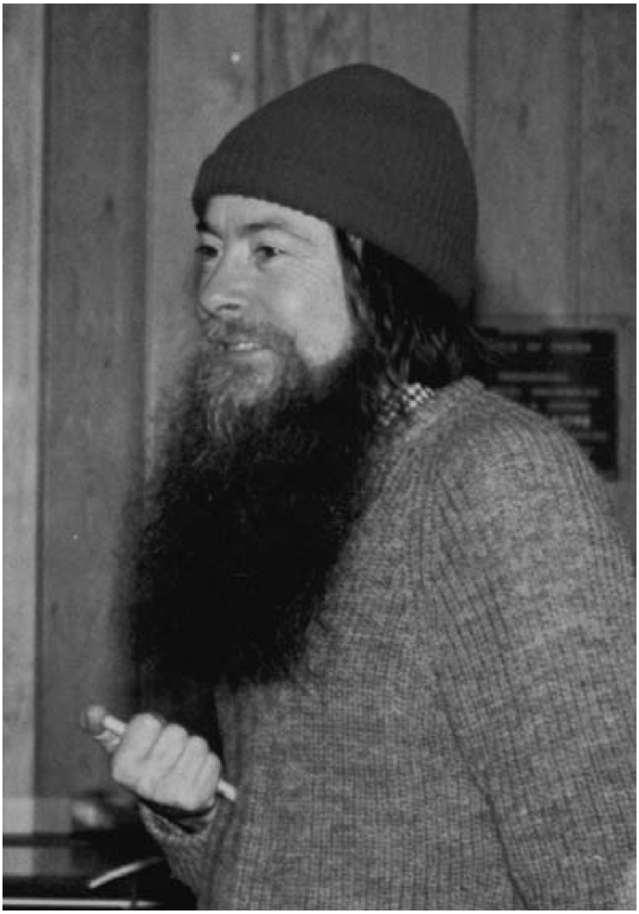
Bull. London Math. Soc. 36 (2004) 695–710
DOI: 10.1112/S0024609304003406

OBITUARY

JOHN HAWKES (1944–2001)



In his curriculum vitae, John Hawkes lists his research interests as geometric measure theory, probability (Lévy processes), and potential theory (probabilistic). In fact, he made significant contributions to all three areas, and there are strong relationships between them. He used both geometric measure theory and potential theory as tools for his study of the trajectories of particular Lévy processes, but in many cases he needed to develop the tool before it was ready to be used. We will summarise his research later, but first we discuss what is known of his life history.

John was born in Leeds on 28 February 1944; little is known of his early life. He attended a primary school near Elland Road football ground, and he was the only pupil in his year at that school to pass his 11-plus selection test; this gave him entry to Leeds Grammar School (which at that time was still a state grammar school). As a teenager he played chess for his school and entered the adult regional chess competitions; some of those against whom he competed in chess recognised his outstanding talents and suggested that he should go to university, a possibility that had not previously occurred to him. One of his mathematics teachers suggested he should do a mathematics degree, and his application to Kings College, London was successful. There he was influenced by Rex Tims, a student of Littlewood, who taught him analysis and inspired him to become interested in precise analytical thinking. He graduated from Kings College in 1966 with a first class honours degree.

From Kings, John moved to Westfield College, and I took him on as a research student working towards a PhD. Conditions in London were good for students in Analysis/Probability at this time, because they could be introduced to a working analysis seminar at University College under the leadership of Ambrose Rogers, and a seminar in probability at Imperial College under the leadership of Harry Reuter; in addition, increasing numbers of active mathematicians from the USA and elsewhere were coming to London for their sabbatical years. In most respects, John was a model research student. He was a natural mathematician who set himself high standards of precise thinking. He had good taste in choosing interesting problems and conjectures, which he would then try to prove or disprove. I shared responsibility for his supervision with Harry Reuter; John wrote an outstanding thesis and was awarded a PhD by London University in 1969.

John's first teaching post was at the University of Michigan, Ann Arbor; when his contract was completed, he was encouraged to stay on to contend for a tenured post, but decided to return to the UK in 1971. He held temporary posts first at Imperial College and then at University College, London. (This was the first of many periods of financial austerity in UK universities, which made tenured posts almost unavailable.) John was married in 1973 to Jennifer Le Ruez, whom he had met when they were both students at Westfield. They moved to the University College of Swansea in 1974, where John first had an appointment in the Department of Statistics before moving to Mathematics in 1983, first as Reader and then as Professor from 1988 until his (early) retirement on health grounds in 1997, at which point he moved back to London.

Throughout his career, John developed relationships with other working researchers who shared a common interest. He was a welcome contributor to Symposia, and often visited other departments to give colloquium talks. During his time in Swansea he had two sabbaticals; the first in 1978–79, when he was employed as research scientist at the IBM research centre near New York, working with Benoit Mandelbrot on the mathematics of fractals, and the second in 1989–90, when he was Visiting Professor at the University of British Columbia, working with Ed Perkins on the theory of branching processes. He made numerous shorter visits including: Stanford University (1976) to visit Kai Lai Chung, Moscow State University (1977) to visit B. V. Gnedenko, Free University of Amsterdam (1981) to visit H. Berbee, University of New South Wales (1981) to visit Gavin Brown, University of Warwick (1985, 1987) for the Stochastic Analysis Summer School, and University of Paris Sud (1992) to visit Jean-Pierre Kahane. Many of these contacts produced collaborative research; several of his collaborators have revealed that they despaired of ever getting John to agree that their carefully crafted paper was ready for submission. John remained a perfectionist with regard to his publications, and many of his ideas remain unpublished because of his desire for complete results with

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no avenue unexplored. For this reason his research output, though very impressive, did not match his talents or potential. It is worth remarking that John had a remarkable gift for communicating mathematics, both in print and in lectures. He could explain deep and technical results with lucidity, and his talks were always carefully structured and beautifully delivered.

