HANNA NEUMANN

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1. Biography*

Fellowship of the Australian Academy of Science and Fellowship of the Australian College of Education are formal recognition of Hanna Neumann's impact on a country she had first set foot in only in August 1963. But then Hanna Neumann was a remarkable person. Throughout her life she had won the love and respect of many people. The extent of this can not really be measured, however some indications can be given. A memorial meeting held in Canberra overflowed a large lecture theatre even though it was virtually vacation time. A collection of papers dedicated to her memory and a fund to provide some form of memorial to her have both drawn quite overwhelming support from many parts of the world. One finds a tremendous list of words describing her memorable qualities: warm, enthusiastic, inspiring, energetic, firm, tactful, sympathetic, efficient, patient, shrewd, humble, peace-loving, courageous, gracious, ... No words can hope to evoke more than a pale shadow of such a person; this story must be read in such a light.

A description of Hanna's life (she was not a formal sort of person and much preferred this simple style of address) divides rather naturally into three parts: Germany 1914–1938; Britain 1938–1963; Australia 1963–1971.

Hanna was born in Berlin on 12 February 1914 the youngest of three children of Hermann and Katharina von Caemmerer. Her father was the only male descendant of a family of Prussian officer tradition. He broke the tradition to become an historian. He had a doctorate and his venia legendi (right to lecture) and was well on the way to establishing himself as an archivist and academic historian when he was killed in the first days of the 1914–18 war. Her mother was descended from a Huguenot family which had settled in Prussia in the second half of the eighteenth

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century. The older children were a brother Ernst (1908) and a sister Dora (1910). Her brother is Professor of Law at Freiburg i.Br. — he was for a time Rektor (Vice-chancellor). Her sister (who also has a doctorate) is in Berlin working in the re-training of social workers.

As a result of her father's death the family lived impecuniously on a war pension which had to be supplemented by other earnings. Already at the age of thirteen Hanna contributed to the family income by coaching younger school children. By the time she reached the final years at school she was coaching up to fifteen periods a week. This presumably helped teach her to organize her time efficiently.

After two years in a private school she entered the Augusta-Victoria-Schule, a girls' grammar school (Realgymnasium), in 1922. She graduated from there early in 1932. Her school report for university entrance lists fifteen subjects taken. She attained a grade of 'good' or higher in all but one of these in the Abiturium (final examination); the exception was music which she none-the-less liked and maintained an interest in throughout her life. The report comments that she showed independence of judgement, acute thinking (well beyond the requirements of the school) in mathematics and natural science and a noteworthy willingness to help. Only one teacher stood out in Hanna's memory of her school days—Fräulein Otto, her form mistress and French teacher for the final two years. This woman, who was to become a trusted friend in the turbulent Nazi years ahead, by the example of her fortitude, sense of humour, tolerance and wisdom, strongly influenced Hanna's view of people and events; her lack of hatred and bitterness, more than anything else, convinced Hanna that they have no place, ever, in human relations.

Her early hobby was botany. She collected plant specimens and built up voluminous herbaria for about four years until at about the age of fourteen this interest was superseded by her interest in mathematics.

Hanna entered the University of Berlin at the Easter of 1932. Her first year, the summer semester of 1932 and the winter semester of 1932–3, was all that she had dreamt it would be. The lecture courses in mathematics she took that year were: Introduction to Higher Mathematics given by Feigl; Analytical Geometry and Projective Geometry both given by Bieberbach; Differential and Integral Calculus, E. Schmidt; and The Theory of Numbers, Schur. The first of these courses eventually appeared in print in 1953 under the names of Feigl and Rohrbach [A]; in the introduction one finds an acknowledgement to use made of notes taken by Hanna in that summer semester of 1932. She was introduced to physics by the Nobel Laureate Nernst in a course of lectures on Experimental Physics. She also attended a course, Introduction to the Theory of Physics, by Orthmann. As well as these formal courses she took full advantage of the German tradition of attending lectures on a wide variety of topics. She listened to Köhler, one of the originators of Gestalt theory, on Psychology; to the well-known Roman

Catholic theologian Guardini on Dante; and to Wolff, the leading academic lawyer in Germany, on Common Law (his popularity was such that he always had overflow audiences in the biggest lecture theatre in Berlin University).

Bieberbach, Schmidt and Schur, all full professors of mathematics, were to have strong mathematical and personal effects on her life. Bieberbach was the first strong mathematical influence. He was, to her, an inspiring mathematician in spite of disorganized lecturing. He nearly turned her into a geometer. In fact she seems to have had quite a strong geometrical bent. Schmidt and Schur were, respectively, responsible for her introduction to Analysis and Algebra.

In this first year at university besides the excitement of study and the inevitable coaching there were, because lectures started early (8 a.m. and sometimes in summer 7 a.m.) and finished late, coffee breaks. Hanna soon found herself in a group of people, all senior to her — some already with doctorates — many of whom were later to make their mark in mathematical circles. It included Werner Fenchel and his future wife Käte (both professors at Copenhagen), Kurt Hirsch (recently retired as Professor of Pure Mathematics at Queen Mary College, London), Rudolf Kochendörffer (Professor at Dortmund, for a time Professor of Pure Mathematics in the University of Tasmania), Erika Pannwitz (formerly Chief Editor of the *Zentralblatt für Mathematik*), Richard Rado (Emeritus Professor at Reading), Helmut Wielandt (Professor at Tübingen and longtime editor of *Mathematische Zeitschrift*) and, in particular, her future husband Bernhard H. Neumann (Professor of Mathematics at the Australian National University).

The friendship between Hanna and Bernhard started in January 1933 and quickly blossomed into something special. In August 1933 Bernhard left for Cambridge in England; it had become clear that Germany would be no place for Jews for some time to come. At the Easter of 1934 Hanna visited Bernhard in London and they became secretly engaged; already the climate in Germany, and soon the law, was against such 'mixed' marriages. Then Hanna returned to her studies.

As a result of her work in her first year, Hanna won three-quarters remission of fees and got a job as a part-time assistant in the library of the Mathematical Institute. This meant not only a lighter load of coaching but also, very importantly, an earlier than usual introduction to a wider range of mathematical books and to mathematical journals.

In Germany at that time the first university degree was a doctorate (of philosophy). However university study could also lead to the Staatsexamen which was a necessary prerequisite for entry into the public service including the teaching service. The formal requirements for both were similar. There were certain attendance requirements: at lecture courses, at exercise classes, at practical classes, at seminars and at a physical education course (swimming in Hanna's case). There was also for each a final examination. The examination for the

Staatsexamen laid more stress on breadth, it consisted of two essays and an oral examination in two major fields of study and one subsidiary (the example of Hanna's examination will be given a little later). The examination for the doctorate laid more stress on depth; it consisted of a thesis usually embodying some original results and an oral examination in two major fields (for example, Algebra and Analysis), a minor field (say, Experimental Physics) and a subsidiary (say, Philosophy). As a consequence, in the first couple of years the final goal was relatively unimportant.

In her second year Hanna attended lectures on Higher Geometry, Differential and Integral Calculus, Differential Equations, The Theory of Functions, Ideal Theory, Mechanics, and General Experimental Chemistry. She took part in the exercise classes associated with some of these courses, in the beginners' practical classes in Astronomy and Physics, and in a junior (pro-)seminar.

There is a story about the practical Physics class which illustrates a significant feature of Hanna's make-up. During the course the students, working in pairs, were required to use a theodolite to measure the height of a distant chimney stack. Hanna and her partner made the measurements, did the appropriate calculations and took the work for marking. They were told their result was significantly wrong and to repeat the work. This they did with essentially the same result. They were then told how far short their result was and to try again. They did with again much the same result. They then managed to persuade the demonstrator to check the measurements. Much to his surprise his agreed with theirs. Investigations revealed that a few years earlier the stack had been lowered by several courses of bricks!

