KRULL DIMENSION OF AFFINOID ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

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To Kenny Brown and Toby Stafford, on the occasion of their sixtieth birthdays

Abstract. Using Beilinson–Bernstein localisation, we give another proof of Levasseur's theorem on the Krull dimension of the enveloping algebra of a complex semisimple Lie algebra. The proof also extends to the case of affinoid enveloping algebras.

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1. Introduction.

1.1. Krull dimension of classical enveloping algebras.
Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, and let $U(\mathfrak{g})$ be its enveloping algebra. The Krull–(Gabriel–Rentschler) dimension $\mathcal{K}(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ is a non-negative integer bounded above by $\dim \mathfrak{g}$ that gives a rough measure of how close $U(\mathfrak{g})$ is to being commutative; for example, this upper bound is attained whenever $\mathfrak{g}$ is solvable, but in general $\mathcal{K}(U(\mathfrak{g}))$ is strictly smaller than $\dim \mathfrak{g}$.

The problem of showing that $\mathcal{K}(U(\mathfrak{g}))$ is equal to the dimension of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ when $\mathfrak{g}$ is semisimple was considered by Paul Smith in [**20**], [**21**] and has been open until relatively recently. In 1981, Thierry Levasseur made the observation [**14**] that if G is the semisimple simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$ and U is a maximal unipotent subgroup of G, then the Krull dimension of $U(\mathfrak{g})$ is bounded above by the Krull dimension of the ring of global differential operators $\mathcal{D}(X)$ on the "basic affine space" $X = G/U$. The problem with this strategy is that X is only quasi-affine, and that $\mathcal{D}(X) = \mathcal{D}(\overline{X})$ for some singular affine variety $\overline{X}$. The algebra of differential operators on a singular variety can behave rather badly: for example, it need not even be Noetherian. Levasseur [**15**] was eventually able to deduce that $\mathcal{K}(U(\mathfrak{g})) = \dim \mathfrak{b}$ from deep work of Bezrukavnikov, Braverman and Positselskii [**9**], which established that $\mathcal{D}(G/U)$ *is* Noetherian, and even has finite self-injective dimension. This algebra was subsequently studied in more depth by Levasseur and Stafford [**16**].

1.2. Another approach.
In this paper, we give another proof of the inequality $\mathcal{K}(U(\mathfrak{g})) \leqslant \dim \mathfrak{b}$, using Beilinson–Bernstein localisation [**4**]. Let B be a Borel subgroup

of G containing U, and let $\xi : G/U \to G/B$ the natural projection. Then, ξ is a Zariski locally trivial $H := B/U$-torsor, and $\widetilde{\mathcal{D}} := (\xi_* \mathcal{D}_{G/U})^H$ is a sheaf of "enhanced" differential operators on the flag variety G/B. Letting $V_1, \ldots, V_m$ be the Weyl-group translates of a big cell in G/B, the infinitesimal action of $\mathfrak{g}$ on G/B gives an algebra homomorphism

$$U(\mathfrak{g}) \longrightarrow \oplus_{i=1}^{m} \widetilde{\mathcal{D}}(V_i),$$

and each $\widetilde{\mathcal{D}}(V_i)$ is a polynomial algebra in $\dim H$ variables over a Weyl algebra A_n, where $n = \dim V_i = \dim G/B = \dim U$. An application of Bernstein's Inequality shows that the Krull dimension of this algebra is $\dim U + \dim H = \dim \mathfrak{b}$, so we would be done if we knew that $\oplus_{i=1}^{m} \widetilde{\mathcal{D}}(V_i)$ was a faithfully flat $U(\mathfrak{g})$-module. This is in fact not the case (see Example 4.4), but using Beilinson–Bernstein localisation it is still possible to show that there is a morphism from the lattice of left ideals in $U(\mathfrak{g})$ to the corresponding lattice in $\oplus_{i=1}^{m} \widetilde{\mathcal{D}}(V_i)$, which preserves strict inclusions. This is sufficient for the intended application – see Corollary 4.3.

1.3. Affinoid enveloping algebras.

Recently, a new class of non-commutative Noetherian rings has emerged from the study of non-commutative Iwasawa algebras [2]. Let R be a complete discrete valuation ring with field of fractions K, let $\pi \in R$ generate the unique maximal ideal of R, and let $\mathfrak{g}$ be an R-Lie algebra, free of finite rank over R. Form the π-adic completion of the R-enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, and then invert π; the result is the *affinoid enveloping algebra*

$$\widehat{U(\mathfrak{g})_K} := \left(\varprojlim U(\mathfrak{g})/\pi^a U(\mathfrak{g}) \right) \otimes_R K.$$

For example, when $\mathfrak{g} = R^n$ is abelian, its affinoid enveloping algebra $\widehat{U(\mathfrak{g})_K}$ can be identified with the *Tate algebra* $K\langle x_1, \ldots, x_n \rangle$ consisting of formal power series $\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{x}^\alpha \in K[[x_1, \ldots, x_n]]$ such that λ_α converges to zero in K as $\alpha_1 + \ldots + \alpha_n$ approaches infinity.