John had various physical ailments in the last years of his life, but he never complained or discussed these with his friends, always remaining an amusing and lively companion. He is survived by his wife, Jennifer, who nursed him devotedly through his final illness. He died of cancer of the pancreas on 11 April 2001.

Research

In order to give the flavour of John Hawkes's mathematics, we will need quite a bit of notation. Before describing his work, we establish some notation and definitions in three distinct areas that appear unrelated but are used frequently to define the problems on which he focused his research. Actually, some of his papers require input from all three areas. His interest in geometric measure theory and probabilistic potential theory was motivated by natural questions concerning the trajectories of processes with stationary independent increments (now called Lévy processes because their deep study was initiated by Paul Lévy).

1. *Lévy processes*

A random process $X_t = X(t) = X(t,\omega)$ taking values in $\mathbb{R}^d$, has an increment

$$I(t,h) = X_{t+h} - X_t,$$

which is a random vector in $\mathbb{R}^d$ whose distribution does not depend on t, the variable that we think of as time. Since increments are independent and we can write, for each n,

$$I(t,h) = \sum_{j=0}^{n-1} I\left(t + \frac{jh}{n}, \frac{h}{n}\right)$$

with identically distributed summands, the distribution of the increment $I(t,h)$ has to be *infinitely divisible*, so that its Fourier transform satisfies the Lévy–Khintchine formula

$$\mathbb{E}e^{i\langle u, I(t,h)\rangle} = e^{-h\varphi(u)}$$

with the exponent $\varphi(u)$ satisfying

$$\varphi(u) = i\langle b, u\rangle = \frac{1}{2}uSu' + \int \left[1 - e^{i\langle x,u\rangle} + \frac{i\langle x,u\rangle}{1+|x|^2}\right]\nu(dx) \tag{1.1}$$

where b is a constant vector in $\mathbb{R}^d$, S is a non-negative symmetric $d \times d$ matrix, and ν is a Borel measure on $\mathbb{R}^d$ satisfying

$$\int \frac{|x|^2}{1+|x|^2}\nu(dx) < \infty. \tag{1.2}$$

The three terms in $\varphi(u)$ correspond to:

(i) a deterministic drift in the direction b;

(ii) a linear transformation of Brownian motion, the process for which $\varphi(u) = e^{-|u|^2}$;

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(iii) a jump process in which the measure ν (now called the *Lévy measure*) determines the rate of jumps; the number of jumps in a Borel set A is a Poisson process with rate $\nu(A)$ when this is finite.

When A has 0 as a limit point, $\nu(A)$ may be infinite, and then the order in which the jumps are added may make a difference; in this case, we add the jumps in decreasing absolute size, but it may be necessary to compensate with a drift term which is unbounded as we include smaller and smaller jumps. Condition (1.2) ensures that the measure ν is finite outside any neighbourhood of the origin, and does not grow too quickly near 0.

Whenever the second (Brownian motion) term is present in (1.1), it dominates most local properties of the trajectory. The sample path properties of Brownian motion are fascinating but were mostly well understood by the 1960s. It was natural for Hawkes to concentrate his efforts on Lévy processes in which the first two terms in (1.1) are missing, so that the Lévy measure ν determines the process. Some of his more important papers relate to subordinators, that is, the class of monotone Lévy processes. Now the increment $I(t,h)$ is a non-negative random variable determined by its Laplace transform

$$\mathbb{E}\exp(-\lambda I(t,h)) = \exp[-hg(\lambda)],$$

with

$$g(\lambda) = a\lambda + \int_0^\infty (1-e^{-\lambda\mu})\nu(du),$$

where $a \geqslant 0$, and the Lévy measure ν now charges the positive reals and satisfies

$$\int \min(u,1)\nu(du) < \infty. \tag{1.3}$$

Condition (1.3) limits the growth of ν near 0 more strongly than (1.2). There are several indices determined by the Lévy measure. The upper Blumenthal–Getoor index β is given by

$$\beta = \inf\{\alpha \geqslant 0 : |u|^{-\alpha}|\varphi(u)| \to 0 \text{ as } |u| \to \infty\}. \tag{1.4}$$

We have $0 \leqslant \beta \leqslant 2$ for any Lévy process, while $0 \leqslant \beta \leqslant 1$ for any subordinator. The lower index for a subordinator is given by

$$\sigma = \sup\{\alpha \geqslant 0 : |u|^{-\alpha}g(u) \to \infty \text{ as } u \to \infty\}. \tag{1.5}$$

A Lévy measure on the positive reals satisfying (1.2) but not (1.3) will require an infinite correction term, as the sum of jumps will not be convergent. If we assume that ν satisfies (1.3) and $a=0$, then the value of the process X_t at time t will be the sum of the jumps occurring up to t. The special case $g(\lambda)=\lambda^\alpha$ corresponds to a stable subordinator of index α, where $0<\alpha<1$. We remark that symmetric stable processes of index α in $\mathbb{R}^d$ result whenever (1.1) reduces to $\varphi(u) = \exp(-|u|^\alpha)$ with $0<\alpha\leqslant 2$.

