CONSTRUCTION OF PERIODIC ORBITS, PROBLEMS OF STABILITY AND PERIOD DETERMINATION, IN THE ELLIPTICAL NON-PLANAR RESTRICTED PROBLEM

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ABSTRACT

Periodic orbits in a fixed frame are constructed in the vicinity of non-periodic solutions of the non perturbed problem. In a first phase, approximate initial conditions are found and in a second phase more accurate initial conditions obtained are used in order to check the periodic orbit by numerical integration of the three-body problem. Some peculiar solutions are found, for example, orbit with nearly zero angular momentum. A study of stability of periodic solutions is proposed with an approximation of the monodromy matrix Φ (T,o), not requiring numerical integration of the 6x6 variational linear system. Finally, some numerical problems of period determination are outlined.

INTRODUCTION

This paper refers to the construction of solutions of period T of a non-linear integrable system disturbed by T-periodic or autonomous perturbation, leading to an implicit system for the initial state vector. The non-integrable and non-linear perturbed system may be related to a quasi-linear one by Taylor expansion near a reference orbit. Then we can obtain rigorous periodicity conditions (Roseau 1966). Suppose now that the generating solution verifies the unperturbed integrable system, thus it is a polyparametric family where time occurs explicitely. But if this orbit is T-periodic, then periodicity conditions are singular. It is the critical case as it has been studied by I. Stellmacher (1976, 1977, 1979, 1981). Here a non-T-periodic solution is chosen to avoid singularities. It is the non critical case. The periodicity conditions lead to an implicit system for the initial state vector. For such a system solution series, relative to a small parameter of perturbation, exist if certain conditions are satisfied as demonstrated by Poincaré (1892). The method is semi-numerical; in a first step approximate initial conditions are found, in a second step more accurate conditions

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are obtained by an iterative differential corrector. Finally to each particular non-T-periodic solution of the unperturbed system corresponds a T-periodic one of the perturbed motion. Therefore, a "polyparametric" family of T-periodic perturbed orbits is obtained.

This method is applicable to various perturbed problems with simplification for Hamiltonian systems. Here, numerical investigations are made to the elliptical spatial restricted three-body problem for the Sun-Jupiter system in a heliocentric (or planetocentric for lunar case) inertial frame. The generating orbit is an hyperbolic of parabolic comet.Later we will study capture orbits for these comets by this method. Several "curious" solutions are obtained: orbits almost rectilinear, near collisions, orthogonal to the orbit of the perturbing body, orbit of period T/k where k is an integer. A study of linear stability is proposed; the monodromy matrix is analytically approximated to avoid numerical integration of variational equations. Finally, some problems of period determination are outlined.

We need analytical solutions for the variational system:

1) of the unperturbed motion for a non T-periodic generating unperturbed orbit, to construct the periodic perturbed solution

2) of the perturbed system with a T-periodic generating perturbed orbit for the linear stability analysis.

We have analytical solutions for the unperturbed integrable system while for the perturbed non integrable system only an approximation is available.

2. EQUATIONS

The perturbed system has the form:

$$\dot{x} = X\ (x,t,\varepsilon), \tag{1}$$

and the unperturbed integrable system is

$$\dot{z} = X\ (z,t,o), \tag{2}$$

where $X(x,t,\varepsilon)$ is a T-periodic function of time t satisfying certain regularity conditions with respect to the state vector x of R^n and to the small parameter ε describing the perturbation (Roseau 1966). $x\ (x_o,t,\varepsilon)$ represents the solution of (1) with initial state vector $x(o) = x_o$; thus $z(t) = x(z_o,t,o)$ is the solution of the integrable unperturbed system (2) with starting vector $z(o)=z_o$; we put $x = z+\Theta$, Θ is the solution of the system written in the form:

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$$\dot{\Theta} = \overline{X}'_x(t)\Theta + \overline{X}'_\varepsilon(t)\varepsilon + F(\Theta, t, \varepsilon) \qquad (3)$$

where the surlining symbol means that the derivatives are evaluated for $(\Theta,\varepsilon) = (o,o)$ and where $F(\Theta,t,\varepsilon)$ are the terms of order $\geqq 2$ in the Taylor expansion of $X(z+\Theta,t,\varepsilon)$ with respect to (Θ,ε) near (o,o); we want to study now the existence and the characterization of T-periodic solutions of (1).

