

A NOTE ON THE UNCLOUDING THE SKY OF NEGATIVELY CURVED MANIFOLDS

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Abstract

The problem of finding geodesics that avoid certain obstacles in negatively curved manifolds has been studied in different situations. In this note we give a generalization of the unclouding theorem of J. Parkkonen and F. Paulin: there is a constant $s_0 = 1.534$ such that for any Hadamard manifold M with curvature ≤ -1 and for any family of disjoint balls or horoballs $\{C_a\}_{a \in A}$ and for any point $p \in M - \bigcup_{a \in A} C_a$ if we shrink these balls uniformly by s_0 one can always find a geodesic ray emanating from p that avoids the shrunk balls. It will be shown that in the theorem above one can replace the balls by arbitrary convex sets.

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0. Introduction

The problem of finding geodesics that avoid certain obstacles in negatively curved manifolds has been studied in different situations. It found applications in several areas like the existence of bounded geodesics in manifolds with finite volume [3–5] and the construction of proper closed invariant subsets of the unit tangent bundle with large footprints [1, 6].

An interesting theorem [4, Theorem 1.1] of this kind was proved recently by Parkkonen and Paulin for $CAT(-1)$ metric spaces (an earlier version of the theorem appeared in [3]). The statement in the Riemannian setting is as follows.

THE UNCLOUDING THEOREM (J. Parkkonen and F. Paulin). *There is a constant $s_0 = 1.534$ such that for any Hadamard manifold M with curvature ≤ -1 and for any family of disjoint balls or horoballs $\{C_a\}_{a \in A}$ and for any point $p \in M - \bigcup_{a \in A} C_a$ if we shrink these balls uniformly by s_0 one can always find a geodesic ray emanating from p that avoids the shrunk balls.*

Our goal is to show that in the theorem above one can replace the balls by arbitrary convex sets. To state our theorem first we have to define the meaning of ‘shrinking a convex set’.

Let M be an n -dimensional ($n \geq 2$) complete and simply connected Riemannian manifold with sectional curvatures ≤ -1 . For a convex subset $C \subset M$ and a positive number $s > 0$ we define the s -kernel of C as

$$C(s) = \{x \in C : B(x, s) \subset C\},$$

where $B(x, s)$ denotes the open geodesic ball of radius s and centre x . It is easy to see that the s -kernel is always closed, and possibly empty. Then we have the following result.

THEOREM. *There is a constant $s_0 > 0$ such that for any $s > s_0$ and for any family of disjoint open convex sets $C_n \subset M$, $n \in A$, and for any point $x \in M - \bigcup_{n \in A} C_n$ there is a geodesic ray emanating from x and avoiding $C_n(s)$ for all $n \in A$.*

The constant s_0 is chosen such that the area of the geodesic ball $B(s_0)$ of radius s_0 in $\mathbb{H}^2(-1)$ is 2π . Since the area of

$$B(r) = 2\pi(\cosh(r) - 1)$$

(see [2, Chapter 3]), this yields

$$s_0 = \cosh^{-1}(2) < 1.4,$$

which is slightly better than the constant given in [4].

To understand why such a statement may be true it is helpful to consider the following example. Let us assume that M is a surface. Fix a point p in M and let us try to construct a counterexample, that is a family of disjoint open and convex sets not containing p such that the s -kernels block all the rays from p . It is logical to try to find ‘large’ convex sets that block a lot of rays. The best candidates are half-spaces (one side of geodesics). Their s -kernels block the same rays as the half-spaces themselves. So let us assume that we have n points q_1, \dots, q_n , on the ideal boundary S arranged in a cyclic order (set $q_{n+1} = q_1$) and let $\gamma_{i,j}$ be the geodesic connecting q_i to q_j . Let D be the connected component of $M - \bigcup_{i=1}^n \gamma_{i,i+1}$ that contains p and assume that the boundary of D contains all the geodesics $\gamma_{1,2}, \gamma_{2,3}, \dots, \gamma_{n,1}$. Set C_i be the convex half-space bounded by $\gamma_{i,i+1}$ that does not contain p . Then with the exception of the rays γ_{p,q_i} all the rays from p are blocked by $C_i(s_0)$. To finish the counterexample we need to find disjoint convex sets in D that will block the rays γ_{p,q_i} , $i = 1, 2, \dots, n$. The smallest convex sets are geodesic balls; therefore we need to place a geodesic ball of radius s_0 (so that the s_0 -kernels of the balls are not empty) centred on the rays γ_{p,q_i} , $i = 1, 2, \dots, n$. But that is clearly impossible. From the Gauss–Bonnet theorem we have $\text{Area}(D) \leq (n-2)\pi$. On the other hand the area comparison theorem (see for example [2, Theorem 3.7]) yields that the total area of the balls is at least $2n\pi$.