Here, $\alpha=2$ corresponds to Brownian motion, and $\alpha=1$ to the symmetric Cauchy process. Note that (1.2) forces $0<\alpha<2$ when there is a Lévy measure. A more general stable process of index α in $\mathbb{R}^d$ results when the characteristic function (1.1) is given by

$$\varphi(u) = -c|u|^\alpha \int_{\mathbb{S}^d} w_\alpha(u,\theta)\mathrm{m}(d\theta) \qquad \text{with } c>0,$$

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where

$$
\begin{aligned}
w_\alpha(u,\theta) &= \left[1 - i\,\mathrm{sgn}(u,\theta)\tan\frac{\pi\alpha}{2}\right]\left|\left\langle\frac{u}{|u|},\theta\right\rangle\right|^{\alpha}, \qquad \alpha \neq 1,\\
w_1(u,\theta) &= \left|\left\langle\frac{u}{|u|},\theta\right\rangle\right| + \frac{2i}{\pi}\left\langle\frac{u}{|u|},\theta\right\rangle \log\left|\left\langle\frac{u}{|u|},\theta\right\rangle\right|,
\end{aligned}
\tag{1.6}
$$

and m is a probability measure on $\mathbb{S}^d$, the unit sphere in $\mathbb{R}^d$. The symmetric stable process results when m is the uniform measure on $\mathbb{S}^d$. The scaling property held by all stable processes with index $\alpha \neq 1$, and the symmetric Cauchy process can be expressed as

$$r^{-1/\alpha}X(rt) \text{ is another version of } X(t), \qquad \text{for any } r > 0. \tag{1.7}$$

This property greatly simplifies many arguments. It was first observed for Brownian motion.

2. *Geometric measure theory*

The early years of the twentieth century saw the establishment of Lebesgue measure as an appropriate tool for real analysis. Then Carathéodory defined *metric outer measure*, and used this to define measurability. In 1914, Hausdorff exploited the Carathéodory method to define an outer measure h-m for all subsets of a metric space, starting with any monotone increasing function $\mathrm{h}:(0,1) \longrightarrow \mathbb{R}$ with $\mathrm{h}(0+) = 0$. His definition is

$$\text{h-m}(E) = \liminf_{\delta\downarrow 0}\left[\sum_{i=1}^{\infty} h(\mathrm{diam}\ C_i) : E \subset \bigcup_{i=1}^{\infty} C_i, \mathrm{diam}\ C_i < \delta\right]. \tag{2.1}$$

Now h-m(E) may be 0, finite and positive, or $+\infty$. Because it defines a metric outer measure, all Borel (and analytic) sets are measurable. In $\mathbb{R}^d$, if $h(s) = s^d$, h-m is known to be a multiple of Lebesgue measure, but other powers of s lead to measures that classify the size of subsets of zero Lebesgue measure. The (Hausdorff–Besicovitch) dimension is defined by

$$\dim\ E = \inf\{\alpha > 0 : s^\alpha - \mathrm{m}(E) = 0\} = \sup\{\alpha > 0 : s^\alpha - \mathrm{m}(E) = \infty\}. \tag{2.2}$$

Clearly, $0 \leqslant \dim\ E \leqslant d$ for any $E \subset \mathbb{R}^d$, but even if $\dim\ E = \alpha$, we can have $s^\alpha - \mathrm{m}(E)$ taking the value 0 or $+\infty$. We say that E has *exact dimension* h if $0 < \text{h-m}(E) < +\infty$. Many random sets generated by the trajectory of a stochastic process can be shown to have zero Lebesgue measure, and so calculating the h-measure of such sets gives information about their size.

There are several other ways of examining 'small sets'. Suppose that E is a bounded subset in $\mathbb{R}^d$ and $\varepsilon > 0$; let $N_\varepsilon(E)$ denote the smallest number of sets of diameter less than 2ε that can be used to cover E. The ε-entropy of E, $H_\varepsilon(E)$ is defined by $H_\varepsilon(E) = \log_2 N_\varepsilon(E)$. The upper and lower entropy dimensions of E are given by

$$\bar{\mathrm{h}}(E) = \limsup_{\varepsilon\downarrow 0}\{H_\varepsilon(E)/\log(1/\varepsilon)\}, \quad \underline{\mathrm{h}}(E) = \liminf_{\varepsilon\downarrow 0}\{H_\varepsilon(E)/\log(1/\varepsilon)\}. \tag{2.3}$$

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If they are equal, their common value is called the *entropy dimension* and denoted $\mathrm{h}(E)$. It is easy to see that

$$\dim E \leqslant \underline{\mathrm{h}}(E) \leqslant \bar{\mathrm{h}}(E),$$

and the inequalities may be strict.

3. *Probabilistic potential theory*

If $q(x, y)$ is a function on $\mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^+$ and μ is a Borel measure on $\mathbb{R}$, we call

$$h(x) = h_q^\mu(x) = \int q(x, y)\mu(dy)$$

the *q-potential at x generated by the measure μ*. If a closed set E is such that $h_q^\mu(x)$ is unbounded for each positive measure μ concentrated on E, we say that the *q-capacity of E* is zero. On the other hand, if $h(x)$ is bounded for some measure μ on E, we can choose the largest such measure for which

$$0 \leqslant h_q^\mu(x) \leqslant 1 \qquad \text{for all } x.$$

This special measure μ is called the *capacitory distribution* on E for the potential q. Classical potential theory in $\mathbb{R}^d (d \geqslant 3)$ comes from the Newtonian kernel

$$q(x, y) = |x - y|^{2-d}.$$

Riesz potentials of index α come from $q_\alpha(x, y) = |x - y|^{\alpha-d}$, for $\alpha \leqslant d$.