During Hanna's first year at university the Nazis came to power and Hanna was outspokenly critical of them. The Nazis tried to stop the lectures of Jewish staff by organizing protests and violence in them. In her second year Hanna was active in a group of students who tried to protect the Jewish lecturers by ensuring that only genuine students attended their lectures. In spite of this student support the objective was achieved. Moreover people with Jewish ancestry were prevented from studying in universities. Hanna lost her job in the Mathematical Institute, presumably as a result of these activities. However she had by then won, and continued to earn for the rest of her course, full remission of fees.

In her third year Hanna attended lectures on Set Theory, Elliptic Functions, Groups of Linear Transformations, The Theory of Functions, The Theory of Invariants, the Theory of Electricity and Magnetism, Logic and Fundamental Questions of Metaphysics. She did practical physics and attended the Analysis, Geometry and Algebra seminars and also the Philosophy of Religion seminar of Guardini (this latter she regarded as a particular honour as attendance was by invitation only and it was not one of her major studies).

Early in the third year Hanna was invited to become a reviewer for the *Jahrbuch über die Fortschritte der Mathematik*. A couple of years later she had a vacation job in the editorial office; employed, as she was much later to describe it, 'rather like a superior office boy'.

In her fourth year she attended courses in the Theory of Functions, Additive Number Theory, Galois Theory, The Philosophy of History, The Principal Problems of Systematic Philosophy and The History of the Development of German Education. She attended exercise classes in the Introduction to Philosophy and on Plato's Republic, a Philosophy colloquium, further practical physics classes and again the Analysis and Algebra seminars and the pro-seminar of A. Brauer.

The Nazi terror had the effect of polarizing people; it was almost impossible to remain neutral. Hanna was fascinated and frightened by this process — fascinated by the way she and others developed a sixth sense for detecting the direction in which people had become polarized, frightened by the way some people reacted (one eminent mathematician started writing in all seriousness about the differences between Aryan and Jewish mathematics).

There was also a direct effect on her studies. Hanna had by now set her sights directly on a doctorate. However in her fourth year she was warned that in the oral examination the above-mentioned mathematician would personally examine her on "political knowledge" which was by now compulsory. She was advised to switch quickly to the Staatsexamen for which, though it had a similar requirement, the oral might be arranged with a different examiner. She could then go on and do a doctorate at another university.

As remarked earlier the Staatsexamen had requirements which placed more emphasis on breadth than those for the doctorate. Hanna chose to be examined in Mathematics, Physics and Philosophy. This involved an oral examination in all three subjects and extended essays in Mathematics and Philosophy. The switch also involved some last minute changes in her course for the eighth semester to meet the requirements in Philosophy. Fortunately she was able to find a Philosophy lecturer who was sympathetic to her difficulties. He suggested the essay topic: The epistemological basis of number in Plato's later dialogues. Though this work was intended as a make-weight, Hanna tackled it with commendable thoroughness. In order to be able to compare the translations of critical passages she acquired a rudimentary knowledge of Greek in a couple of months of private study. The mathematical essay was: The construction of relative cyclic fields. The summer semester of 1936 was spent on leave from courses preparing for the orals in August. Preparation was seriously disrupted by an attack of scarlet fever. Nevertheless she obtained distinctions in both Mathematics and Physics and good in Philosophy for an over-all award with distinction.

During all this time Hanna and Bernhard kept in contact by correspondence. It was, in the circumstances, not an easy correspondence; it was conducted anonymously through various friendly channels. They met only once during this period

— in Denmark for a couple of weeks in 1936 when Bernhard was travelling from the International Congress of Mathematicians in Oslo.

With the Staatsexamen completed and through the good offices of Hans Rohrbach, a lecturer at Göttingen and former Assistant at Berlin (now Emeritus Professor at Mainz), Hanna was accepted as a research student by Hasse, one of the professors in Göttingen. He also found her a minor tutoring and assistant's job with which she could finance her stay. Before taking up studies there in the summer of 1937, Hanna spent six months working in the statistics department of an institute of military economics. Göttingen was very active though it was no longer the outstanding centre that it had been before the advent of the Nazis [B]. As well as Hasse and his team, there was Siegel and his co-workers. Hasse believed in team work: he assigned each of his school some task towards a common goal. At that time it was the Riemann conjecture in algebraic function fields of characteristic p. Séminars were used to ensure that everyone retained an overall picture of the project. The most powerful members of the team were Witt, H. L. Schmid, and Deuring; fellow students were Günther Pickert and Paul Lorenzen.

In Göttingen Hanna found time for some chess and some gliding. She also found time to attend a course on Czech — this because a friend wanted to learn the language and the minimum class size was two. The course was no hardship as Hanna had a flair for learning languages, one that she put to good use later in her professional career in reading papers in a wide variety of languages.

Early 1938 saw the annexation of Austria and summer the Czechoslovak crisis. Hanna decided it would be impossible to complete her course without risking a prolonged delay in her marriage plans. So, after three semesters, she gave up her course and in July 1938 went to Britain. Hanna never harboured any bitterness or resentment against Germany and was later to enjoy a number of visits there.

The first years in Britain were far from easy. Yet they saw the beginning of her family, and the begining of productive research. Hanna and Bernhard felt they could not openly marry until his parents were safe from possible reprisals. Bernhard was a Temporary Assistant Lecturer in Cardiff. Hanna went to live in Bristol. There she started working on a problem, suggested to her by Bernhard, that was to be the seed for her first paper, "On the elimination rule" [1]. The opening two paragraphs of the paper tell the story:

> "Chess matches are often decided according to the following elimination rule. The team with the higher score wins, of course. If both teams score the same number of points, the one that lost at the last board at which the game was not drawn wins the match. The problem is to find an arithmetical equivalent of this rule, i.e., to attribute to the single boards positive integral weights (which then have to be chosen as small as possible) such that the result is in accordance with this rule.

We solve this problem as a special case of the following more general problem."

It was also then that she started working on finite plane geometries, an interest that was to remain with her throughout her life. The interest was inspired by a report Bernhard gave her of a lecture describing the connection between Graeco-Latin squares and finite planes that he heard at the British Association meeting in August 1938. Her work on finite planes, though rarely a major interest, provided material for several lecture courses and occasional lectures, and in 1954 a paper "On some finite non-desarguesian planes" [14]. In a memorial lecture in Toronto the leading geometer Coxeter described this as an important contribution. She showed the existence of finite planes with two types of quadrangles: some whose diagonal points are collinear, and some whose diagonal points are not (the Fano configuration). She made the bold (according to Coxeter) conjecture that a finite plane in which all quadrangles are of the same type is desarguesian. This conjecture is still unresolved.

Late in 1938 Hanna and Bernhard were secretly married in Cardiff. They finally set up house together in Cardiff early in 1939 when Bernhard's parents joined them. Later that year their first child, Irene, was born. During this time in Cardiff Hanna's earlier interest in botany was turned to practical use. The family were able to vary and supplement their diet with the use of such plants as sorrel which could be found growing wild.

Both Hanna and Bernhard were classified as 'least restricted' aliens. This meant that at first they were not affected by restrictions on aliens. However, after Dunkirk a larger part of the coast was barred to all aliens and they were required to leave Cardiff. They moved to Oxford — because it was a university town. Within a week Bernhard was interned and a few months later released into the British army. Meanwhile Hanna, expecting a second child, made arrangements to complete a doctorate (D. Phil.). This was made possible by the Society of Oxford Home Students (later St Anne's College) through which she enrolled, and a generous waiver of fees that Oxford University granted to all refugee students whose courses had been interrupted. Just after Christmas the second child, Peter, was born (he has become a mathematics don at Oxford after himself gaining a D.Phil. from there).