As far as this example is concerned we could have chosen the constant s_0 such that $\text{Area}(B(s_0)) = \pi$ in $\mathbb{H}^2(-1)$ but in the actual proof we need the larger constant.

1. Proof of the theorem

Denote by S the sphere at infinity (viewed as the set of equivalence classes of geodesic rays) and equip $M \cup S$ with the usual truncated cone topology. Then S is homeomorphic to the sphere and $M \cup S$ is homeomorphic to the closed ball.

Define the projection map $\pi : M \cup S - \{p\} \rightarrow S$ such that $\pi(q) = [\gamma_{pq}]$, where γ_{pq} denotes the geodesic ray starting from p and passing through q , and for a geodesic ray γ , $[\gamma] \in S$ denotes the equivalence class. The following is easy to show.

PROPOSITION 1. *If $D \subset M$ is open, then $\pi(D)$ is an open set in S , and if $F \subset M$ is closed and bounded, then $\pi(F)$ is closed in S .*

The proof is obvious.

We shall need the following technical definition. Let $p \in M$ be a fixed point and we call a subset $D \subset M \cup S - \{p\}$ a cone (with vertex p) if $q \in D$ implies that

$$\gamma_{pq} \cup \{\gamma_{pq}\} \subset D \cup \{p\}.$$

Since $p \in M$ will be fixed throughout the paper we shall sometimes call it a cone without referring to the vertex p .

Similarly for any set $D \subset M \cup S$ not containing p we define the cone $\text{Cone}(D)$ of the set D as

$$\text{Cone}(D) = \{q \in M \cup S - \{p\} : \gamma_{pq} \cap D \neq \emptyset\}.$$

Clearly the cone of an open set is also open. For of any set $D \subset M \cup S - \{p\}$ the cone naturally decomposes into three disjoint sets:

$$\begin{aligned} \text{Cone}^+(D) &= \{q \in \text{Cone}(D) : [p, q] \cap D = \emptyset\}, \\ \text{Cone}^-(D) &= \{q \in \text{Cone}(D) - D : [p, q] \cap D \neq \emptyset\} \quad \text{and} \quad D. \end{aligned}$$

Here $[p, q]$ denotes the geodesic segment connecting p and q .

Since the intersection of a cone and a convex set may not be convex or connected even if the cone is a cone over some convex set, the following technical proposition will be helpful.

PROPOSITION 2. *Let D be a convex set not containing the point p and K be some cone based at p . Let*

$$D \cap K = \bigcup_{i \in A} D_i,$$

where A is some index set and D_i are the connected components of $D \cap K$. Then

$$\pi(D_i) \cap \pi(D_j) = \emptyset,$$

for all $i \neq j$.

PROOF. Suppose it is not true and

$$\pi(D_i) \cap \pi(D_j) \neq \emptyset,$$

for some i and j . Let

$$q \in \pi(D_i) \cap \pi(D_j)$$

be any point and let $q_i \in D_i$ and $q_j \in D_j$ be points such that $\pi(q_i) = \pi(q_j) = q$. But then the geodesic segment $[q_i, q_j] \subset D \cap K$ and connects D_i and D_j ; therefore $D_i \cup D_j$ is connected and that is a contradiction. \square

The next proposition is also simple.