In the 1940s, Kakutani ⟨**7**⟩ had obtained the connection between the hitting probabilities of Brownian motion and classical potential. If E is a Borel set of positive (Newtonian) capacity in $\mathbb{R}^d$ $(d \geqslant 3)$, then for a Brownian motion $B(t)$ starting at x, the probability that $B(t)$ will enter E for some $t > 0$ is given by

$$\Phi(x, E) = \int |x - y|^{2-d}\mu(dy),$$

where μ is the capacitory distribution on E; whenever E has zero capacity the probability that $B(t)$ will hit E is zero. Hunt ⟨**5**⟩ extended Kakutani's results to a general Markov process. Lévy processes are a special type of Markov process for which the corresponding kernel can be calculated. For example, the kernel for a symmetric stable process of index α in $\mathbb{R}^d$ is exactly the Riesz kernel $q_\alpha(x, y)$ defined above. Given a set E, we can define the set of accessibility $A(E)$ to be the set x of points from which there is positive probability that the Lévy process started at x will hit the set E. We say that E is *polar* for X if $A(E)$ is empty. It is *essentially polar* if $A(E)$ has zero Lebesgue measure. Thus polar sets have zero capacity in the appropriate potential theory. Given a set E of positive capacity, we can identify the *regular* points x to be such that the process $X(t)$ starting at x will hit E immediately with probability 1, or

$$\inf\{t > 0; X(t) \in E\} = 0 \qquad \text{almost surely.} \tag{3.1}$$

We can use the process to define hitting probabilities in terms of the total time spent in E. Put

$$U(E) = \mathbb{E}\int_0^\infty 1_E(X_t)dt; \quad U^\alpha(E) = \mathbb{E}\int_0^\infty e^{-\alpha t}1_E(X_t)\,dt. \tag{3.2}$$

Then $U(E)$ will be finite for bounded sets E whenever X_t is transient; U^α is always defined.

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We are now ready to look at Hawkes's research contributions, which I will arrange in sections as his interests developed.

4. *Sample path properties of stable processes* [**2–6**]

Hawkes's early work developed from his PhD. In [**2**], Hawkes looks at properties of the asymmetric Cauchy process on the line, the simplest stable process which does not have the scaling property (1.7). Since the unit sphere in $\mathbb{R}^d$ contains only two points $+1$ and -1, the exponent given by (1.6) reduces to

$$\varphi(u) = |u|\{1 + ih\,\mathrm{sgn}(u)\log|u|\} \tag{4.1}$$

with $h = (2/\pi)(p-q)$, where p and q are the weights on $+1$, -1, such that $p+q = 1$.

Here, $h = 0$ corresponds to the symmetric case, which was already well understood. Hawkes proves that, when $h \neq 0$, the range has positive Lebesgue measure, while the zero set $\mathbf{Z}$ has Hausdorff dimension 0 and logarithmic dimension 1. The paper [**3**] obtains precise answers to potential-theoretic questions that can be asked about stable processes of index α in $\mathbb{R}$, for $0 < \alpha < 1$. He obtains the surprising result that the class of recurrent sets, $E \subset \mathbb{R}^+$, is the same for all stable processes with index α, including the stable subordinator of index α. (We call a set E *recurrent* for X if the set $\{t > 0 : X_t \in E\}$ is unbounded.) It was already known that all strictly stable processes of a given index have the same class of polar sets. Hawkes improves this result by showing that, for a non-polar set E, the question of whether x is regular for E determined by (3.1), again has the same answer for all stable processes with the same index.

The paper [**4**] formulates and proves a precise analogue for stable subordinators of a result of P. Lévy on the uniform nature of the size of small oscillations on a Brownian path. This uniform lower rate of escape is then used to establish that, see (2.2),

$$\dim C(\omega) = \alpha(1 - 1/\beta) \ \text{ almost surely},$$

where the random collision set $C(\omega)$ is the set of points $x \in \mathbb{R}$ for which $x = X_t = Y_t$ for some $t > 0$, where $\alpha < 1 < \beta < 2$, X_t is a stable subordinator of index α, and Y_t is any stable process of index β. Similar ideas are used in [**5**] to find the Hausdorff dimension of the intersection between a fixed Borel set E and the range of a stable process in $\mathbb{R}^d$. Some results are obtained for general d, but a complete result is given for $d = 1$: suppose that dim $E = \gamma$, and X_t is a symmetric stable process of index $\alpha \geqslant 1$; then the occupation set

$$S(\omega) = \{t > 0 : X_t(\omega) \in E\} \text{ almost surely satisfies } \dim S = \frac{\alpha + \gamma - 1}{\alpha},$$

and the intersection set $E \cap R(\omega)$ satisfies

$$\min[\gamma, \alpha + \gamma - 1] = \sup\{\theta > 0 : \mathbb{P}^x[\dim E \cap R(\omega) > \theta] > 0\}.$$

The complete solution to the dimension problem for general stable processes in $\mathbb{R}^d$ as defined by (1.6) is given in [**6**]. Here, Hawkes proves that there is a uniform connection between the dimension of E and its image $X(E, \omega)$, which is valid almost surely for *all* Borel sets $E \subset \mathbb{R}$, including sets E that may depend on ω, and any

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stable process of index $\alpha \neq 1$ or the symmetric Cauchy case $\alpha = 1$. Provided that $\alpha \leqslant d$,

$$\dim X(E,\omega) = \alpha \dim E \quad \text{for } \textit{all} \text{ Borel sets } E, \text{ with probability 1.}$$

This gives, for a Borel set E with dim $E = \gamma$,

$$\mathbb{P}^x[\dim E \cap R(\omega) = \alpha + \gamma - d] = 1,$$

whenever x is a regular point for E, and $R(\omega)$ is the range of a strictly stable X_t of index α in $\mathbb{R}^d$. (Note that, if $\alpha < d - \gamma$, E is polar for X, and so it will have no regular points.)