On leaving Germany Hanna had abandoned her research on algebraic function fields feeling that it was not fruitful to continue this line outside the team. (She was not aware till after the war that Weil [C] had solved the problem in 1940). For her D. Phil. thesis she chose the problem of determining the subgroup structure of free products of groups with an amalgamated subgroup. This had been suggested in the paper of Kuroš in which he solved the corresponding problem when there is no amalgamation. Her research supervisor was Olga Taussky-Todd (then a lecturer at Westfield College, London, which had been evacuated to Oxford; now a professor at the California Institute of Technology). The supervision was largely a formality as Hanna made good progress and her supervisor was not especially interested in the topic of research. Hanna also had once or twice a term to visit her College Tutor. On these occasions fellow students would mind the children in the common room. The children used to travel in a side-car attached to Hanna's bicycle. The combination became well-known throughout Oxford.

The major problem during this time was accommodation. The original flat became unavailable towards the end of 1941. It was not easy to find accommodation with two young children and was made no easier by having to compete with refugees from the bombing of London. All Hanna could find was a subletting of part of a house — with shared facilities. A year later another move became necessary. This time Hanna found a brilliant solution. She rented a caravan and got permission from a market gardener to park it on his farm. She also, as was necessary, had it declared 'approved rooms' by the Oxford Delegacy of Lodgings.

It was then that the thesis was largely written; in a caravan by candlelight. The typing was done on a card-table by a haystack when the weather permitted. The thesis was submitted in mid-1943. Soon after, restrictions on aliens were eased and Hanna was able to return to Cardiff. In November of that year the third child, Barbara, was born (she graduated in Mathematics from Sussex University and went on to teach mathematics).The thesis was examined by two Fellows of the Royal Society — Philip Hall (later Sadleirian Professor of Pure Mathematics at Cambridge) and Henry Whitehead (later Wayneflete Professor of Pure Mathematics at Oxford). The oral examination took place in Oxford in April 1944. Hanna returned to Cardiff with her D. Phil.

A year later the war in Europe was over. Bernhard was demobilized from the army and resumed his university career at the beginning of 1946 with a Temporary Lectureship at the University College in Hull. At the same time the fourth child, Walter, was born (after studying at universities in New York, Adelaide and Bonn, he gained a doctorate and is now active in mathematical research). For the next academic year Bernhard was made a Lecturer. Hanna was offered a Temporary Assistant Lectureship which she took and thus began her formal teaching career.

Hanna was to stay in Hull for twelve years rising through the ranks to be by the end of her time there a Senior Lecturer. She also saw the transformation from a college of about 500 students being prepared for London external degrees to an autonomous university of about 1400 students. Bernhard, on the other hand, received an invitation to a Lectureship at Manchester and from October 1948 spent his terms in Manchester.

The curriculum of British universities was not one which Hanna's training had specifically equipped her to teach. In reviewing the book of Feigl-Rohrbach [D], she regretted that a course of that kind was not suitable for use in British universities "where so much more time is spent on enabling a student to solve problems

— or perhaps: so much more care is taken to turn out students not worried by an integral or a differential equation". With characteristic energy, and she would no doubt say because of her more mathematical training, she learnt the requisite techniques and was able to give lectures which students found clear and illuminating though demanding. The head of the department in Hull was an applied mathematician. So Hanna, with her (by British standards) very pure background, became the focus for moves to change the curriculum to introduce some of the more recent developments in pure mathematics. Here her ability to argue a case clearly, firmly and with tact was invaluable in getting changes made.

She took an active interest in her students. She was a strong supporter of the student mathematical society. She gave lectures to it on a number of occasions on such topics as: Dissection of rectangles into incongruent squares; Difficulties in defining the area of surfaces; and Prime numbers. Her aim was to exhibit some of the facets of mathematics for which there was not enough time in the regular courses and, as always, to convey her joy in mathematics. It was one of Hanna's striking qualities that she found joy in so much. The model building group also had her active support; in particular she participated in the making of paper models of regular and other solids. The outstanding feature, though, was her coffee evening. She often invited staff and students to meet at her house over coffee. This turned into a regular weekly open house at which her students were always welcome and, as one of her colleagues of those times says, "many benefited greatly from being able to drop in for company, discussion and often help with personal affairs". She was very interested in people and in seeing that they made the most of their abilities. One finds over and over that her interest in someone's work and her encouragement of it played a significant role.

A number of people now teaching in British universities received significant help from Hanna. One of the undergraduates, John Britton, stayed on to take a Master's degree under Hanna's direction. This involved preparing him for two examination papers; he chose Group Theory and Analysis. The latter involved Hanna in learning a lot of hard analysis by working through Whittaker and Watson [E]. He then went to Manchester to work for a doctorate under Bernhard's supervision and is now a professor at Queen Elizabeth College, London. One of her young colleagues, John Shepperd, who had a Master's degree for work of an applied nature, became interested in Group Theory, and, under Hanna's guidance, gained a doctorate for work in it. John Bowers, who is now lecturing at Leeds, took a Master's degree under Hanna and went on to London to do a doctorate.

Meanwhile the family thrived and grew with the addition of a fifth child, Daniel, born in 1951 (he has completed a university course in Mathematics and Greek). This was, of course, a very busy time for Hanna. Even with a home-help (in whom she invariably inspired intense loyalty), she had to be well-organized

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and call on all her resources of stamina, will-power and self-discipline. Visitors were always struck by the organization of the children: all had tasks to do and carried them out with responsibility and efficiency.

Research continued too. Two papers [2, 3] were prepared from material of the thesis and published in the *American Journal of Mathematics*. In Manchester Bernhard shared an office with Graham Higman (now Whitehead's successor at Oxford) and this led to the joint paper [4] in 1949 which is much quoted and has led to certain groups being called HNN-groups. Her own research and joint research with Bernhard also progressed well and resulted in a number of papers. In 1955 her published work was submitted to Oxford and judged worthy of a D.Sc. A lecture given by H. Hopf, a very distinguished topologist, to the fourth British Mathematical Colloquium in 1952 helped revive interest in a group-theoretic problem of his which is related to the structure of certain manifolds. Hanna was invited to lecture at the sixth British Mathematical Colloquium in 1954 and chose to report on Hopf's problem. The problem involved a property of groups which is now called the Hopf property. Hanna reported on the state of knowledge about Hopf groups and went on to ask a number of questions about them. One of these, whether the free product of finitely many Hopf groups is again a Hopf group, was to concern her for quite a number of years. In 1954 she attended the International Congress of Mathematicians in Amsterdam and reported on some work on near-rings [12]. This led on to work on varieties of groups [15] which was to be a very significant part of her mathematical career and in which she was to be a leading figure.

As if she didn't already have a full load, Hanna also took on for a time the job of Secretary of a local United Nations Association branch.

At various times from 1948 on Hanna looked for a suitable position in Manchester so that the family could lead a life under one roof. This search finally succeeded in 1958, when the Faculty of Technology of the University of Manchester (now The University of Manchester Institute of Science and Technology) decided to set up an honours program in mathematics and were looking for a relatively senior pure mathematician to be responsible for that aspect of the courses. (There was and is no formal contact between the Department of Mathematics in the Faculty of Techonology and that in the other part of the university in which Bernhard was by then a Reader; they are also physically quite separated.) Hanna applied for and was appointed to a Lectureship in the Faculty of Technology — with the understanding that the drop from Senior Lecturer would be short-lived; and indeed it was. It was considered by some that this was not only a drop in rank but also a drop to a lower kind of institution. Hanna did not feel this and in a lecture to her former colleagues at Hull a year later was able to report from experience that she saw no justification for that view.

Before taking up the appointment in Manchester in October, Hanna and Bernhard fitted in a stay at the International Congress of Mathematicians in Edinburgh and a cycling holiday with the family. Longish cycling trips with the children had become very much part of their life and cycling remained an important recreation with Hanna.