1.4. Main result.

We may form the affinoid enveloping algebra of any R-Lie lattice in a finite dimensional K-Lie algebra. As one may expect, "canonical" lattices arising from semisimple algebraic groups are better behaved than others, so we restrict our attention to these lattices. Our main result, Theorem 4.3, reads as follows.

THEOREM. *Let $\mathbf{G}$ be a connected, simply connected, split semisimple, affine algebraic group scheme over R, let $\mathbf{B}$ be a closed and flat Borel R-subgroup scheme of $\mathbf{G}$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. Suppose that the characteristic of K is zero, the residue characteristic p of R is very good for $\mathbf{G}$ and that $n > 0$. Then*

$$\mathcal{K}\left(\widehat{U(\pi^n \mathfrak{g})_K} \right) \leqslant \dim \mathbf{B}.$$

We refer the reader to [2, Section 6.8] for a precise definition of what it means for a prime number p to be a *very good prime* for $\mathbf{G}$ and simply remark here that this condition is satisfied by any $p > 5$ if $\mathbf{G}$ is not of type A. Both this theorem and its classical analogue follow from a general result, Theorem 2.3. We have carefully given all the details in the affinoid case, which requires many more technicalities than the

classical enveloping algebra. For this reason, the reader may find it easier to begin with the remarks following Corollary 4.3.

The interest in Theorem 1.4 is threefold. Firstly, it breaks down completely if K had positive characteristic, since in this case enveloping algebras are known to be finite modules over their centre – it is genuinely a mixed characteristic phenomenon. Secondly, it is nice to have a proof of Levasseur's theorem using only the classical Beilinson–Bernstein theorem. However, what is of most interest is to contrast affinoid enveloping algebras with Iwasawa algebras. When $R = \mathbb{Z}_p$, the affinoid enveloping algebra $\widehat{U(p^{n-1}\mathfrak{g})}_{\mathbb{Q}_p}$ arises as a microlocalisation of the Iwasawa algebra $\mathbb{Q}_p G_n$ of the nth congruence kernel $G_n = \ker(\mathbf{G}(\mathbb{Z}_p) \to \mathbf{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p))$ of the p-adic Lie group $\mathbf{G}(\mathbb{Z}_p)$, and we expect [3] the Krull dimension of this algebra to be equal to $\dim B + \dim H$. We hope to compute the Krull dimension of $\mathbb{Q}_p G_n$ as a consequence of work in progress (K. Ardakov and I. Grojnowski, in preparation), which has been ongoing concurrent with [2].

2. Localisation and Krull dimension.

2.1. Coherently $\mathcal{D}$-acyclic spaces.
We refer the reader to [12, Section 0.5.3.1] for the definition of *coherent $\mathcal{D}$-modules* over a sheaf $\mathcal{D}$ of not necessarily commutative rings over a topological space X. We write $\mathrm{coh}(\mathcal{D})$ for the abelian category of coherent sheaves of $\mathcal{D}$-modules on X, and $\mathrm{mod}(\mathcal{D})$ for the abelian category of *all* sheaves of $\mathcal{D}$-modules.

Recall [2, Section 5.1] that X is said to be *coherently $\mathcal{D}$-acyclic* if $\mathcal{D}$ is a coherent sheaf of rings on X and every coherent $\mathcal{D}$-module is $\Gamma(X, -)$-acyclic and has coherent global sections as a $\mathcal{D}(X)$-module. If this is the case, then $\Gamma(X, -)$ is exact on coherent $\mathcal{D}$-modules. We say that X is *coherently $\mathcal{D}$-affine* if X is coherently $\mathcal{D}$-acyclic and every coherent $\mathcal{D}$-module is generated by its global sections as a $\mathcal{D}$-module. In this case, $\Gamma(X, -)$ induces an equivalence of categories between $\mathrm{coh}(\mathcal{D})$ and the category of coherent $\mathcal{D}(X)$-modules (see [2, Proposition 5.1]).

2.2. The left ideal sheaf I°.
Let $\mathcal{D} \to \mathcal{D}'$ be a map of sheaves of rings on X. We assume throughout Section 2 that:

(a) X is coherently $\mathcal{D}'$-acyclic,
(b) $D' := \Gamma(X, \mathcal{D}')$ is left Noetherian, and is a flat right $D := \Gamma(X, \mathcal{D})$-module.
Since we do not consider any other space apart from X in this section, we will abbreviate $\Gamma(X, \mathcal{M})$ to $\Gamma(\mathcal{M})$ for any sheaf $\mathcal{M}$ on X.