3. PERIODICITY CONDITIONS

We have to solve an implicit system for the starting vector x_o of the T-periodic solution of (1):

$$x_o = x(x_o, T, \varepsilon) \qquad (4)$$

that leads to the implicit system, for $\Theta(o) = \Theta_o$,

$$\Delta\Psi^* \quad \Theta_o = \int_o^T \Psi^*(s) \left\{ \varepsilon \overline{X}'_\varepsilon(s) + F\left[\Theta(\Theta_o, s, \varepsilon), s, \varepsilon\right] \right\} ds \qquad (5)$$

where $\Psi(t)$ is a nxn fundamental matrix solution of the joint variational unperturbed system relative to a non T-periodic generating solution $z(t)$ of the unperturbed system (2). The symbol * means matricial transposition and ΔU is $U(T)-U(o)$; if we can solve the implicit system (5) for Θ_o, then we obtain the initial state vector x_o of the T-periodic solution of (1) by : $x_o = z_o + \Theta_o$, z_o being a known vector; this system (5) will be solved semi-numerically.

4. APPROXIMATE INITIAL CONDITIONS

We put $\Theta_o = \overset{o}{\Theta}_o + \beta$ (Roseau 1966), where $\overset{o}{\Theta}_o$ satisfies the explixit system

$$\Delta\Psi^* \quad \overset{o}{\Theta}_o = \int_o^T \Psi^*(s) \quad \varepsilon \overline{X}'_\varepsilon(s)\, ds + \Psi^*(T) \quad \Delta z. \qquad (6)$$

As $z(t)$ is non T-periodic, $\Delta\Psi^*$ is non singular, thus $\overset{o}{\Theta}_o$ is easily obtained from (6); the associated initial state vector $\overset{o}{x}_o$ is $z_o + \overset{o}{\Theta}_o$; if $z(t)$ is T-periodic (critical case) then $\Delta\Psi^*$ is singular (Szebehely 1967).

5. MORE ACCURATE INITIAL CONDITIONS

We now have to solve an implicit system for β:

$$\Delta\Psi^* \quad \beta = \int_o^T \Psi^*(s) \quad F\left[\Theta\ (\overset{o}{\Theta}_o+\beta, s, \varepsilon), s, \varepsilon\right] ds. \qquad (7)$$

This system has solutions provided that F satisfies certain conditions (Poincaré 1892). β is obtained by an iterative differential corrector. For a hamiltonian system Ψ^* (t) is replaced by Φ^* (t) E, where Φ (t) is a fundamental matrix solution of the unperturbed variational system relative to the generating orbit z(t), and E is the nxn matrix such that $E^2 = -I$, I being the nxn unit matrix.

6. LINEAR STABILITY ANALYSIS

The linear stability of the periodic solution x(t) is related to the eigen values of the monodromy matrix, (Hennawi 1980, Wiesel 1980). This matrix is approximated by a time averaging; the approximation is an analytical expression of the components of x(o) and x(T) (for the 3-body restricted problem); therefore we dont need numerical integration of the variational system of the perturbed motion.

7. APPLICATION TO THE RESTRICTED THREE-BODY PROBLEM

For this particular case: n=6, (2) is the two-body problem, ε is the mass of the disturbing body for the planetary case, and the mean motions ratio for the lunar case. The two primaries are the Sun and Jupiter, therefore T is Jupiter's period.

Finally, to each hyperbolic or parabolic cometary orbit z(t) is associated a T-periodic solution x(t) of the three-body problem. It should be a capture orbit for only quasiperiodic solution x(t) if:
1) $x(-\infty) = z(-\infty)$, 2) x(T)-x(o) is small, 3) x(t) is linearly stable. Several kinds of periodic solutions are found: orbits of period T/k where k is an integer, with various inclinations and eccenctricities. For these orbits, the osculating elements change slowly with time near the mean elements. Other orbits are almost rectilinear and orthogonal to Jupiter's orbit. Their osculating elements drastically vary; but for these orbits the periodicity of the first approximation is good.

Period determination

Solutions of period slightly different from T can be obtained by the procedure below: by (6) we have an approximated state vector x_o and the corresponding osculating elements give a period T_o. Now, we apply (6) with this period T_o+T and so on; we have a series of periods: T_o, T_1, ..., T_n; this series is convergent for some solution and we get by (7) an initial state vector x_o for a perturbed solution of the three-body restricted problem of period T_n + T.

Numerical investigations

Many numerical investigations have been done for various generating non T-periodic orbits z(t). The numerical integration procedure of the

equations of motion have been performed by P. Rocher (Rocher, 1981). The differential corrector used to solve (8) is similar to that of Markellos (Markellos, 1980). For parabolic orbit we utilize Subbotin's formulae (Subbotin, 1968) to obtain an analytical expression for a fundamental solution of the variational equations of the two-body problem.
The numerical results and further details will be published elsewhere.

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