During Hanna's first year in Manchester Bernhard took his first study leave. In the nine months of it he visited India and Australia. Hanna took over the supervision of one of his research students (MFN).

Hanna set about organizing courses which would show the students something of mathematics as she saw it. She was able to introduce into the first year course, which had till then been entirely problem-oriented, a small strand of one lecture a week of an introduction to mathematics in the style of Feigl-Rohrbach. The later-year algebra courses much more thoroughly reflected her own interests and views. She continued to develop a style of teaching which aimed at making the acquisition of very abstract ideas accessible through judicious use of more concrete examples and well-graded exercises. Through the use of books like those of Kemeny and others [F, G], she was able to emphasize to undergraduates that parts of mathematics other than calculus were being applied to branches of human endeavour other than physics. Hanna also set about building up an active teaching and research team around her. After a year John Shepperd came from Hull and is still there. He soon became involved with and solved a problem raised by a braid manufacturer which was first taken to the textile engineers and was brought by them to the mathematicians. The sloution of this used some quite deep group theory. Hanna was delighted with this application and built it into a lecture for non-specialist audiences [H]. In the following year (1960–1) Jim Wiegold, a former research student of Bernhard's with whom Hanna had started joint work on certain products of groups which they called linked products, joined the staff. That year Hanna started supervising her first research students: Ian Dey who is now a Senior Lecturer at the Open University, and Chris Houghton who is now a Lecturer at Cardiff. Ian Dey worked on the problem of whether the free product of finitely many finitely-generated Hopf groups is again Hopf and settled a number of special cases.

Life thus continued very busy. Hanna would sometimes work all night reading manuscripts or preparing lectures, take a good long shower and appear in the office seemingly as fresh as if she had had a night's sleep. She did not allow this pressure of work to interfere with her contact with fellow staff and students nor with taking an interest in their work. There were regular coffee sessions at which they would discuss problems of interest. She was not beyond getting new experiences such as that of wall-papering.

In the summer of 1959 she went on a fortnight's tour of universities in

Hungary lecturing on various aspects of her research. Hanna also received an invitation to address the twelfth British Mathematical Colloquium in 1960. On this occasion she talked about Wreath Products — a group construction which had implicitly seen the light of day in the work of Frobenius in the 1890s but which had really burst into prominence in the 1950s. It had played a key part in some work with Bernhard [18] and was to play a key part in some other work a year later [23].

Group theory is studied by mathematicians largely for the fascination of its problems and the appeal of its ideas. However certain aspects of it have proved useful in the application of mathematics to various fields but especially physics. While Hanna was always at pains to stress that she saw the intrinsic motivations of beauty and joy as quite crucial, she was also interested in exploring such applications. Therefore she agreed to take part in a postgraduate course run by mathematicians and physicists on representations of groups. The mathematicians were to begin by giving a detailed account of those parts of the theory of interest to the physicists and then the physicists were to take over and explain how the theory was used. Hanna gave the mathematical lectures during 1960–1; the physical part never eventuated.

During 1960–1 preparations were made for a joint study leave by Hanna and Bernhard at the Courant Institute of Mathematical Sciences in New York in 1961–2; Hanna was a Visiting Research Scientist. It was also then that an offer came to Bernhard to set up a research department of mathematics at the Australian National University. Hanna was offered a post as Reader (now called Professorial Fellow) in that department. They accepted, with Bernhard to take up his appointment after the year in New York and Hanna a year later after discharging her obligations to her research students in Manchester.

The year in New York was very successful. They were accompanied on the trip by their three sons. The eldest (by then an undergraduate at Oxford) started an active interest in research under the guidance of one of the professors there, Gilbert Baumslag (another former student of Bernhard's), and was soon involved in his parents' research. During the year Bernhard, Hanna and Peter solved the problem of the structure of the semigroup of varieties of groups, showing that it is free [23]. Together with Baumslag the three of them also made a significant study of varieties of groups that are generated by a finitely-generated group [27]. Hanna gave a number of invited lectures in the course of the year.

While Hanna was away, the group at Manchester grew with the addition of another staff member. Laci Kovács (yet another former student of Bernhard's who is now at the Australian National University) and three more research students (supervised by some of the other staff). One of these, Carl Christensen, was a recent graduate of the department who had been inspired to do further work in mathematics by Hanna. The year of winding up was also a hectic year. Hanna was invited to give quite a number of lectures around the country on the New York work. She also gave a graduate course on varieties of groups: notes were taken by her two students [26]. The course was to be very influential in stimulating the growth of interest in this part of group theory. It was during this year that the so-called finite basis problem for varieties generated by a finite group (first posed by Bernhard in 1935) was solved in stages in Oxford. Hanna reported on progress as it happened. Very typically she kept tabs on what was happening and by her interest encouraged the people making the progress. It also involved a lot of effort on Hanna's part working into an area of group theory with which she was not very familiar. Soon after she reached Australia she was able to report the successful completion of the solution.

While in Manchester Hanna took an active role in the Mathematical Society, a group of people interested in mathematics in a wider and to some extent non-professional sense.

In August 1963 Hanna left Britain to face new challenges in Australia. Hanna came to a research post in which she hoped to pursue her research interests and guide some research students to doctorates. In fact two students were waiting for her when she arrived. They were Martin Ward and Bob Burns; both successfully completed doctorates and are now in university teaching posts at the Australian National University and York University (Canada) respectively. Her first goal was to polish the lectures on varieties of groups into a monograph [29].

Instead Hanna found herself heading into major teaching responsibility. She was invited to take the newly created chair of Pure Mathematics in the National University's School of General Studies (that is the part of the university which is responsible for the teaching of undergraduate students and in which the academic staff are expected to devote a significant part of their time to teaching duties). With the chair went the headship of the Department of Pure Mathematics which, together with the Department of Applied Mathematics, had grown out of the fission of the former Department of Mathematics. She accepted the invitation and took up the appointment in April 1964.

She also quickly became involved with helping teachers in secondary schools with some of the problems being created by the introduction of the Wyndham scheme into secondary schooling in New South Wales. This scheme involved a radical restructuring which forced the creation of new syllabuses. In mathematics these new syllabuses reflected some of the changes that were taking place in the teaching of mathematics in other parts of the world. Many teachers found that their training had not prepared them to teach some aspects of these syllabuses. In the first term of 1964 Hanna and Ken Mattei, one of the mathematics masters in Canberra, ran (under the auspices of the Canberra Mathematical Association) a once-a-week course for teachers entitled "The language of sets in school mathematics". This was Hanna's first excursion into this kind of activity, however her experience and sensitivity enabled her to hit the right note and she was thanked "...for the lessons and guidance given so cheerfully and efficiently". This direct involvement with secondary teachers was, as will be seen, to continue for the rest of her life.

Meanwhile Hanna set about building up a department of pure mathematics under difficult circumstances. Most of the more experienced staff of the former Department of Mathematics had, because of their research interests, joined the Department of Applied Mathematics. At that time experienced staff was almost impossible to come by. Fortunately Hanna was, in time, able to attract some of her former students to join her (Martin Ward, Carl Christensen and Ian Dey) and by her guidance and enthusiasm to build up an active and keen young department round her. In this she was helped by being able to draw on some of the people in Bernhard's department for occasional advanced courses, by being able to attract some more senior people as visitors (including M. Stone, professor at Chicago, and Coxeter, professor at Toronto, for a term each and Jim Wiegold for two years), and to use some of the research students to help out with part-time tutoring.