If I is a left ideal in D, then we define a left ideal sheaf I° of $\mathcal{D}$ as follows:

$$I^\circ := \ker \left(\mathcal{D} \longrightarrow \mathcal{D} \otimes_D \frac{D}{I} \right).$$

Equivalently, I° is the image of $\mathcal{D} \otimes_D I$ in $\mathcal{D}$ under the natural multiplication map. This left ideal sheaf fits into the short exact sequence

$$0 \to I^\circ \to \mathcal{D} \to \mathcal{D} \otimes_D \frac{D}{I} \to 0.$$

Taking global sections gives a commutative diagram of D-modules with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & \frac{D}{I} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\varphi_I} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(I^\circ) & \longrightarrow & \Gamma(\mathcal{D}) & \longrightarrow & \Gamma(\mathcal{D} \otimes_D \frac{D}{I}). & &
\end{array}
\tag{1}
$$

Since $\Gamma(\mathcal{D}) = D$ by assumption, the middle vertical map is an isomorphism and φ_I is an injection. Thus, we may view $\Gamma(I^\circ)$ as a left ideal of D containing I.

Similarly, whenever J is a left ideal in D', we can define an ideal sheaf

$$
J^\circ := \ker\left(\mathcal{D}' \longrightarrow \mathcal{D}' \otimes_{D'} \frac{D'}{J} \right).
$$

Since D' is left Noetherian by Section 2.2(b), D'/J is a finitely presented D'-module. Since $\mathcal{D}'$ is coherent by Section 2.2(a) and $\mathcal{D}' \otimes_{D'} -$ is right exact, it follows that $\mathcal{D}' \otimes_{D'} \frac{D'}{J}$ is a coherent $\mathcal{D}'$-module. Therefore, J° is also a coherent $\mathcal{D}'$-module. Thus, we obtain a similar diagram of D'-modules:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & D' & \longrightarrow & \frac{D'}{J} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\psi_J} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(J^\circ) & \longrightarrow & \Gamma(\mathcal{D}') & \longrightarrow & \Gamma(\mathcal{D}' \otimes_{D'} \frac{D'}{J}) & \longrightarrow & 0
\end{array}
\tag{2}
$$

and its bottom row is exact because Γ is exact on $\operatorname{coh}(\mathcal{D}')$ by [**2**, Proposition 5.1].

2.3. Lemma.   For every finitely generated D-module M, the natural map

$$
\gamma_M : D' \otimes_D M \longrightarrow \Gamma(\mathcal{D}' \otimes_D M)
$$

is an isomorphism in $\operatorname{coh}(D')$.

Proof. The D'-module $N := D' \otimes_D M$ is finitely generated, and $\mathcal{D}' \otimes_D M \cong \mathcal{D}' \otimes_{D'} N$ naturally in M. Now D' is left Noetherian by Section 2.2(b) so N is a coherent D'-module. Since X is coherently $\mathcal{D}'$-acyclic by Section 2.2(a), the result follows from the proof of [**2**, Proposition 5.1]. $\square$
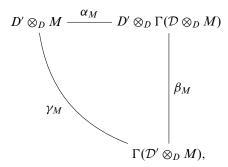
2.4.   Since $\Gamma(\mathcal{D}) = D$, the functor Γ is right adjoint to $\mathcal{D} \otimes_D - : \operatorname{mod}(D) \to \operatorname{mod}(\mathcal{D})$. The counit of this adjunction induces a natural transformation

$$
\alpha_M : D' \otimes_D M \longrightarrow D' \otimes_D \Gamma(\mathcal{D} \otimes_D M)
$$

of D'-modules. Since $\Gamma(\mathcal{D}' \otimes_D M)$ is naturally a left $\Gamma(\mathcal{D}') = D'$-module, we also have a natural transformation of D'-modules

$$
\beta_M : D' \otimes_D \Gamma(\mathcal{D} \otimes_D M) \longrightarrow \Gamma(\mathcal{D}' \otimes_D M).
$$

When M is a finitely generated D-module, α_M and β_M fit into a commutative diagram
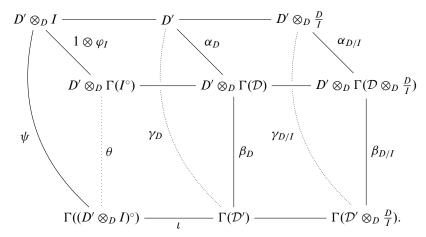
$$D' \otimes_D M \xrightarrow{\ \alpha_M\ } D' \otimes_D \Gamma(\mathcal{D} \otimes_D M)$$

with curved arrow γ_M down to $\Gamma(\mathcal{D}' \otimes_D M)$, and vertical arrow β_M,

$$\Gamma(\mathcal{D}' \otimes_D M),$$

where the curved arrow γ_M is the isomorphism in $\mathrm{coh}(D')$ given by Lemma 2.3.

2.5. Proposition. Let I be a left ideal in D, and suppose that the hypotheses of Section 2.2 are satisfied. Then, the natural map

$$1 \otimes \varphi_I : D' \otimes_D I \longrightarrow D' \otimes_D \Gamma(I^\circ)$$

is an isomorphism.