PROPOSITION 3. *Let $C \subset M$ be a convex open set not containing p , let K be some cone with vertex p and set $D = K \cap C$. Suppose there are points $x \in \text{Cone}^+(D)$ and $y \in \text{Cone}^-(D)$ such that there is a curve $\alpha : [0, 1] \rightarrow \text{Cone}(D)$ with $\alpha(0) = x$ and $\alpha(1) = y$. Then α intersects D .*

PROOF. For simplicity denote by $[q, \infty)$ the infinite geodesic segment $\gamma_{p,q} - [p, q)$. Suppose α does not intersect D . Then we can sort the points of $[0, 1]$ into two sets defined as follows:

$$R_1 = \{z \in [0, 1] : [p, \alpha(z)] \cap D \neq \emptyset\} \quad \text{and} \quad R_2 = \{z \in [0, 1] : [\alpha(z), \infty) \cap D \neq \emptyset\}.$$

From the definition above we have that $x \in R_2$ and $y \in R_1$. Since $\alpha([0, 1]) \subset \text{Cone}(D)$ it is clear that $R_1 \cup R_2 = [0, 1]$. From the assumption that $\alpha([0, 1]) \cap D = \emptyset$ and from the convexity of D we obtain that $R_1 \cap R_2 = \emptyset$. Finally since D is open (in the topology of K) and the conditions defining the sets R_1 and R_2 are open conditions we conclude that R_1 and R_2 are open subsets of $[0, 1]$. But this contradicts the fact that $[0, 1]$ is connected. \square

The following elementary lemma is crucial to the proof.

LEMMA 4. *Let M be an n -dimensional complete and simply connected Riemannian manifold with sectional curvatures ≤ -1 . Consider the geodesic triangle Δ_{pqr} , where $p \in M$ and $q, r \in M \cup S$. Suppose that there is a point $x \in [p, q]$ such that $B(x, s_0) \cap [p, r] = \emptyset$. Then the geodesic segment $[r, q]$ intersects $B(x, s_0)$.*

There are probably many different ways to prove this but let us argue in the spirit of the example given in the Introduction.

PROOF. For every $t \in [r, q]$ consider the segment $[p, t]$. The union of these segments is a ruled surface $F = \bigcup_{t \in [r, q]} [p, t]$. Since the second fundamental form of F cannot be definite with respect to any normal, by the Gauss theorem we conclude that the sectional curvature of F (with the metric inherited from M) is ≤ -1 .

Let us assume that on the contrary $[r, q] \cap B(x, s_0) = \emptyset$. Let

$$B_F(x, s_0) = \{y \in F : \text{dist}_F(x, y) \leq s_0\}$$

be the geodesic ball of radius s_0 and centre x on the surface F . Observe that

$$B_F(x, s_0) \subset B(x, s_0) \cap F.$$

Therefore we have $[r, q] \cap B_F(x, s_0) = \emptyset$, where the geodesic segment $[r, q]$ considered as a geodesic segment in M is also a geodesic segment in F . From the Gauss–Bonnet formula applied for the surface F we have that the F -area of the triangle Δ_{pqr} is less than π . On the other hand the triangle Δ_{pqr} contains a half-ball $B_F(x, s_0)$ whose F -area is greater than or equal to π by the standard area comparison principle [3, Theorem 3.7] and that is a contradiction. \square

The following proposition also plays an important role.

PROPOSITION 5. *Let M be an n -dimensional complete and simply connected Riemannian manifold with sectional curvatures ≤ -1 . Let C be an open convex set with $C(s) \neq \emptyset$ for some $s \geq s_0$. Suppose that there is a disjoint open convex set D such that there is a point $q \in D$ with the property that the geodesic segment $[p, q]$ intersects $C(s)$. Then $\text{dist}(p, D) > \text{dist}(p, C)$.*

PROOF. Suppose the statement is not true and let us assume that

$$\text{dist}(p, D) \leq \text{dist}(p, C).$$

Let $x \in [p, q] \cap C(s)$ be any point and $r \in \partial D$ be a point such that

$$\text{dist}(r) \leq \text{dist}(p, C).$$

Consider the geodesic ball $B(x, s)$ and the cone $\text{Cone}(B(x, s))$. Then

$$\text{either } r \in \text{Cone}(B(x, s)) \text{ or } r \notin \text{Cone}(B(x, s)).$$

If $r \notin \text{Cone}(B(x, s))$, then the geodesic segment $[p, r]$ does not intersect $B(x, s)$ and Lemma 4 implies that the geodesic segment $[r, q]$ intersects $B(x, s)$. But that is a contradiction since the geodesic segment $[r, q] \subset D$ and $B(x, s) \subset C$ and the sets D, C were disjoint.