Hanna was concerned to see that all students got courses suited to their needs. On the one hand she wanted the better students to get a real appreciation of mathematics so that they could sensibly decide whether they wanted to make a career within mathematics and be well prepared to do so. In this respect, besides making available an intensive course of study through lectures, she instituted forms of examining, especially take-home assignments, which encouraged more sustained use of the ideas and techniques involved than the conventional short closed-book examination. She also made a supervised project an important component of the final honours year. While this was not intended, these projects occasionally produced original research some of which has been published. On the other hand she was deeply concerned that students with a limited background who were intending only one year's study of mathematics should get as clear an understanding as possible of the nature of the subject because many of these people would be required to make some use of mathematics later in their lives. She was keen to get over the idea that doing and thinking about mathematics can be joyous human activities, though it needed effort to get the rewards. She conveyed this by her own obvious joy in the subject and her willingness to work hard. It is not really possible to assess how successful these shorter courses were in achieving their aims; certainly the classes seem quite happy with them. The success of the intensive course is more easily measurable: at least a dozen students have gone on to complete doctorates in such widely scattered places as Cambridge, Edinburgh and Oxford in Great Britain, Chicago and Seattle in U.S.A. and Kingston in Canada as well as in Australia; mostly in mathematics but also in computing, physics and the history of science. These doctorates have been attained by graduates from the honours classes of 1965 to 1968 and represent about half the graduates from those classes. Quite a few of the later graduates are at present working towards doctorates.

Not only did she have these ideas about teaching which she put into operation, she also created an atmosphere in which her staff were encouraged to have ideas about teaching and to discuss, plan and execute them. Some of these won their way into wide acceptance. For instance a suggestion by a part-time tutor was the seed from which a course on distributions (in the sense of Schwartz) to third year pass students grew. Hanna gave this course a number of times and the lecture notes have been published [31]. These notes were used by Erdélyi, professor at Edinburgh, in connection with a course he gave at the tenth Summer Research Institute of the Australian Mathematical Society and have been used for a course at the University of New South Wales. A short course on computing designed by Bill Steiger and Martin Ward has been made available to the first year students.

As well as creating the course on distributions, Hanna designed some of the details of the more problematic elementary courses and used courses to the final year honours students to work up a knowledge of important areas related to her research interests such as: cohomology of groups and Lie methods in group theory. She supervised the project work of a number of fourth year students on these topics but also on normal numbers and Hilbert's tenth problem.

Hanna believed in making herself available: as far as formal commitments allowed, she was always in her office with the door open. She encouraged students to seek help with their difficulties and she was often to be seen explaining a point at her blackboard. She also found herself helping students with non-mathematical problems. Her impact here is best summed up by the following extract from a letter by two students published in the local paper just after her death:

> "We will remember her not only as a mathematician; she was a friend who always had a sympathetic ear for any student, and was never too busy.
>
> We will always miss her tremendous dedication and sincerity, and the friendliness of her presence."

Of course the price was paid in much midnight oil.

The new responsibilities drastically reduced the time Hanna had for research and research related activities. Production of the monograph slowed down. In 1965 she helped organize in Canberra a very successful international conference on the theory of groups. At the conference she gave one of the major survey talks — on varieties of groups [30] — in which she was able to report on some of the work that had been inspired by her original course. In 1966 she attended the International Congress of Mathematicians in Moscow and reported on recent work in Canberra on varieties [I]. The monograph was finished towards the end of 1966 and appeared early in 1967. It showed quite clearly the influence of the earlier course in developing interest in the subject. The monograph listed some of the unsolved problems, many of Hanna's own devising, about varieties. Many have been by now solved, quite a number of these by people in Canberra who have been inspired by Hanna to take up an interest in the subject. Almost immediately a Russian translation was started by Šmel'kin in Moscow — this was ready with a couple of appendices a year later but did not appear till 1969. She was invited to give talks on this work at various Australian universities and had been invited to give one of the major lectures at the Australian Mathematical Society meeting in 1972. In 1966 her first two research students in Australia completed their courses and Hanna took on two new students, Chau and Itqan Farouqi, who also went on to take doctorates and take university appointments in Sudbury (Canada) and Karachi (Pakistan) respectively. These two were followed in 1969 by two more, Bill Haebich and George Ivanov, who have both recently completed their doctorates.

Of course family life continued. Only one child was still living at home. However the family was supplemented by a year long visit by a niece and a longish visit by Hanna's mother. There was also quite a lot of entertaining of a wide range of colleagues and students, of visitors to Canberra and of friends from their other activities. Hanna served her term on the executive of her local Parents' and Citizens' Association. Hanna's recreations were listed in *Who's Who* as cycling and photography. The former continued unabated: it was a common sight to see Hanna and Bernhard cycle to and from their offices or to their lunch-time coffee in the city. They also developed a fondness for four-wheel travel and saw much of Australia, especially the back-blocks which so many city-dwellers never see. The photography, which had been a brief interest during student holidays, was revived by coming across some old photos that she had taken. The royalties of the monograph bought a new camera. This interest was combined with the old interest in botany to build up an impressive collection of photographs of flowers and trees of all sorts but especially of many varieties of acacia. The chase for these involved much use of four wheels. It also resulted in bodily damage, and at least one broken rib is directly attributable to a chase after an elusive acacia. Such ailments had no noticeable effect on her work, and even a leg in plaster could do no more than keep her away from classes for a week — she still prepared the lectures for a colleague to give.

The interest in secondary education that had been kindled continued to grow. Later in 1964 Hanna gave an in-service course to teachers in Goulburn (a city about sixty miles from Canberra) on the new emphases in mathematics in the junior secondary school. That year also saw her taking an active part in the discussions on the new syllabuses for the senior forms. It was undoubtedly her work in evaluation of the draft proposals and her energetic work on suggestions for improvements which earned for the Canberra Mathematical Association a reputation for trenchant and constructive criticism. The following year when the syllabuses had been published she gave in-service courses on aspects of them in both Canberra and Goulburn. She visited Armidale and Newcastle in New South Wales and lectured to the Mathematical Associations there. In Armidale she also gave an intensive course to honours students on group representations. In Canberra she continued her support for the new spirit in the junior forms by giving a lecture (for the Canberra Mathematical Association) to parents "Learning Mathematics and learning Chinese", a title she borrowed from the introduction to a book by W. W. Sawyer [J]. She set out to explain to (an overflow audience of over two hundred) parents the ideas behind the new syllabuses and to enlist their co-operation in making them a success. She believed that the community had to be educated to create a more favourable climate (one in which mathematics is not feared) for the learning of mathematics — especially among girls. At the beginning of 1966 she lectured to the University of New South Wales' Summer School for Mathematics Teachers on Évariste Galois and the theory of equations [K].

January of 1966 also saw the meeting which finally, after four years of discussion, set up the Australian Association of Mathematics Teachers. Hanna was immediately elected to be one of the foundation Vice-Presidents. In that role she had, in September of that year, to deliver the first presidential address in the absence overseas of the President (Bernhard). In her address "Education in Semut" she described a semi-utopia in which mathematical education had reached the stage of incorporating all the best features of mathematical education that she had personally observed in various parts of the world. She admitted that no one system had all these features but felt that their existence somewhere made the achievement of the system she described realizable. She also had to chair the lengthy meetings of the first council and succeeded in moulding into a group this collection of individuals from all over Australia many of whom were meeting each other for the first time.

A little later in 1966 Hanna was elected Vice-President of the Canberra Mathematical Association as a prelude to becoming its President for 1967–8. It was during this time that the Canberra Mathematical Association pamphlets for teachers were largely prepared. This is a series of notes intended to provide teachers with background to the new topics in the senior forms which was inspired by some of the misunderstandings which showed up in many of the first text-books written for these syllabuses. Hanna wrote a pamphlet on Probability [28] which is the best-seller in the series. It has been described by one recent text-book author as the best account available anywhere of an introduction to this topic. The pamphlet is used as a text for first year students at La Trobe University. This series of pamphlets spawned the series of *Notes in Pure Mathematics* published by Bernhard's and Hanna's departments in which her notes on distributions are published.