Proof. Consider the following diagram of D'-modules:

$$
\begin{array}{ccccc}
D' \otimes_D I & \longrightarrow & D' & \longrightarrow & D' \otimes_D \frac{D}{I} \\
\end{array}
$$

with maps $1 \otimes \varphi_I$, α_D, $\alpha_{D/I}$,

$$
D' \otimes_D \Gamma(I^\circ) \longrightarrow D' \otimes_D \Gamma(\mathcal{D}) \longrightarrow D' \otimes_D \Gamma(\mathcal{D} \otimes_D \tfrac{D}{I})
$$

with maps ψ, θ, γ_D, $\gamma_{D/I}$, β_D, $\beta_{D/I}$,

$$
\Gamma((D' \otimes_D I)^\circ) \xrightarrow{\ \iota\ } \Gamma(\mathcal{D}') \longrightarrow \Gamma(\mathcal{D}' \otimes_D \tfrac{D}{I}).
$$

The two squares on the top are obtained by applying the functor $D' \otimes_D -$ to the diagram Section 2.2(1), and the two squares at the back with curved sides together form a special case of the diagram Section 2.2(2) with $J := D' \otimes_D I$ and $\psi = \psi_{D' \otimes_D I}$. Thus, these squares commute, and top two rows are exact since D' is a flat right D-module by assumption Section 2.2(b).

The right front square commutes because β is a natural transformation. This induces the map θ which makes the left front square commute. Note that θ is an injection because β_D is an isomorphism. The middle curved triangle commutes by Section 2.3; since ι is an injection, we see that the curved triangle on the left also commutes

$$\psi = \theta \circ (1 \otimes \varphi_I).$$

But ψ is an isomorphism because γ_D and $\gamma_{D/I}$ are isomorphisms by Lemma 2.3, and $1 \otimes \varphi_I$ is injective because φ_I is injective and $D' \otimes_D -$ is exact. Therefore, $1 \otimes \varphi_I$ is an isomorphism. $\qquad\square$

2.6. An application to Krull dimension. Now, let $\{V_1, \ldots, V_m\}$ be an open cover of X and let U be a subring of D. Then, we have a function

$$\varrho : I \mapsto \oplus_{i=1}^m \Gamma(V_i, (D \cdot I)^\circ)$$

from the set of left ideals in U to the set of left ideals in the ring $\oplus_{i=1}^m \mathcal{D}(V_i)$.

LEMMA. $\varrho(I) \subseteq \varrho(J)$ whenever $I \subseteq J$ are left ideals in U.

Proof. Clearly $D \cdot I \subseteq D \cdot J$. There is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (D \cdot I)^\circ & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D} \otimes_D \frac{D}{D \cdot I} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & (D \cdot J)^\circ & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D} \otimes_D \frac{D}{D \cdot J} & \longrightarrow & 0
\end{array}
$$

inducing an injective map $(D \cdot I)^\circ \hookrightarrow (D \cdot J)^\circ$ of left ideal sheaves of $\mathcal{D}$. Now, apply the left exact functor $\oplus_{i=1}^m \Gamma(V_i, -)$. $\qquad\square$

THEOREM. *Let $\mathcal{D}$ be a coherent sheaf of rings on X, let $\{V_1, \ldots, V_m\}$ be an open cover of X, and let U be a subring of $D = \Gamma(X, \mathcal{D})$. Suppose that*
 (1) D is left Noetherian, and a faithfully flat right U-module,
 (2) each V_i is coherently $\mathcal{D}$-affine,
 (3) for any simple left U-module M, there exists a map $\mathcal{D} \to \mathcal{D}'$ such that
   (a) X is coherently $\mathcal{D}'$-acyclic,
   (b) $D' := \Gamma(X, \mathcal{D}')$ is left Noetherian, and a flat right D-module,
   (c) $D' \otimes_U M \neq 0$.
Then, $I \overset{\varrho}{\mapsto} \oplus_{i=1}^m \Gamma(V_i, (D \cdot I)^\circ)$ preserves strict inclusions, and consequently

$$\mathcal{K}(U) \leqslant \mathcal{K}(\oplus_{i=1}^m \mathcal{D}(V_i)).$$

Proof. Let $I \subset J$ be two left ideals in U. Assumption (1) forces U to be left Noetherian, so we may assume that $M := J/I$ is a simple U-module. Since D is a faithfully flat right U-module by (1), $N := D \cdot J/D \cdot I \cong D \otimes_U M$ is non-zero. Using (3), choose $\mathcal{D} \to \mathcal{D}'$ such that $D' \otimes_U M \neq 0$. By Proposition 2.3,

$$D' \otimes_D \frac{\Gamma((D \cdot J)^\circ)}{\Gamma((D \cdot I)^\circ)} \cong D' \otimes_D \frac{D \cdot J}{D \cdot I} = D' \otimes_D N \cong D' \otimes_U M \neq 0$$