If $r \in \text{Cone}(B(x, s))$ then $r \in \text{Cone}^+(B(x, s))$ since

$$\text{dist}(p, r) \leq \text{dist}(p, C) \leq \text{dist}(p, B(x, s)),$$

and obviously $q \in \text{Cone}^-(B(x, s))$. Now, if the geodesic segment $[r, q] \subset \text{Cone}(B(x, s))$, then by Proposition 3 we have that $[r, q]$ intersects $B(x, s)$ and the contradiction arises the same way as before. On the other hand if the geodesic segment $[r, q]$ leaves the cone then there is a point $r' \in [r, q] \subset D$ such that $r' \notin \text{Cone}(B(x, s))$ and we can repeat the previous argument with r' in place of r to obtain a contradiction. This concludes the proof of the proposition. \square

PROOF OF THEOREM. First we show that it is enough to prove the theorem for finitely many bounded convex sets.

Let us assume on the contrary that we have a family $C_n, n \in A$, of disjoint open convex sets, a constant $s > s_0$ and a point $p \in M - \bigcup_{n \in A} C_n$ such that every geodesic ray emanating from p intersects $C_n(s)$ for some $n \in A$.

Let $s_0 < s' < s$. Then clearly for any set D and for any $q \in D(s)$,

$$B(q, s - s') \subset D(s').$$

Since

$$\bigcup_{n \in A} \pi(C_n(s)) = S,$$

we have

$$\bigcup_{q \in \bigcup\{C_n(s) : n \in A\}} \pi(B(q, s - s')) = S.$$

Since the projection of an open set is open and S is compact there are finitely many points $q_i, i = 1, \dots, m$, such that $q_i \in C_n(s)$ for some $n \in A$ and

$$\bigcup_{i=1}^m \pi(B(q_i, s - s')) = S.$$

Let

$$l = \max\{\text{dist}(p, q_i) : i = 1, \dots, m\} + s + 1,$$

and $A' \subset A$ such that $n \in A'$ if $q_i \in C_n(s)$ for some $i = 1, \dots, m$. Clearly A' is a finite subset of A .

For $n \in A'$ set $C'_n = C_n \cap B(p, l)$. Clearly C'_n is bounded open and convex, and if $q_i \in C_n(s)$, then $q_i \in C'_n(s)$ as well. Therefore $B(q_i, s - s') \subset C'_n(s')$ and

$$\bigcup_{n \in A'} \pi(C'_n(s')) = S.$$

To summarize: we have produced a finite family of disjoint bounded open convex sets $C'_n, n \in A'$, and a positive number $s_0 < s'$ such that every geodesic ray emanating from p intersects $C'_n(s')$ for some $n \in A'$.

Therefore it is enough to show that the theorem is valid for such a family and from now on we assume that we have a finite family $C_i, i = 1, \dots, n$, of disjoint, bounded, open and convex sets, a constant $s > s_0$ and a point $p \in M - \bigcup_{i=1}^n C_n$ such that every geodesic ray emanating from p intersects $C_i(s)$ for some $i = 1, \dots, n$, that is $\bigcup_{i=1}^n \pi(C_i(s)) = S$.

We are going to select different convex sets

$$C_{a_1}, C_{a_2}, \dots, C_{a_i}, \dots$$

from the family $\{C_j\}_{j=1}^n$ and construct subsets

$$S \supset S_1 \supset S_2 \cdots \supset S_i \supset \cdots$$

that are open and connected in the topology of S such that the following properties are satisfied:

- (1) $\pi(C_{a_i}(s)) \cap S_{i-1} \neq \emptyset$;
- (2) $S_i \subset S_{i-1} \cap (\pi(C_{a_i}) - \pi(C_{a_i}(s)))$ and there are points $q_i \in C_{a_i}(s)$ and $r_i \in \partial S_i$ such that $\pi(q_i) \in \partial S_i$ and $[p, r_i] \cap B(q_i, s) = \emptyset$;
- (3) $D_i = C_{a_i} \cap \text{Cone}(S_i)$ is open, $\text{Cone}(D_i) = \text{Cone}(S_i)$ and $\text{Cone}^+(D_i) \cap C_j(s) = \emptyset$ for all $j = 1, \dots, n$; and
- (4) $S_i \cap \pi(C_{a_j}(s)) = \emptyset$ for all $j \leq i$.