In 1967 she gave the Canberra Mathematical Association lecture to school pupils on her much favoured topic of "Braids". These lectures which had been started early in the life of the Canberra Mathematical Association were at that time being replaced by the Friday evenings [L] of which Hanna was a very active supporter. She often attended and took an active part in the discussions over refreshments. When the A.N.U. — A.A.M.T. National Summer School for talented high school students was started in 1969 she was an enthusiastic supporter of it and on two occasions gave lectures on geometry which proved very popular.

In November of 1968 she was invited to give the inaugural address to the Riverina Mathematical Association. Under the title "Who wants Pure Mathematics?" she illustrated her view that the range of mathematics which is being applied had broadened a lot as have the fields of human endeavour to which it is being applied.

Hanna went on study leave in August 1969. Her first stop was at the First International Congress on Mathematical Education at Lyons. This provoked her into writing a letter (one of a very few) to the editors of several Australian newspapers which it seems appropriate to quote here:

> "The proceedings of this congress have confirmed my impression that the development of mathematical education in Australia is lagging behind that of the rest of the world to a frightening extent.
>
> Typically, while we in Australia are asking whether to teach computer programming in schools, the discussion here takes it for granted that this is done and goes on to consider the question of how the (new!) mathematics programmes have to be changed and reorganised to take account of the impact of computers on the content of mathematics.
>
> It is clear that the great advances in other countries stem from experimentation made possible by the enlightened flexibility of examining bodies and their clientele (for example, employers, universities) and the availability of funds.
>
> Certainly, mathematical education in Australia is changing, but the rate of change has to increase vastly if we want to catch up with the progress made elsewhere."

Because of Hanna's known interest in educational matters she was proposed for membership of the Australian College of Education early in 1968, was elected to Fellowship (F.A.C.E.) in 1970 and was a member of the A.C.T. Chapter committee in 1971.

Hanna's post as Professor of Pure Mathematics involved her in committee work within the university. On these committees her qualities of commonsense, balance, fairness and impartiality won her respect and her views were listened to. She was asked to take on some of the more demanding administrative tasks but usually felt she could not accept them without putting an unfair load on her young staff. She did, however, accept the position of Dean of Students from January 1968 till August 1969. In this she played an important role in maintaining good relations between the student body and the university authorities. The students appreciated the time and effort that she put into acting on their behalf and, though she could not always agree with their position, she was respected for her integrity and the soundness of her judgement.

The Australian Mathematical Society also made use of Hanna's organizational abilities. She was invited to be the director of the ninth Summer Research Institute held in January 1969. She invited Mac Lane, professor at Chicago, and Gaschütz, professor at Kiel, as main lecturers. This attracted the greatest attendance ever at a Summer Research Institute. Bonuses were visits by Erdős and Hirsch.

In March 1969 her academic excellence was given further recognition by her election to a Fellowship of the Australian Academy of Science (F.A.A.).

The next stop after Lyons in the year long study leave (taken with Bernhard) was a meeting on Decision Problems in Group Theory held in California. Then they went on to a five-month stay at Vanderbilt University in Nashville, Tennessee, where Hanna was on a National Science Foundation Senior Foreign Scientist Fellowship. Another visitor to Vanderbilt at that time was their eldest son Peter. Hanna gave a course to graduate students on Varieties of Groups. Into this she was able to incorporate a solution to one of the fundamental questions in the theory: the finite basis problem. News of a negative solution of the problem by a young Russian Ol'šanskiĭ (a student of Šmel'kin) reached them early in their stay. A better solution was found by Vaughan-Lee who was then also at Vanderbilt, having just completed a doctorate at Oxford under Peter's supervision. The course was concluded by Vaughan-Lee presenting his solution. However, the highlight of the stay was the solution of the problem on the free product of finitely many finitely-generated Hopf groups. Hanna and Ian Dey had been continuing work on the problem making some progress. Now with Hanna having more time to devote to the problem the final difficulty was overcome. It turned out that such free products are indeed Hopf groups [32]. The solution required almost all the techniques of this area of group theory often in specially sharpened form. This work was indeed a fitting climax to Hanna's research career. However with typical modesty Hanna's report to the university on the leave apologizes for her having only achieved this. During her time in Nashville Hanna was invited to give many lectures in other parts of North America. In spite of declining some invitations she still gave at least fifteen lectures in places as far apart as Atlanta, Houston and Toronto; usually on varieties or the Hopf problem. They then moved on to Cambridge in England where they stayed for the next four months, Hanna as Honorary Bye-Fellow at Girton College and as a Visiting Professor to the University. In the latter capacity she gave a course of lectures on varieties of groups. Here again she gave invited lectures up and down the country including one to the London Mathematical Society. She also managed to visit her (at that time) nine grandchildren. This stay was followed by six weeks at the Mathematisches Forschungsinstitut of the University of Freiburg delightfully situated in the Black Forest in Germany. Hence, refreshed, they did a three-week lecture tour of the Indian sub-continent spending the main time in Lahore, Madras and Madurai before returning to Australia in August 1970.

No sooner was Hanna back than she was invited to make a lecture tour of Canada under the Commonwealth Universities Interchange Scheme. This was arranged for the (Canadian) winter of 1971–72. In her department she found that the tightening financial position was making it more difficult to continue to offer the same services to students. This together with some changes in the structure of the university and problems which were becoming more clearly visible with some of the courses convinced her that a major new planning of courses would be needed and she set about initiating it.

During 1971 she was invited to give two talks. First in Adelaide to a joint meeting of the Mathematical Association of South Australia and the Australian Mathematical Society in which she talked on "Teaching first year undergraduates: fads and fancies" [33]; and second at Wodonga Technical High School to a regional meeting of teachers on "Modern Mathematics — Symbolism and its importance at the secondary and tertiary levels".

At the end of October Hanna set off on her Canadian lecture tour. She visited in quick succession the University of British Columbia, the University of Calgary, the University of Alberta, the University of Saskatchewan and the University of Manitoba. She arrived at Carleton University, Ottawa, on the 8th November for a somewhat longer stay. On the evening of the 12th she felt ill, admitted herself to hospital and quickly went into a coma. She died on the 14th without regaining consciousness.

2. Mathematical Papers

Some of the papers have already been discussed briefly in their biographical context. The aim of the present section is to describe them more fully and to relate them to the work of others. For convenience, the papers are grouped roughly under subject headings.

A. *Free products with amalgamations.* ([2], [3], [5], [6], [8], [11]). The papers under this heading arose, directly or indirectly, out of Hanna's D. Phil. thesis and as such represent her earliest work in group theory. Papers [2] and [3] are fundamental. The subsequent papers contain various developments and refinements. An excellent account of the general area will be found in the essay [a].

Roughly speaking, the *free product* G of groups G_α is the group generated, in a purely formal way, by their set-theoretical union; it is assumed that the unit

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element is common to all G_α but that no other element is common to any pair of groups G_α and G_β ($\alpha \neq \beta$). The famous *subgroup theorem* of Kuroš [b] asserts that every subgroup H of the free product G is itself a free product, each factor of which is either an infinite cyclic group or a conjugate $x^{-1}Kx$ of a subgroup K of one of the G_α.

The problem solved in [2] and [3] (and suggested by Kuroš himself) is to extend the subgroup theorem to a free product of groups G_α *with an amalgamated subgroup* U. The latter is a direct generalization of the ordinary free product: one assumes that the intersection of any two factors G_α and G_β is the amalgamated subgroup U rather than the unit subgroup. The free product with an amalgamated subgroup arises naturally in combinatorial topology as well as in group theory. If X, Y are (connected, overlapping) simplicial complexes, then the fundamental group $\pi(X \cup Y)$ is the free product of $\pi(X)$ and $\pi(Y)$ amalgamating $\pi(X \cap Y)$.