so $(D \cdot I)^\circ \subset (D \cdot J)^\circ$. Since D is left Noetherian by (1), $(D \cdot I)^\circ$ is the image of a morphism between two coherent $\mathcal{D}$-modules and is therefore a coherent $\mathcal{D}$-module. Since $\{V_1, \ldots, V_m\}$ is an open cover of X, $(D \cdot I)^\circ_{|V_j} \subset (D \cdot J)^\circ_{|V_j}$ for some j. But V_j is coherently $\mathcal{D}$-affine by (2), which implies that $\Gamma(V_j, -)$ is exact and faithful on coherent $\mathcal{D}_{V_j}$-modules by [2, Proposition 5.1]. Therefore,

$$\varrho(I) = \oplus_{i=1}^m \Gamma(V_i, (D \cdot I)^\circ) \subset \oplus_{i=1}^m \Gamma(V_i, (D \cdot J)^\circ) = \varrho(J)$$

as claimed. The last statement follows from [**18**, Proposition 6.1.17(ii)] applied to the poset map ϱ with $\gamma = \delta = 0$. $\square$

3. The sheaf $\widetilde{\mathcal{D}_{n,K}}$ on the flag variety.

Throughout, we will work over a complete discrete valuation ring R with uniformiser π, residue field k of characteristic $p \geqslant 0$, and field of fractions K. We begin by briefly recalling relevant definitions and notation from [**2**].

3.1. Crystalline differential operators on the flag variety.

Let X be a scheme over $\mathrm{Spec}(R)$ which is smooth, separated and locally of finite type. The sheaf of *crystalline differential operators* $\mathcal{D}$ on X [**2**, Section 4.2] is the sheaf of associative R-algebras generated by $\mathcal{O}$ and the tangent sheaf $\mathcal{T}$, subject only to the relations
- $f\partial = f \cdot \partial$ and $\partial f - f\partial = \partial(f)$ for each $f \in \mathcal{O}$ and $\partial \in \mathcal{T}$;
- $\partial\partial' - \partial'\partial = [\partial, \partial']$ for $\partial, \partial' \in \mathcal{T}$.

Let $\mathbf{G}$ be a connected, simply connected, split semisimple, affine algebraic group scheme over R. Let $\mathbf{B}$ be a closed and flat Borel R-subgroup scheme of $\mathbf{G}$, let $\mathbf{N}$ be its unipotent radical and let $\mathbf{H} := \mathbf{B}/\mathbf{N}$ be the abstract Cartan group. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}$ and $\mathfrak{h}$ be the corresponding R-Lie algebras.

Let $\mathcal{B} = \mathbf{G}/\mathbf{B}$ be the flag variety and $\widetilde{\mathcal{B}} = \mathbf{G}/\mathbf{N}$ the *base affine space* of $\mathbf{G}$. The natural projection $\xi : \widetilde{\mathcal{B}} \to \mathcal{B}$ is a Zariski locally trivial $\mathbf{H}$-torsor, and we define

$$\widetilde{\mathcal{D}} := (\xi_* \mathcal{D}_{\widetilde{\mathcal{B}}})^{\mathbf{H}}$$

to be the *relative enveloping algebra* of ξ. We write $\mathcal{S}$ for the basis of $\mathcal{B}$ consisting of open affine subschemes V on which ξ is trivial – see [**2**, Section 4.6] for more details.

3.2. The Harish-Chandra homomorphism.

Since our group $\mathbf{G}$ is split by assumption, we can find a Cartan subgroup $\mathbf{T}$ of $\mathbf{G}$ complementary to $\mathbf{N}$ in $\mathbf{B}$. Let $i : \mathbf{T} \xrightarrow{\cong} \mathbf{H}$ denote the natural isomorphism, and let $i : \mathfrak{t} \xrightarrow{\cong} \mathfrak{h}$ be the induced isomorphism between the corresponding Lie algebras. The adjoint action of $\mathbf{T}$ on $\mathfrak{g}$ induces a root space decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$$

and we will regard $\mathfrak{n}$, the Lie algebra of $\mathbf{N}$, as being spanned by *negative* roots. This decomposition induces an isomorphism of R-modules

$$U(\mathfrak{g}) \cong U(\mathfrak{n}) \otimes U(\mathfrak{t}) \otimes U(\mathfrak{n}^+)$$

and a direct sum decomposition

$$U(\mathfrak{g}) = U(\mathfrak{t}) \oplus \left(\mathfrak{n} U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+ \right).$$

