 13c, and 13d indicate that

$$|uu'| + |ww'| + |E| + |F| \geqq 2h + r - |A|/2 - |B|/2 - [|B|/2] - [|C|/2] - 1.$$

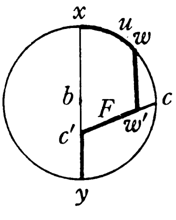

FIG. 13a

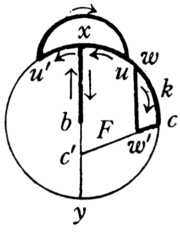

FIG. 13b

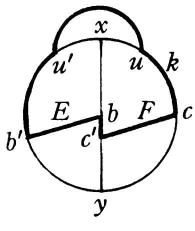

FIG. 13c

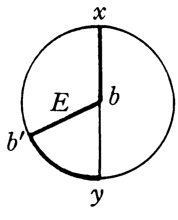

FIG. 13d

Therefore, at least $2h + r - |A|/2 - |B| - [|C|/2] - 5$ vertices are added by uu', ww', E, and F. If $h \leqq r - 2$, this means at least $3h + r - |A| - |B| - |C| - 3$ vertices are added. This suffices unless there are two or three vertices of $A \cup B \cup C$ of distance h from b. If this is the case and two or three such vertices lie on $xc \subset C$, at least one or two vertices respectively must be added to the previous number of those of E, F, or ww'. Otherwise, a vertex of distance h from b lies elsewhere on $A \cup C$ and another case holds, thus necessitating more paths and so more vertices. If $h = r = |A| = |C|$, k is the only vertex of distance d from b, and the paths above are of the minimal length given above, then the vertex next to b on $by \subset B$ is of distance $r - 1$ from all vertices of G, a contradiction since the radius of G is assumed to be r. There are then at least two or more vertices added here and the theorem holds in this case. When $u' \in b'y$, paths indicated by figures 13a, 14a, 14b, and 14c are used similarly.

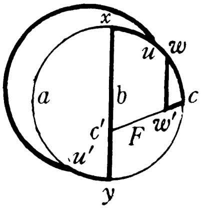

FIG. 14a

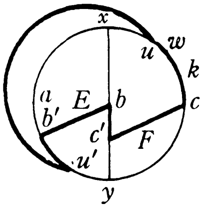

FIG. 14b

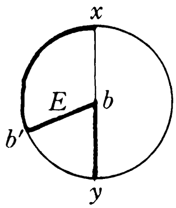

FIG. 14c

When $c' \in bx \subset B$, let w be the vertex of $cy \subset C$ closest to y along C which is connected by a proper path to F (w may be c). Let u be the vertex adjacent to w on $wy \subset C$. Then u is connected by a proper path to $c'y \subset B$, $b'x \subset A$, or $b'y \subset A$. In the first case, the paths indicated by figures 15a through 15d imply that

$$|F| + |ww'| + |uu'| \geqq 2h + r - |B|/2 - [|B|/2] - |C|/2 - [|C|/2] - 1.$$

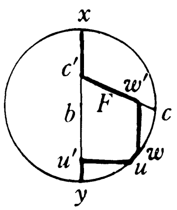

FIG. 15a

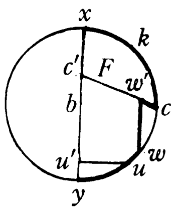

FIG. 15b

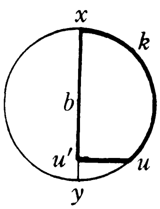

FIG. 15c

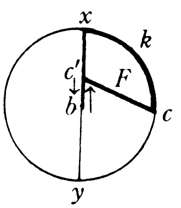

FIG. 15d

In the second case, paths as indicated by figures 13d, 15d, 16a, 16b, and 16c indicate that

$$|E| + |F| + |ww'| + |uu'| \geqq 2h + 3r/2 - |A|/2 - 2[|B|/2] - |C|.$$

https://doi.org/10.4153/CJM-1983-002-3 Published online by Cambridge University Press

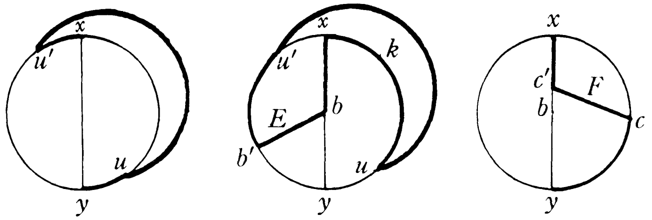

FIG. 16a FIG. 16b FIG. 16c

In the third case, the paths indicated by figures 17a, 17b, and 17c indicate that

$$|E| + |F| + |uu'| + |ww'| \geqq 2h + r/2 - |A|/2 - [|B|/2] - [|C|/2] - 1.$$

As before, note that the number of vertices added is at least $3h + r - |A| - |B| - |C| - 1$. The arguments in the other cases are similar to those above.

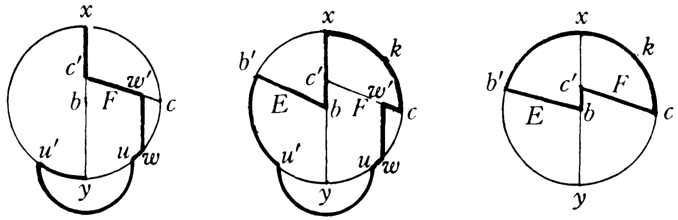

FIG. 17a FIG. 17b FIG. 17c

Finally note that in cases II and III the subcase of $k \in cy \subset C$ may be handled using similar paths and inequalities. This completes the proof of Lemma 2 as in each case it has been shown that G must have a total of at least $4r - 4$ vertices.

The fact that any polyhedron of radius r, $r \geqq 2$, must have at least $4r - 4$ vertices implies that any polyhedron with n vertices, $n \geqq 6$, must have a radius of at most $[n/4 + 1]$. This concludes the proof of the main theorem of this paper.

REFERENCE

1. E. Jucovic and J. W. Moon, *The maximum diameter of a convex polyhedron*, Mathematics Magazine (1965), 31–32.

San Diego Mesa College,
San Diego, California