Our goal is to show that this process never stops and this is a contradiction since we have only finitely many C_i to select from.

We start by selecting the first set. Let C_{a_1} be the set of the family $\{C_j\}_{j=1}^n$ with the smallest distance from p . Since the family is finite there has to be a set with the smallest distance, and if there is more than one set with minimal distance, choose one of them.

Next we construct the set S_1 satisfying the second property. Since C_{a_1} is open and $C_{a_1}(s)$ is a closed bounded set from Proposition 1 we obtain that

$$\pi(C_{a_1}) - \pi(C_{a_1}(s))$$

is open and since S is locally connected all connected components are open as well. Let S_1 be a connected component of

$$\pi(C_{a_1}) - \pi(C_{a_1}(s))$$

such that

$$\partial(S_1) \cap \pi(C_{a_1}(s)) \neq \emptyset \quad \text{and} \quad \partial(S_1) \cap \partial\pi(C_{a_1}) \neq \emptyset.$$

Naturally we have to show that such component exists.

To this end we show that if S_1 is any connected component of $\pi(C_{a_1}) - \pi(C_{a_1}(s))$ we have $\partial S_1 \cap \pi(C_{a_1}(s)) \neq \emptyset$. Suppose it is not true. Since $\pi(C_{a_1}(s))$ is a closed subset of S and therefore compact there is an open set $B \subset S$ such that $B \cap S_1 = \emptyset$ and $\pi(C_{a_1}(s)) \subset B$. Since S_1 was a connected component of

$$\pi(C_{a_1}) - \pi(C_{a_1}(s)),$$

we have

$$(\pi(C_{a_1}) - \pi(C_{a_1}(s))) - S_1$$

is also open, and therefore

$$B' = B \cup ((\pi(C_{a_1}) - \pi(C_{a_1}(s))) - S_1)$$

is open and disjoint from S_1 . Since

$$\pi(C_{a_1}) \subset B' \cup S_1$$

and B' and S_1 are disjoint we have decomposed $\pi(C_{a_1})$ into the union of two disjoint open sets,

$$\pi(C_{a_1}) = (\pi(C_{a_1}) \cap S_1) \cup (\pi(C_{a_1}) \cap B'),$$

and that is a contradiction since $\pi(C_{a_1})$ as the projection of a connected set must be connected.

We also have to show that there is a connected component S_1 of

$$\pi(C_{a_1}) - \pi(C_{a_1}(s))$$

such that

$$\partial(S_1) \cap \partial\pi(C_{a_1}) \neq \emptyset.$$

This is quite straightforward. Let $r_1 \in \partial\pi(C_{a_1})$ be any point. Since C_{a_1} is bounded the infinite geodesic segment $[p, r_1]$ intersects the boundary of C_{a_1} , and if $x \in [p, r_1] \cap \partial C_{a_1}$ is an intersection point then there is an $\varepsilon > 0$ such that

$$\pi(B(x, \varepsilon)) \cap \pi C_{a_1}(s) = \emptyset \quad (C_{a_1}(s) \text{ is a compact set}).$$

Then $B(x, \varepsilon) \cap C_{a_1}$ is open and convex, $r_1 \in \partial\pi(B(x, \varepsilon) \cap C_{a_1})$ and

$$\pi(B(x, \varepsilon) \cap C_{a_1}) \subset \pi(C_{a_1}) - \pi(C_{a_1}(s))$$

is an open and connected set (the projection of a connected set is connected). Choose S_1 to be the connected component that contains $\pi(B(x, \varepsilon) \cap C_{a_1})$.

Since

$$\partial S_1 \cap \pi(C_{a_1}(s)) \neq \emptyset,$$

let $q_1 \in C_{a_1}(s)$ be any point such that

$$\pi(q_1) \in \partial S_1 \cap \pi(C_{a_1}(s)).$$

Such a point must exist because $C_{a_1}(s)$ is a compact set. Then $[p, r_1] \cap B(q_1, s) = \emptyset$ otherwise r_1 is in the interior of $\pi(B(q_1, s))$, and since $B(q_1, s) \subset C_{a_1}$ the point r_1 would belong to the interior of $\pi(C_{a_1})$ and that contradicts the choice of r_1 . Therefore the set S_1 and the points q_1, r_1 satisfy the second property.