Now, the adjoint action of the group $\mathbf{G}$ induces a rational action of $\mathbf{G}$ on $U(\mathfrak{g})$ by algebra automorphisms, so we may consider the subring $U(\mathfrak{g})^{\mathbf{G}}$ of $\mathbf{G}$-invariants. We call the composite of the natural inclusion of $U(\mathfrak{g})^{\mathbf{G}} \hookrightarrow U(\mathfrak{g})$ with the projection $U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{t})$ onto the first factor defined by this decomposition the *Harish-Chandra homomorphism*:

$$\phi : U(\mathfrak{g})^{\mathbf{G}} \longrightarrow U(\mathfrak{t}).$$

LEMMA. *Let* $\mathbf{W}$ *be the Weyl group of* $\mathbf{G}$*, and suppose that* p *is a very good prime for* $\mathbf{G}$*. Then* $\operatorname{gr} U(\mathfrak{t})$ *is a free graded* $\operatorname{gr}(U(\mathfrak{g})^{\mathbf{G}})$*-module of rank* $|\mathbf{W}|$ *via* $\operatorname{gr}\phi$*.*

Proof. There is an analogous factorisation $S(\mathfrak{g}) \cong S(\mathfrak{n}_-) \otimes S(\mathfrak{t}) \otimes S(\mathfrak{n})$, and a corresponding decomposition $S(\mathfrak{g}) = S(\mathfrak{t}) \oplus (\mathfrak{n}S(\mathfrak{g}) + S(\mathfrak{g})\mathfrak{n}^+)$. Let $\psi : S(\mathfrak{g})^{\mathbf{G}} \to S(\mathfrak{t})$ be the composition of the inclusion $S(\mathfrak{g})^{\mathbf{G}} \hookrightarrow S(\mathfrak{g})$ with the projection $S(\mathfrak{g}) \twoheadrightarrow S(\mathfrak{t})$ along this decomposition. Then

$$\operatorname{gr}\phi = \psi.$$

It has been shown in [**2**, Proposition 6.9] that ψ is injective, and that the image of ψ is precisely the ring of invariants $S(\mathfrak{t})^{\mathbf{W}}$. Since p is a very good prime, the result now follows from [**10**, Corollaire du Théorème 2 and Théorème 2(c)]. $\qquad\square$

3.3. Deformations. Let A be a positively $\mathbb{Z}$-filtered R-algebra with F_0A an R-subalgebra of A. Recall [**2**, Section 3.5] that A is said to be a *deformable R-algebra* if $\operatorname{gr} A$ is a flat R-module. A *morphism* of deformable R-algebras is an R-linear filtered ring homomorphism. The *nth deformation* of A is

$$A_n := \sum_{i \geqslant 0} \pi^{in} F_i A \subseteq A.$$

This is actually an R-subalgebra of A. It becomes a deformable R-algebra when we equip A_n with the subspace filtration arising from the given filtration on A, and multiplication by π^{in} on graded pieces of degree i extends to a natural isomorphism of graded R-algebras

$$\sigma_A : \operatorname{gr} A \xrightarrow{\cong} \operatorname{gr} A_n$$

by [**2**, Lemma 3.5]. The assignment $A \mapsto A_n$ is functorial in A.

LEMMA. *Let* $B \xrightarrow{\alpha} A$ *and* $B \xrightarrow{\gamma} C$ *be morphisms of deformable R-algebras with central images. Suppose that* $\operatorname{gr} C$ *is a free graded* $\operatorname{gr} B$*-module via* $\operatorname{gr}\gamma$*. Equip* $A \otimes_B C$ *with the tensor filtration. Then*

(a) there is a natural isomorphism $\operatorname{gr} A \otimes_{\operatorname{gr} B} \operatorname{gr} C \xrightarrow{\cong} \operatorname{gr}(A \otimes_B C)$*,*
(b) $A \otimes_B C$ *is a deformable R-algebra,*
(c) there is a natural isomorphism of deformable R-algebras

$$A_n \otimes_{B_n} C_n \xrightarrow{\cong} (A \otimes_B C)_n.$$

Proof. (a) This follows from [**17**, I.6.14].

(b) Since $\operatorname{gr} C$ is a free graded $\operatorname{gr} B$-module, $\operatorname{gr} A \otimes_{\operatorname{gr} B} \operatorname{gr} C$ is free as a $\operatorname{gr} A$-module. Since A is deformable, $\operatorname{gr} A$ is flat over R and therefore $\operatorname{gr} A \otimes_{\operatorname{gr} B} \operatorname{gr} C$ is also flat over R. Now, apply part (a).

(c) There are natural maps $A \longrightarrow A \otimes_B C$ and $C \longrightarrow A \otimes_B C$ of deformable R-algebras which send $a \in A$ to $a \otimes 1$ and $c \in C$ to $1 \otimes c$, respectively. Applying the deformation functor to these maps, we obtain a filtered R-algebra homomorphism $A_n \otimes_R C_n \to (A \otimes_B C)_n$ which descends to a filtered R-algebra homomorphism

$$\theta : A_n \otimes_{B_n} C_n \longrightarrow (A \otimes_B C)_n.$$

The associated graded of this map fits into the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{gr}(A_n \otimes_{B_n} C_n) & \xrightarrow{\ \mathrm{gr}\,\theta\ } & \mathrm{gr}(A \otimes_B C)_n \\
\uparrow & & \uparrow{\scriptstyle \sigma_{A \otimes_B C}} \\
\mathrm{gr}\,A_n \otimes_{\mathrm{gr}\,B_n} \mathrm{gr}\,C_n \xleftarrow[\sigma_A \otimes \sigma_C]{} \mathrm{gr}\,A \otimes_{\mathrm{gr}\,B} \mathrm{gr}\,C & \longrightarrow & \mathrm{gr}(A \otimes_B C)
\end{array}
$$

where all the other maps are isomorphisms either by part (a) above or by [**2**, Lemma 3.5]. Hence, θ is an isomorphism. $\qquad\square$