Hanna made the critical observation that in order to solve this problem the free product with *one* amalgamated subgroup must be generalized to a free product with *arbitrary* amalgamations. One is given a system of groups G_α and, for each pair of distinct indices α and β, isomorphic subgroups $U_{\alpha\beta}$ of G_α and $U_{\beta\alpha}$ of G_β with an "amalgamating" isomorphism $i_{\alpha\beta}$ between them. In the generalized free product with these amalgamations, which we may denote by $G = \{\Pi_\alpha^* G_\alpha;\ i_{\alpha\beta}\colon U_{\alpha\beta} \to U_{\beta\alpha}\}$, it is required that, for each pair α, β, every element x of $U_{\alpha\beta}$ be identified with the corresponding element $i_{\alpha\beta}(x)$ of $U_{\beta\alpha}$ and *there be no other identifications between the elements of* G_α *and* G_β. Such a free product need not exist, for it may be impossible to ensure that the final condition is fulfilled (even when the obvious necessary conditions are assumed to hold). In [2], the generalized free product is introduced, conditions for its existence are found and the difficulties charted by examples. In particular, the following reduction theorem is proved: let U_α denote the subgroup of G_α generated by the $U_{\alpha\beta}$ as β varies over all indices different from α; then the free product G above exists if, and only if, the free product $U = \{\Pi^* U_\alpha;\ i_{\alpha\beta}\colon U_{\alpha\beta} \to U_{\beta\alpha}\}$ exists. A much simpler proof of this was given later in the joint paper [5].

The actual generalization of Kuroš' theorem is stated and proved in [3]. Let G, U be as above, U being regarded as a subgroup of G in the obvious way. Then the theorem asserts that every subgroup of G is itself a generalized free product $\{\Pi_\lambda^* H_\lambda;\ j_{\lambda\mu}\colon V_{\lambda\mu} \to V_{\mu\lambda}\}$, where each amalgamated subgroup $V_{\lambda\mu}$ is conjugate to a subgroup of U and each factor H_λ satisfies one of the following three conditions: (a) it is conjugate to a subgroup of some G_α; (b) it is conjugate to a subgroup of U; (c) V_λ (the subgroup of H_λ generated by the $V_{\lambda\mu}$ for varying μ) is a normal subgroup of H_λ such that H_λ/V_λ is an infinite cyclic group. This is a deep theorem and the proof is both long and difficult. In contrast to the situation for Kuroš' theorem itself, Hanna's work has remained in its original form and it is only now that another treatment, based on the theory of groupoids and

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developed by Higgins [c], [d], may perhaps lead to a more accessible account of the general case. The results are frequently applied in the literature.

The union of the groups G_α after making the identifications indicated by the amalgamating isomorphisms $i_{\alpha\beta}$ is called a *group amalgam*. To say that the corresponding free product exists is to say that the group amalgam can be embedded in a group. Papers [6], [8], [11] are concerned with such embeddings, particularly in the case where the constituent groups G_α are abelian. The following two examples will indicate the kind of result proved. The statement that every amalgam of n abelian groups can be embedded in an abelian group is true for n at most 3, but false for n greater than 3. The statement that every amalgam of n abelian groups can be embedded in a group is true for n at most 4, but false for n greater than 4.

B. *Embeddings*. ([4], [9], [13], [19], [20]). The importance of group-theoretical constructions for the theory of infinite groups is now fully appreciated. The papers in the present section (all written in collaboration, chiefly with Bernhard) were among the first to demonstrate the striking results that can be obtained by systematic application of such, by now standard, constructions.

The power of the amalgamated free product as a tool for constructing groups is seen in the celebrated paper [4] of Higman and the Neumanns. Here it is proved, for example, that a group in which every element except the unit element has infinite order can be embedded in a group in which all non-unit elements are conjugate. Another result, also proved independently by topological methods by H. Freudenthal, is that every countable group G can be embedded in a 2-generator group G^*; moreover, G^* can be defined by as few relations as G. The subject of "HNN Constructions" still flourishes at the present.

A crucial step in [4] is to extend a partially defined automorphism of a group to a fully defined automorphism of some larger group. In [9], [13], an elegant solution is given to the corresponding problem for partially defined endomorphisms.

The paper [18] takes up again the embedding of countable groups in 2-generator groups but uses a different construction, *viz*. the wreath product of groups, to get sharper results. The main theorem asserts that every countable group belonging to a variety* $\mathfrak{V}$ can be embedded in a 2-generator group belonging to the variety $\mathfrak{V}\mathfrak{A}^2$. For example, taking $\mathfrak{V} = \mathfrak{A}$, we conclude that every countable abelian group can be embedded in a 2-generator soluble group of derived length at most 3. Another achievement of the paper is to construct a finitely generated *soluble* group which is non-Hopf, that is, isomorphic to a proper quotient group of itself.

The papers [19] and [20] belong to a rather different circle of ideas. A group

* The notation used here is explained under heading C below.

G generated by subgroups A, B such that the normal closure of B meets A in X and the normal closure of A meets B in Y is called a linked product of A and B with kernels X and Y. The problem is to construct such a linked product given A, B, and their normal subgroups X, Y. The general question of the existence of the linked product remains open, although a number of cases are settled in the two papers.

C. *Varieties.* ([7], [12], [15], [23], [26], [27], [29], [29′], [30]). The study of varieties of groups, initiated by Bernhard [e] in the 1930's, became established as an independent discipline during the 1950's. Hanna's book [29], published in 1967 and arising out of an earlier course of lectures [26], describes the very considerable advances made by the mid-1960's. There has been further progress since then but many problems remain unsolved.

The subject is concerned with identical relations, or laws, in groups. The terms "variety" means simply the class of all groups governed by some fixed code of laws. Going into more detail, we may conveniently define a *law in a group* G to be a word $w(x_1, \cdots, x_n)$ which is equal to the unit element 1 for all choices of the elements x_i in G; for example, since $x_1^{-1}x_2^{-1}x_1x_2 = 1$ is equivalent to $x_1x_2 = x_2x_1$, the word $x_1^{-1}x_2^{-1}x_1x_2$ is a law in G if, and only if, G is abelian. If W is a set of words, then the class of all groups in which the elements of W are laws is the *variety* $\mathfrak{V}$ *defined by* W; for example, the set W consisting of the single word $x_1^{-1}x_2^{-1}x_1x_2$ defines the variety $\mathfrak{A}$ of all abelian groups. Other common varieties are the "Burnside" variety $\mathfrak{B}_n$ defined by the law x_1^n, the variety $\mathfrak{N}_c$ of all nilpotent groups of class at most c and the variety $\mathfrak{S}_l$ of all soluble groups of derived length at most l. $\mathfrak{O}$ (the class of all groups) and $\mathfrak{E}$ (the class of all groups of order 1) are trivially varieties.

We have just introduced the variety defined by a given set of words. It is convenient to introduce also the variety *generated by a given set* S *of groups*. This is simply the smallest variety containing S. There is another, more constructive, definition which need not be given here.

Each variety $\mathfrak{V}$ has an *n-generator free group* $F_n(\mathfrak{V})$ with the property that the n-generator members of $\mathfrak{V}$ are precisely its homomorphic images. This group can be represented very concretely as follows. Let F_n be the free group (in the usual sense) on n free generators $x_1, \cdots, x_n$ and let V_n denote the subset of all n-generator laws $w(x_1, \cdots, x_n)$ which hold in $\mathfrak{V}$. Then V_n is in fact a normal (indeed, fully invariant) subgroup of F_n and $F_n(\mathfrak{V}) \cong F_n/V_n$.

In the development of the theory of varieties, Hanna's book [29] has played a central part. It has been an invaluable source-book—authoritative, timely, concise and yet most readable. Moreover, it presents a *living* subject still in the process of development. As an illustration, we may cite the 25 problems posed in the book—many, but not all, of which have now been solved. Here are two of them.