5. *Properties of more general Lévy processes* [**8**–**10**, **16**, **19**, **20**, **22**, **24**]

The exact Hausdorff measure function $\mathrm{f}(s)$ for the range $R(s,\omega)$, $0 \leqslant t \leqslant s$, of a general subordinator was found by Fristedt and Pruitt ⟨**4**⟩. This implies that

$$\text{f-m}\, R(s,\omega) = cs, \qquad \text{for } \textit{all } s \text{ almost surely for some constant } c > 0.$$

In [**8**], Hawkes shows that $\frac{1}{2} \leqslant c \leqslant 1$, and by computing c exactly for a stable subordinator of index α he shows that these bounds are sharp. He uses these results to obtain information about the local time of a stable process with index $\alpha > 1$, as well as the asymmetric Cauchy process.

Blumenthal and Getoor ⟨**3**⟩ in 1960 defined several indices [see (1.5), (1.6)] for a general Lévy process $X_t(\omega)$, and used these to compare dim B and the image $\dim X(B)$ for a fixed Borel set B. However, in many applications the set B is random, depending on the sample path, so that results uniform in B are needed. This is the problem attacked successfully in [**9**]. The main results obtained in [**9**] are as follows.

For any Lévy process X_t with upper index β,

$$\mathbb{P}[\dim X(B) \leqslant \beta \dim B \text{ for } \textit{all} \text{ B}] = 1.$$

For a general subordinator with indices β and σ,

$$\mathbb{P}[\sigma \dim \mathrm{B} \leqslant \dim X(B) \leqslant \beta \dim B \text{ for } \textit{all } B] = 1,$$

and there is a special class of sets $\mathcal{D}_\sigma$ involving a regularity condition for which

$$\mathbb{P}[\dim X(B) = \sigma \dim B \text{ for } \textit{all } B \in \mathcal{D}_\sigma] = 1.$$

Indices coincide for stable processes, so it follows that, if X_t is strictly stable of index $\alpha \leqslant d$, than

$$\mathbb{P}[\dim X(B) = \alpha \dim B \text{ for } \textit{all} \text{ Borel sets } B] = 1.$$

Examples are given to show that the results are best possible.

In [**10**], Hawkes exploits, for a Lévy process X_t that hits points, connections between the modulus of continuity of the local time, its subordinator inverse, and the exponent of X_t. He defines a new parameter $b = b(X)$ directly in terms of the exponent of X, and proves that $\dim Z(\omega) = 1 - 1/b$ almost surely, where $Z(\omega) = \{t \geqslant 0 : X(t,\omega) = 0\}$ is the zero set. He further shows that the corresponding local time $L(t)$ belongs to Lip_α for $\alpha < 1 - 1/b$, but not for $\alpha > 1 - 1/b$. He gives examples to show that b is a genuinely new parameter for a Lévy process X_t.

Markov random sets, considered by Hawkes in [**16**], are really the image of subordinators. If X_1, X_2 are subordinators with ranges R_1, R_2 such that $R_1 \cap R_2$

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is non-empty almost surely, he proves that

$$\int_0^1 u_1(t)dU_2(t) < \infty,$$

where u_1 is the density of the potential kernel of X_1, and U_2 is the potential kernel of X_2, as defined in (3.2). The converse holds if u_1 is monotone. Hawkes then finds interesting properties of $R = R_1 \cap R_2$ when this is non-empty.

Takeuchi ⟨**13**⟩ obtained precise results for the moments of the last exit time of a transient symmetric stable process in $\mathbb{R}^d$ of index $\alpha < d$. The objective of [**19**] is to obtain the corresponding results for any symmetric Lévy process. The paper [**20**] studies multiple points for symmetric processes. Note that a process that misses points may still hit some points more than once. The set $M_k(\omega)$ consists of k-multiple points $x \in \mathbb{R}$ such that

$$x = X(t_i, \omega), \qquad i = 1, 2, \ldots, k \text{ with } 0 < t_1 < t_2 < \ldots < t_k.$$

The existence of $M_k(\omega)$, and its size, were already known for special processes; Hawkes obtains corresponding results based on the integrability properties of the potential kernel $U(x)$, which are valid for any symmetric Lévy process.

The paper [**22**] obtains precise information about the Hausdorff measure of the image $X(B)$ of a Borel set B, and the intersection $B \cap R$, where $R = X(0, \infty)$ is the range of a general subordinator X_t. In [**24**], Hawkes considers the dimension properties of $A \cap X^{-1}(B)$ and $B \cap X(A)$, where A is a time set, B is a Borel set and X_t is a stable process of index α. The new idea needed in this paper is the use of a special metric on $\mathbb{R}^+ \times \mathbb{R}^d$, which corresponds to the scaling properties of the graph process $t \mapsto (t, X_t)$.

6. *Analysis of small sets* [**7**, **11**–**13**, **15**, **31**]

In his study of sample path properties, Hawkes needed to use results from both the theory of fractal measures and general potential theory. When the results needed were not available, he was stimulated to answer the unanswered questions. At times he did so within the context of sample path properties, but these papers are independent of sample path problems and deal directly with specific problems in analysis.