Combining this result together with Lemma 3.2, we obtain the following:

COROLLARY. *Suppose that p is a very good prime for* **G**. *Then*
(a) $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathbf{G}}} U(\mathfrak{t})$ is a deformable R-algebra,
(b) its associated graded is isomorphic to $S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^{\mathbf{G}}} S(\mathfrak{t})$, and
(c) $(U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathbf{G}}} U(\mathfrak{t}))_n \cong U(\mathfrak{g})_n \otimes_{(U(\mathfrak{g})^{\mathbf{G}})_n} U(\mathfrak{t})_n$ for all $n \geqslant 0$.

We will assume from now on that p is a very good prime for **G**.

3.4. π-adic completions. If B is a deformable R-algebra, $\widehat{B} := \varprojlim B/\pi^a B$ will denote its π-adic completion. Recall *almost commutative affinoid K-algebras* from [**2**, Section 3.8]. Such an algebra A has a *double associated graded ring* $\mathrm{Gr}(A)$; when

$$
A = \widehat{B_{n,K}} = \widehat{B_n} \otimes_R K
$$

for some deformable R-algebra B, [**2**, Corollary 3.7] tells us that $\mathrm{Gr}(A)$ can be computed as follows:

$$
\mathrm{Gr}(A) = \mathrm{Gr}(\widehat{B_{n,K}}) \cong \mathrm{gr}\,B/\pi\,\mathrm{gr}\,B.
$$

In this way, we obtain three examples of almost commutative affinoid K-algebras:

$$
U := \widehat{U(\mathfrak{g})_{n,K}}, \quad Z := \widehat{U(\mathfrak{g})^{\mathbf{G}}_{n,K}} \quad \text{and} \quad \widetilde{Z} = \widehat{U(\mathfrak{t})_{n,K}}
$$

by applying this process to the algebras $U(\mathfrak{g})_n$, $(U(\mathfrak{g})^{\mathbf{G}})_n$ and $U(\mathfrak{t})_n$, respectively. Note that $\widetilde{Z}$ becomes a Z-module via the completed, deformed, Harish-Chandra homomorphism $\widehat{\phi} : Z \to \widetilde{Z}$ – see [**2**, Section 9.3].

LEMMA. $U \otimes_Z \widetilde{Z}$ *is an almost commutative affinoid K-algebra, and there is a natural isomorphism*

$$
\mathrm{Gr}(U \otimes_Z \widetilde{Z}) \cong S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathbf{G}_k}} S(\mathfrak{t}_k).
$$

Proof. Let $B := U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^{\mathbf{G}}} U(\mathfrak{t})$. Then B is a deformable R-algebra and

$$
B_n \cong U(\mathfrak{g})_n \otimes_{(U(\mathfrak{g})^{\mathbf{G}})_n} U(\mathfrak{t})_n
$$

by Corollary 3.3. So, $\widehat{B_{n,K}}$ is an almost commutative affinoid K-algebra, with

$$
\mathrm{Gr}(\widehat{B_{n,K}}) \cong \mathrm{gr}\,B/\pi\,\mathrm{gr}\,B
$$

by [**2**, Corollary 3.7]. Now, $\mathrm{gr}\,B \cong S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^{\mathbf{G}}} S(\mathfrak{t})$ by Corollary 3.3(b) and $S(\mathfrak{g})^{\mathbf{G}}/\pi S(\mathfrak{g})^{\mathbf{G}} \cong S(\mathfrak{g}_k)^{\mathbf{G}_k}$ by [**2**, Proposition 6.9], so

$$\mathrm{Gr}(\widehat{B_{n,K}}) \cong \mathrm{gr}\,B/\pi\,\mathrm{gr}\,B \cong S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathbf{G}_k}} S(\mathfrak{t}_k).$$

On the other hand, it follows from Lemma 3.2 and [**2**, Lemma 3.5] that $U(\mathfrak{t})_n$ is a finitely generated $(U(\mathfrak{g})^{\mathbf{G}})_n$-module via ϕ_n, so we may apply [**2**, Lemma 6.5] to deduce that

$$\widehat{B_n} \cong \left(U(\mathfrak{g})_n \otimes_{(U(\mathfrak{g})^{\mathbf{G}})_n} U(\mathfrak{t})_n\right)\widehat{} \cong \widehat{U(\mathfrak{g})_n} \otimes_{\widehat{(U(\mathfrak{g})^{\mathbf{G}})_n}} \widehat{U(\mathfrak{t})_n}.$$