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Problem 3: Can every variety be defined by a finite set of laws?

Problem 15: Is $F_n(\mathfrak{V})$ always a Hopf group?

Problem 3 was solved in the negative by A. Ju. Ol'šanskiĭ [f]. Problem 15 still awaits solution.

Turning to the discussion of the papers, let $V_n[U_n]$ denote the intersection of the kernels at all homomorphisms of F_n into [onto] G, where F_n is an n-generator free group. Then F_n/V_n is the "V-group" of G studied in [e] (it is just the n-generator free group of the variety generated by G). The paper [7] deals with the corresponding U-group F_n/U_n. This amounts to studying the essentially different ways in which G can be presented as an n-generator group. Some rather delicate questions arise. For example, if H is a homomorphic image of G, is the U-group of H a homomorphic image of the U-group of G? Gaschütz [g] proved that this is true when G is finite, but Dunwoody [h] later showed it to be false in general.

In the important paper [15], the n-generator free group $F_n(\mathfrak{V})$ of a variety $\mathfrak{V}$ is examined from several different points of view. Amongst the varieties $\mathfrak{V}'$ such that $F_n(\mathfrak{V}') = F_n(\mathfrak{V})$, there is a largest, say $\mathfrak{V}^{(n)}$, and a smallest, say $\mathfrak{V}_{(n)}$. These are readily identified: $\mathfrak{V}^{(n)}$ is the variety defined by the n-variable laws of $\mathfrak{V}$ while $\mathfrak{V}_{(n)}$ is the variety generated by $F_n(\mathfrak{V})$. Moreover, $\mathfrak{V}$ is both the intersection of the descending chain $\mathfrak{V}^{(1)} \supseteq \mathfrak{V}^{(2)} \supseteq \cdots$, and the union of the ascending chain $\mathfrak{V}_{(1)} \subseteq \mathfrak{V}_{(2)} \subseteq \cdots$. The pertinent question is raised whether these chains become stationary after a finite number of steps. For example, $\mathfrak{V}^{(n)} = \mathfrak{V}$ means that $\mathfrak{V}$ is defined by its n-variable laws—an assertion which is certainly true for large enough n when $\mathfrak{V}$ can be defined by a finite set of laws. In fact, both chains may be infinite, as proved by Higman [j] for the ascending chain and Ol'šanskiĭ [f] for the descending chain.

The second part of the paper is motivated by the analogy between $F_n(\mathfrak{V})$ (with its "relatively free" generating set of n elements) and an n-dimensional vector space (with its basis of n elements). Under suitable definitions of $+$ and $\times$ (not quite the expected definition in the case of $+$), the endomorphisms of $F_n(\mathfrak{V})$ form a *near-ring* R_n, satisfying all ring axioms except commutativity of addition and the left distributive law. Furthermore, R_n satisfies a weak form of the missing left distributive law (it is "distributively generated" in the terminology of Fröhlich [k]). On the basis of these facts, a surprisingly detailed theory of ideals and homomorphisms is developed. While examining products of ideals, the paper draws attention for the first time to the important product operation for varieties (although the matter is not expressed in quite these terms). The *product* $\mathfrak{U}\mathfrak{V}$ of varieties $\mathfrak{U}$, $\mathfrak{V}$ is the variety formed by all extensions of $\mathfrak{U}$-groups by $\mathfrak{V}$-groups, that is, all groups G possessing a normal subgroup N such that $N \in \mathfrak{U}$ and $G/N \in \mathfrak{V}$. It is shown that the product is associative, that cancellation of right hand factors

$\neq \mathfrak{O}$ in equations is permissible and that every variety $\mathfrak{B} \neq \mathfrak{O}$, $\mathfrak{E}$ is a product of indecomposable varieties.

This work is brought to completion by the three Neumanns in [23] and independently by Šmel'kin [*l*]. They prove that the semigroup formed by the varieties $\mathfrak{B} \neq \mathfrak{O}$, $\mathfrak{E}$ is *freely* generated by the indecomposable varieties. The important new idea used in [23] is to relate the wreath product of groups to the product of varieties. For example, it is proved that, if $\mathfrak{U}$, $\mathfrak{B}$ are varieties and $\mathfrak{U}$ is generated by the group H, then $\mathfrak{U}\mathfrak{B}$ is generated by H wr $F_\infty(\mathfrak{B})$, where $F_\infty(\mathfrak{B})$ is the free group of $\mathfrak{B}$ on a countable infinity of generators.

In [27], Baumslag and the three Neumanns consider the conditions under which a variety $\mathfrak{B}$ is generated by $F_n(\mathfrak{B})$. Such questions can be extremely difficult and the very specific and detailed results obtained in this paper testify to the growing sophistication of the theory of varieties. Wreath products are again used to deal with product varieties. The new concept of a set of groups "discriminating" a given variety is put to effective use. One of the main results is that a product variety $\mathfrak{U}\mathfrak{B}$ is generated by its free group of rank $n \geqq 2$ when $\mathfrak{B}$ is generated by an n-generator torsion-free nilpotent group. It follows, for example, that the soluble variety $\mathfrak{S}_l$ is generated by its 2-generator free group. This is in sharp contrast to the situation for nilpotent varieties: it is shown that, if $\mathfrak{N}_c$ can be generated by its k-generator free group, then $k \geqq [\frac{1}{2}c]$. It was subsequently proved by Kovács, Newman, Pentony [m] and independently by Levin [n] that, for c at least 3, $\mathfrak{N}_c$ is generated by $F_{c-1}(\mathfrak{N}_c)$ but not by $F_{c-2}(\mathfrak{N}_c)$; this gives a complete solution to Problem 14 in [29].

D. *Other papers.* ([1], [10], [14], [16], [17], [21], [22], [24], [25], [32]). These deal with a number of topics, several of which will be mentioned quite briefly. [1] (on a combinatorial problem) and [14] (on projective planes) have already received some discussion. [10] deals with a particular problem about bilinear forms on abelian groups. [21] is concerned with associative operations that may be constructed from the multiplication in a group. Problems of this kind have been considered in some detail by subsequent authors; see, for example, Street [p].

Let U, V be subgroups of a free group F. Howson [q] proved the interesting theorem that $U \cap V$ is finitely generated when U and V are. In [16], [17], Howson's bound for the rank of $U \cap V$ in terms of the ranks of U and V is improved.

Let R, S be normal subgroups of a non-cyclic free group F. Auslander and Lyndon [r] proved that $R' \supseteq S'$ implies $R \supseteq S$, where R', S' denote the commutator subgroups of R, S. The attempted elementary proof in [22] misfired; see [25]. A generalized version of the Auslander-Lyndon theorem appears as Problem 18 in [29]. For a variety $\mathfrak{B}$ and group G, let $G/V(G)$ denote the largest quotient group of G which belongs to $\mathfrak{B}$; the problem proposed is to show that when $\mathfrak{B} \neq \mathfrak{O}$, $V(R) \supseteq V(S)$ implies $R \supseteq S$. The special case where $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}_p$

(p prime) is settled in [24]. The general problem has now been solved by Bronštein [s].

It has long been conjectured that the free product A and B of two Hopf groups A and B is a Hopf group. In [32] this is proved when A and B are finitely generated. (However, M. F. Newman and J. Sichler, in a paper accepted for publication by the Mathematische Zeitschrift, have constructed a counterexample to the full conjecture). One gets some appreciation of the depth of such a result by recalling that the modular group of all substitutions $w = \frac{az+b}{cz+d}$, where a, b, c, d are integers and $ad - bc = 1$, is the free product of a group of order 2 and a group of order 3. The proof is technically difficult, resting on earlier work of Bernhard [t] and Dey [u] and requiring elaborate arguments of combinatorial group theory.

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