Set $D_1 = C_{a_1} \cap \text{Cone}(S_1)$. This is open since it is the intersection of two open sets, and since $S_1 \subset \pi(C_{a_1})$ one obtains that $\text{Cone}(D_1) = \text{Cone}(S_1)$. The important part is to show that $\text{Cone}^+(D_1) \cap C_j(s) = \emptyset$ for all $j = 1, \dots, n$. Suppose it is not true and there is a C_j such that $\text{Cone}^+(D_1) \cap C_j(s) \neq \emptyset$. This means that there is

a point $q \in D_1 \subset C_{a_1}$ such that the geodesic segment $[p, q]$ intersects $C_j(s)$. Then Proposition 5 implies that $\text{dist}(p, C_{a_1}) > \text{dist}(p, C_j)$ but that is a contradiction since C_{a_1} was chosen with minimal distance from p . This verifies the third property.

Finally the fourth property follows easily from the fact that

$$S_1 \subset \pi(C_{a_1}) - \pi(C_{a_1}(s)).$$

Let us now assume that we have chosen C_{a_1}, \dots, C_{a_i} and constructed the sets S_1, \dots, S_i satisfying the four properties. Let $A \subset \{1, \dots, n\}$ be the set such that if $a \in A$, then

$$\pi(C_a(s)) \cap S_i \neq \emptyset.$$

Since

$$\bigcup_{i=1}^n \pi(C_i(s)) = S,$$

the set $A \neq \emptyset$. Choose $C_{a_{i+1}}$ from the family $\{C_a\}_{a \in A}$ with the minimal distance from p . Again if there is more than one such set, choose one of them. It is clear that the first property is satisfied.

We now deal with the construction of S_{i+1} and the verification of the second property. Let $C'_{a_{i+1}}$ be a connected component of $C_{a_{i+1}} \cap \text{Cone}(D_i)$ such that

$$C_{a_{i+1}}(s) \cap C'_{a_{i+1}} \neq \emptyset.$$

Since $C'_{a_{i+1}}$ is connected from Proposition 3 we obtain that

$$C'_{a_{i+1}} \subset \text{Cone}^+(D_i) \quad \text{or} \quad C'_{a_{i+1}} \subset \text{Cone}^-(D_i).$$

Taking into consideration the third property we obtain that $C'_{a_{i+1}} \subset \text{Cone}^-(D_i)$.

The set S_{i+1} is going to be a connected component of

$$S_i \cap (\pi(C'_{a_{i+1}}) - \pi(C_{a_{i+1}}(s))).$$

First we show that $S_i \not\subset \pi(C'_{a_{i+1}})$. Suppose it is not true. Then there are points $q_i \in C_{a_i}(s)$ and

$$r_i \in \partial S_i$$

satisfying the second property and

$$r_i, \pi(q_i) \in \text{Cl}(\pi(C'_{a_{i+1}})).$$

Since $C'_{a_{i+1}}$ is bounded one can find points

$$q' \in \text{Cl}(C'_{a_{i+1}}) \quad \text{and} \quad r' \in \text{Cl}(C'_{a_{i+1}})$$

such that $\pi(q') = \pi(q_i)$ and $\pi(r') = \pi(r_i)$. Since $C'_{a_{i+1}} \subset \text{Cone}^-(D_i)$ we have that $q_i \in [p, q']$. By the second property the geodesic segment $[p, r'] \subset [p, r_i]$ does not intersect the ball $B(q_i, s)$. Applying Lemma 4 we obtain that

$$[r', q'] \subset \text{Cl}(C'_{a_{i+1}}) \subset \text{Cl}(C_{a_{i+1}})$$

intersects $B(q_i, s) \subset C_{a_i}$. Therefore the convex sets $C_{a_{i+1}}$ and C_{a_i} intersect and that is a contradiction.