The paper [**7**] is based on work of Kahane ⟨**6**⟩. For a fixed sequence $\{\ell_n\}$ of reals decreasing to zero, consider the intervals $I_n(\omega) = (X_n, X_n + \ell_n)$ reduced modulo 1, where $\{X_n\}$ is a sequence of independent random variables uniformly distributed in [0, 1). Let $E_\infty(\omega)$ be the set of points covered infinitely often by $\{I_n(\omega)\}$ and $F_\infty(\omega)$ its complement. For the particular sequence $\ell_n = \alpha/n, 0 < \alpha < 1$, Kahane shows that $\dim F_\infty = 1 - \alpha$ and, if $\dim A < \alpha$ for a fixed set $A \subset [0, 1]$, then $A \subset E_\infty$ almost surely; if $\dim A > \alpha$, then $A \cap F_\infty$ is non-empty almost surely. Hawkes obtains similar results for more general sequences, but gives examples to prove that some regularity conditions are needed. The main result for the Kahane case $\ell_n = \alpha/n$ is that, for a fixed set B with $\dim B = \beta > \alpha$,

$$\beta - \alpha = \dim(B \setminus E_\infty) \text{ almost surely.}$$

The classical definition of independence for two subsets A, B in a probability space is

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

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This forces any set A of zero probability to be independent of every set B; so we need some more useful method of obtaining independence for sets of zero probability. For example, we would like some condition which implies that

$$\dim(A \cap B) = \dim A + \dim B - d$$

for sets A, B 'in general position'. The paper [**11**] considers many candidates for such a condition. Hawkes starts by showing that the range of any Lévy process in $\mathbb{R}$ has the same value for the Hausdorff dimension and the lower entropy dimension as given by (2.3). He then gives a definition of weak independence for subsets of a cube in $\mathbb{R}^d$ which ensures that

$$\mathrm{h}^*(A \cap B) = \mathrm{h}^*(A \times B) - d,$$

where h^* stands for any of the entropy dimensions. In fact, other definitions of independence are explored, which have ramifications relevant to information theory and functional analysis.

For any subset $A \subset \mathbb{R}$, the difference set is given by

$$D(A) = \{x - y : x, y \in A\}.$$

Steinhaus $\langle \mathbf{12} \rangle$ showed that, if A has positive Lebesgue measure, then $D(A)$ contains an open interval about the origin. It is easy to see that the classical Cantor set C satisfies $D(C) = [-1, +1]$ so that $(t + C) \cap C$ is non-empty for t in $[-1, +1]$. In [**12**], Hawkes shows that, for any θ satisfying $0 \leqslant \theta \leqslant \dim C$ $(= \log 2/\log 3)$, the set

$$\{t : \dim[(t + C) \cap C] = \theta\}$$

is non-empty, and

$$\dim[(t + C) \cap C] = \frac{1}{3 \log_2 3}$$

for (Lebesgue) almost all t. He also obtains reasonable regularity conditions on A which ensure that

$$\dim D(A) = \min(1, 2 \dim A).$$

A sequence $\{r_i\}$ of positive numbers is majorising for a set A in a metric space Ω if centres $\{z_i\}$ can be chosen in Ω such that, for each n,

$$A \subset \bigcup_{i=n}^{\infty} S(z_i, r_i).$$

In [**15**], Hawkes and Gardner use the fine connections between Hausdorff φ-measure and the potential theory with kernel $\Phi(x, y) = 1/\varphi(|x - y|)$ to obtain a majorising sequence for any compact set A that is Φ-polar.

Given a compact set K in $[0, 1]$, the complementary intervals have lengths that can be arranged as a sequence $\ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_n \geqslant \ldots \geqslant 0$ with $\sum_{i=1}^{\infty} \ell_i = 1$ if K has zero Lebesgue measure. In 1954, Besicovitch and Taylor $\langle \mathbf{2} \rangle$ defined two indices $\alpha\{\ell_i\}$ and $\beta\{\ell_i\}$, and proved that $0 \leqslant \dim K \leqslant \alpha \leqslant \beta \leqslant 1$. In [**31**], Hawkes defines a natural way of reordering the complementary intervals randomly to produce a random set $K(\omega)$. He shows that α is the lower entropy index of K, and that $\dim K(\omega) = \alpha$, almost surely. In [**37**], Hawkes considers the extension of this result to $\mathbb{R}^d$. Suppose that in the unit ball in $\mathbb{R}^d$ there is a sequence of disjoint open balls whose complement is a residual set R of zero Lebesgue measure. He shows that two constants defined by the sequence of radii of the packing balls coincide with the upper and lower entropy dimensions of R, defined by (2.3).

7. *Potential theory for Lévy processes* [**14**, **25**, **32**]

In [**14**], Hawkes focuses on the question of the polarity of sets $A \subset \mathbb{R}$ for general subordinators with range $R(\omega) = \{x : x = X_t(\omega) \text{ for some } t > 0\}$. If the subordinator has upper and lower parameters β and σ, then $1-\beta < \dim A < 1-\sigma$ yields no information about the polarity of A. He obtains a surprising result, namely, that there is a constant $\eta(A)$, determined by the exponent of X_t, such that

$$\dim\{A \times R(\omega)\} = \eta(A) \text{ almost surely,}$$

and $\eta(A) < 1$ implies that A is polar, while $\eta(A) > 1$ implies that A is non-polar. This allows him to give examples of sets A and B and a subordinator X_t such that $\dim A < \dim B$ while B is polar but A is non-polar for X.