Let

$$r \in S_i - \pi(C'_{a_{i+1}}) \quad \text{and} \quad q \in C'_{a_{i+1}} \cap C_{a_{i+1}}(s)$$

be arbitrary points and let $\alpha : [0, 1] \rightarrow S_i$ be a curve such that $\alpha(0) = r$ and $\alpha(1) = \pi(q)$. Let

$$t_1 = \inf\{t \in [0, 1] : \alpha([t, 1]) \subset \pi(C'_{a_{i+1}})\}$$

and

$$t_2 = \inf\{t \in [0, 1] : \alpha(t) \in \pi(C'_{a_{i+1}} \cap C_{a_{i+1}}(s)) \text{ and } \alpha([t, 1]) \subset \pi(C'_{a_{i+1}})\}.$$

Since $\pi(C'_{a_{i+1}})$ is an open neighbourhood of $\pi(C'_{a_{i+1}} \cap C_{a_{i+1}}(s))$ in the topology of S_i it is clear that $t_1 < t_2$. Let S_{i+1} be the connected component of

$$S_i \cap (\pi(C'_{a_{i+1}}) - \pi(C_{a_{i+1}}(s)))$$

that contains the curve $\alpha(t_1, t_2)$ and set $r_{i+1} = \alpha(t_1)$, and let

$$q_{i+1} \in C'_{a_{i+1}} \cap C_{a_{i+1}}(s)$$

be any point such that $\pi(q_{i+1}) = \alpha(t_2)$. Such a point must exist since $C_{a_{i+1}}(s)$ is compact.

We must show now that the second property is satisfied with this choice. Clearly S_{i+1} is open and connected in S . It is also clear that

$$r_{i+1} \in \partial\pi(C'_{a_{i+1}}),$$

and therefore $r_{i+1} \in \partial S_{i+1}$. If

$$B(q_{i+1}, s) \cap [p, r_{i+1}] \neq \emptyset,$$

that would mean that

$$r_{i+1} \in \pi(B(q_{i+1}, s)).$$

Denote by D' the connected component of $C_{a_{i+1}} \cap \text{Cone}(D_i)$ such that $r_{i+1} \in \pi(D')$. Taking into account the fact that D' and therefore $\pi(D')$ are open we have that

$$\pi(D') \cap \pi(C'_{a_{i+1}}) \neq \emptyset.$$

Since by Proposition 5 the projections of different connected components must be disjoint we conclude that

$$D' = C'_{a_{i+1}} \quad \text{and} \quad r_{i+1} \in \text{Int}(C'_{a_{i+1}}).$$

That contradicts the fact that $r_{i+1} \in \partial\pi(C'_{a_{i+1}})$.

We now move on to verification of the third property. Define the set

$$D_{i+1} = \text{Cone}(S_{i+1}) \cap C_{a_{i+1}}.$$

Since by the construction $S_{i+1} \subset \pi(C_{a_{i+1}})$,

$$\text{Cone}(D_{i+1}) = \text{Cone}(S_{i+1}).$$

Suppose there is a set C_j such that

$$\text{Cone}^+(D_{i+1}) \cap C_j(s) \neq \emptyset.$$

Then on the one hand by Proposition 5

$$\text{dist}(p, C_{a_{i+1}}) > \text{dist}(p, C_j),$$

and on the other hand

$$\pi(C_j(s)) \cap S_i \neq \emptyset.$$

Therefore $j \in A$ and this is a contradiction since $C_{a_{i+1}}$ was chosen out of the sets $\{C_a\}_{a \in A}$ with the minimum distance.

Now we verify the fourth property. The set S_{i+1} was constructed to be a connected component of

$$S_i \bigcap (\pi(C'_{a_{i+1}}) - \pi(C_{a_{i+1}}(s))),$$

which implies that

$$S_{i+1} \cap \pi(C_{a_{i+1}}(s)) = \emptyset.$$

On the other hand since $S_{i+1} \subset S_j$ for $j \leq i$ and from the fourth property we know that

$$S_j \cap \pi(C_{a_j}(s)) = \emptyset,$$

for all $j \leq i$ we obtain

$$S_{i+1} \cap \pi(C_{a_j}(s)) = \emptyset,$$

for all $j \leq i + 1$.

Finally, combining the first and the last properties one obtains that the selected sets are all different. This shows that the construction never stops, which is a contradiction and the proof of the theorem is complete.

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