The potential theory for a general Lévy process was considered by Meyer ⟨**9**⟩ and Orey ⟨**10**⟩. Many of their results required the kernel U^α defined by (3.2) to be absolutely continuous. In [**25**], Hawkes considers relationships between certain classes of exceptional sets for a Lévy process in which the kernel U need not be absolutely continuous. First he shows that absolute continuity of U implies and is implied by what is known as the *strong Feller property* for the semigroup of transition operators defined by the process. For such processes, the class of essentially polar sets is the same as the class of polar sets. He then deduces a sufficient condition for the class of essentially polar sets to be the same for two distinct Lévy processes. Interesting examples, which exhibit unexpected behaviour, are constructed. The paper [**32**] is a survey of various relations between the trajectories of a Lévy process and the potential-theoretic quantities defined by the process. The author's contributions are substantial, though some proofs are omitted and several of the papers cited remain unpublished. However, the survey is valuable in providing an interesting geometric viewpoint.

8. *Miscellaneous problems in probability theory* [**17**, **18**, **21**, **23**, **25**–**30**]

Any random variable X with finite expected value $\mathbb{E}X$ has a characteristic function $\psi(s)$ differentiable at 0. In [**17**], Hawkes shows that, for positive random variables X,

$$i\mathbb{E}X = \lim_{s\to 0} \frac{\psi(s)-1}{s} \tag{8.1}$$

in the sense that if either side exists and is finite, so does the other. He defines a positive X with $\mathbb{E}\,X = \infty$ but

$$\liminf_{s\to 0} \left|\frac{\psi(s)-1}{s}\right| = 0,$$

so (8.1) cannot always be true when $\mathbb{E}X = \infty$.

In [**18**], Hawkes considers a Gaussian process X_t in $\mathbb{R}$ with stationary increments and

$$\sigma^2(t) = \mathbb{E}(X_{s+t} - X_s)^2.$$

Whenever σ is monotone, define

$$\dim_\sigma(E) = \inf\{\alpha \geqslant 0 : \sigma^\alpha - \mathrm{m}(E) = 0\}$$

https://doi.org/10.1112/S0024609304003406 Published online by Cambridge University Press

and two indices

$$\begin{aligned}\alpha(\sigma) &= \sup\{\alpha \geqslant 0 : t^{1/\alpha} = o(1)\sigma(t) \text{ as } t \downarrow 0\},\\ \beta(\sigma) &= \inf\{\beta \geqslant 0 : \sigma(t) = o(1)t^{1/\beta} \text{ as } t \downarrow 0\}.\end{aligned} \tag{8.2}$$

He proves that, if $\beta(\sigma) < \infty$, then

$$\mathbb{P}[\dim X(E) = \min(1, \dim_\sigma(E)) \text{ for } \textit{all} \text{ analytic } E] = 1,$$

extending the result of McKean for Brownian motion. He then uses the methods that he has developed to find precise formulae for the Hausdorff dimension of the level sets and graph of the Gaussian process X, whose indices are given by (8.2).

Hawkes collaborates with John Jenkins in [**21**] to consider sequences of real numbers $\{b_n\}$ and $\{a_n\}$ connected by

$$nb_n = \sum_{j=0}^{n} a_j b_{n-j}, \qquad n = 1, 2, \dots. \tag{8.3}$$

They first characterise such sequences $\{b_n\}$ with $b_0 > 0$ and $a_j \geqslant 0$, and obtain a sufficient condition for b_n to converge to a non-zero limit. This is only possible with $b_0 = 1$ if $\{b_n\}$ is infinitely divisible; that is, for each integer r, there is a non-negative $\{d_n\}$ such that $b_n = d_n^{*r}$. Here that rth convolution power of $\{d_n\}$ is defined inductively by

$$d_n^{*1} = d_n, \qquad d_n^{*r} = \sum_{j=0}^{n} d_{n-j} d_j^{*(r-1)}.$$

These results can be thought of as a generalisation of the renewal theorem with finite mean recurrence time.

The paper [**26**] deals with lacunary series. Suppose that $\{n_k\}$ is a sequence of positive integers with $n_{k+1}/n_k > c > 1$, and $a_k > 0$. Consider

$$Z_n(\theta) = \sum_{k=1}^{n} a_k \exp(2\pi i n_k \theta).$$

If $\sum a_k^2 < \infty$, the asymptotic behaviour of $\{Z_n(\theta)\}$ was previously known. Hawkes shows in [**26**] that under the stronger lacunary condition that $\sum n_k/n_{k+1} < \infty$, the sequence $\{Z_n(\theta)\}$ can be approximated by sums of independent random variables. From this he deduces that $|Z_n(\theta)|$ diverges to ∞ or is dense in the plane according as $\sum B_n^{-1/2} < \infty$ or $\sum B_n^{-1/2} = \infty$, where $B_n = \sum_{k=1}^n a_k^2$. Several other interesting results are obtained. For example, if $n_k = 2^k$, and $C_n = \sum_{k=1}^n a_k$, the set $\{\theta : Z_n(\theta) \sim C_n\lambda\}$ for some λ has Hausdorff dimension 1.

Now consider the unit interval partitioned by a sample of size n from the uniform distribution on $[0, 1]$. Define $Z_n(x)$ to be the length of the sample spacing that contains x, and define

$$Z_n = \max\{Z_n(x) : x \in [0, 1]\}.$$

Hawkes proves in [**27**] that $\lim n\, Z_n/\log n = 1$ almost surely, and $\limsup n\, Z_n(x)/\log\log n = 1$ for Lebesgue almost all x. He also proves that, for $0 \leqslant c \leqslant 1$,

$$\dim\left\{x : \limsup \frac{n\, Z_n(x)}{\log n} = c\right\} = 1 - c.$$

This last result has the flavour of a multifractal decomposition.

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There is a natural metric on the boundary of a Galton–Watson branching process generated by the distribution $\{p_k\}$ with $p_0=0$, $p_k \geqslant 0$, and $\sum_{k=1}^{\infty} p_k = 1$. This process starts from a single ancestor at level 0, and each node at level n independently produces k children with probability p_k at level $(n+1)$. The ergodic theorem implies that if $\mu = \sum kp_k < \infty$, then $W = \lim Z_n/\mu^n$ exists and is random with $\mathbb{E}W = 1$, where Z_n is the number of nodes at level n. In [**28**], Hawkes obtains the Hausdorff dimension of the boundary set K and, in the special case where the distribution of Z_1 is geometric, he shows that, for $\alpha = \log\mu/\log 2$ and $h(s) = s^{\alpha}\log\log 1/s$, $h\text{-m}(K) = W$ almost surely.

If $\{p_k\}$ is an infinitely divisible distribution on the non-negative integers, it will have a corresponding Lévy measure $\{\nu_n\}$. In [**29**], Embrechts and Hawkes obtain a necessary and sufficient condition for the existence of lim p_n/ν_n. Among other results, they also obtain necessary and sufficient conditions for the first positive ladder epoch to belong to the domain of attraction of a positive stable law with index $\alpha \in (1,2]$.

For a probability distribution F which is in the domain of attraction of a stable law F_α of index $\alpha \in (0,2]$, let $S_n = \sum_{i=1}^{n} X_i$ where the X_i are independent with distribution F. Then there exists a sequence $B_n = n^{1/\alpha}L(n)$ with $L(n)$ slowly varying, such that S_n/B_n converges in distribution to F_α.

In [**30**], Bingham and Hawkes prove a local limit theorem in two cases, lattice and non-lattice, from which they derive a central limit theorem for

$$N_n(I) = \text{number of } \{k \leqslant n : S_k \in I\},$$

namely that $N_n(I)/|I|n^{1-1/\alpha}L(n)$ converges in law to the Mittag–Lefler distribution. Kesten $\langle \mathbf{8} \rangle$ had shown the equivalence of the central limit theorem for S_n and $N_n(I)$; in this paper, the authors derive an analogue for symmetric processes, and then extend it to spectrally positive Lévy processes.

9. *Last decade*

In his last decade, Hawkes produced several useful surveys [**34–36**], in which he frequently interjected a new idea to obtain useful examples while expounding known results. He also collaborated with others [**33**, **39**] to produce papers that proved to be seminal, while in [**38**, **40**] he returned to earlier topics to extend or improve on his results. He remained active and involved in the many fields to which he had contributed until shortly before his death.

The paper [**33**], with Martin Barlow, attacks the long-standing problem of joint continuity in x and t of the local time $L(x,t)$ for a 1-dimensional Lévy process X_t for which 0 is regular for $\{0\}$; see (3.1). For fixed x, we can think of $L(x,t)$ as measuring the time spent at x by X_s for $s \leqslant t$. It is known that, as a function of t, $L(x,t)$ is continuous almost surely for each x. The problem is to decide when there is a version of $L(x,t)$ that is jointly continuous in x and t. If the process has exponent $\psi(u)$, we can define

$$\varphi(x) = \frac{1}{\pi}\int (1-\cos ux)\Re\frac{1}{1+\psi(u)}\,du$$

and construct a monotone rearrangement of φ on $[0,1]$, denoted by $\bar{\varphi}$. Hawkes and Barlow show that if

$$I(\bar{\varphi}) < \infty, \qquad \text{where } I(\psi) = \int_0^{1/e} \psi(u)u^{-1}(\log(1/u))^{1/2}\,du,$$

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then $L(x, t)$ has a version jointly continuous in x and t. The fact that $I(\bar{\varphi}) < \infty$ is also necessary for joint continuity was proved shortly afterwards by Barlow ⟨**1**⟩.

In [**34**], Hawkes continues his study of local time $L(x, t)$, now written $L_t(x)$. He shows that, when it exists, it is almost surely square-integrable, even though there are examples (asymmetric Cauchy processes) for which it is known to be unbounded in every interval. For each fixed x, define

$$L(x) = \int_0^\infty e^{-t}\, dL_t(x),$$

and a new process $Z_w = L(w + x)$. He shows that Z_w is second-order stationary, and discusses the possible value of regarding local time as an example of a process with stationary increments.

Physicists became interested in interactions between random fields in the 1970s. For occupation fields related to the intersection of random processes, these had been used by Le Gall and others to investigate the Hausdorff measure of intersection sets. In [**35**], Hawkes gives a unified approach, based on capacity techniques, to many of these interesting random sets. He obtains interesting new results about the intersection of translates of a unit circle in the plane with a fixed set K such that $\dim K > 1$.

The article [**37**] is a survey of results related to, and inspired by Kahane's 1968 book on random series of functions ⟨**6**⟩. He explains results of Sierpinski, Besicovitch, Rademacher, Kolmogorov, Barlow & Hawkes, and Marcus & Rosen in the context of this book.

In [**39**], Hawkes uses some of his ideas from [**28**] to collaborate with Allouba, Durrett and Perkins in examining the natural boundary measure on a Galton–Watson tree, also known as a *supercritical process*. The authors not only obtain new insights into the super α-process, but they construct examples where the random measure has a very smooth density.

In [**40**], Hawkes returns to his work in [**5**] and [**14**], and obtains more precise results about the size of the image $X(B)$ whenever X_t is a stable process of index α. Perkins and Taylor ⟨**11**⟩ had shown that complete precision is impossible using only the Hausdorff measure properties of B. In this paper, Hawkes obtains precise results for stable subordinators and symmetric stable processes X by using the theory of Riesz capacities.

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