Thus, $U \otimes_Z \widetilde{Z} \cong \widehat{B_{n,K}}$ is also an almost commutative affinoid K-algebra, and $\mathrm{Gr}(U \otimes_Z \widetilde{Z}) \cong \mathrm{Gr}(\widehat{B_{n,K}}) \cong S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathbf{G}_k}} S(\mathfrak{t}_k)$ as claimed. $\qquad\square$

3.5. The sheaf $\widetilde{\mathcal{D}}_n$. The actions of $\mathbf{G}$ and $\mathbf{H} = \mathbf{B}/\mathbf{N}$ on $\widetilde{\mathcal{B}} = \mathbf{G}/\mathbf{N}$ can be differentiated to obtain a commutative diagram

$$
\begin{array}{ccc}
U(\mathfrak{g})^{\mathbf{G}} & \xrightarrow{\ \phi\ } & U(\mathfrak{t}) \\
\big\downarrow & & \big\downarrow{\scriptstyle j\circ i} \\
U(\mathfrak{g}) & \xrightarrow[\ U(\varphi)\ ]{} & \widetilde{\mathcal{D}}
\end{array}
$$

of deformable R-algebras – see [**2**, Lemma 4.9].

Fix the deformation parameter n, and let $\widetilde{\mathcal{D}}_n$ be the sheafification of the presheaf obtained by postcomposing $\widetilde{\mathcal{D}}$ with the deformation functor $A \to A_n$. Applying the deformation functor produces the commutative diagram

$$
\begin{array}{ccc}
(U(\mathfrak{g})^{\mathbf{G}})_n & \xrightarrow{\ \phi_n\ } & U(\mathfrak{t})_n \\
\big\downarrow & & \big\downarrow{\scriptstyle (j\circ i)_n} \\
U(\mathfrak{g})_n & \xrightarrow[\ U(\varphi)_n\ ]{} & \widetilde{\mathcal{D}}_n
\end{array}
$$

and a homomorphism

$$\widetilde{\varphi}_n : U(\mathfrak{g})_n \otimes_{(U(\mathfrak{g})^{\mathbf{G}})_n} U(\mathfrak{t})_n \longrightarrow \widetilde{\mathcal{D}}_n.$$

3.6. Global sections of $\widehat{\widetilde{\mathcal{D}}_{n,K}}$. Let $\widehat{\widetilde{\mathcal{D}}_n} := \varprojlim \widetilde{\mathcal{D}}_n/\pi^a \widetilde{\mathcal{D}}_n$ be the π-adic completion of $\widetilde{\mathcal{D}}_n$ and let

$$\mathscr{D} := \widehat{\widetilde{\mathcal{D}}_{n,K}} := \widehat{\widetilde{\mathcal{D}}_n} \otimes_R K$$

be the sheaf of K-algebras on $\mathcal{B}$ obtained from $\widehat{\widetilde{\mathcal{D}}_n}$ by inverting π. The abbreviation $\mathscr{D}$ will be useful because we will need to pass to further completions of this sheaf. The R-algebra homomorphism $\widetilde{\varphi}_n : U(\mathfrak{g})_n \otimes_{(U(\mathfrak{g})^{\mathbf{G}})_n} U(\mathfrak{t})_n \longrightarrow \widetilde{\mathcal{D}}_n$ defined in §3.5 extends to a K-algebra homomorphism

$$\varPhi : U \otimes_Z \widetilde{Z} \longrightarrow \mathscr{D}.$$

PROPOSITION. *The map $\Phi : U \otimes_Z \widetilde{Z} \longrightarrow \Gamma(\mathcal{B}, \mathscr{D})$ is an isomorphism.*

Proof. Let $\{V_1, \ldots, V_m\}$ be an $\mathcal{S}$-cover of $\mathcal{B}$, and let $V := \coprod V_i$. Since each V_i is in $\mathcal{S}$, it follows from [**2**, Proposition 5.10(a)] that $\mathscr{D}(V) \cong \widetilde{\mathcal{D}(V)}_{n,K}$ is an almost commutative affinoid K-algebra, and $\text{Gr}(\mathscr{D}(V)) \cong \mathcal{O}(\widehat{T^*V_k})$. There is a complex

$$0 \to U \otimes_Z \widetilde{Z} \xrightarrow{\Phi} \mathscr{D}(V) \longrightarrow \mathscr{D}(V \times_{\mathcal{B}} V)$$

of almost commutative affinoid K-algebras, and it is enough to show that this complex is exact. Passing to the double associated graded and applying Lemma 3.4, we obtain the complex

$$0 \to S(\mathfrak{g}_k) \otimes_{S(\mathfrak{g}_k)^{\mathfrak{c}_k}} S(\mathfrak{t}_k) \to \mathcal{O}(\widetilde{T^*V_k}) \to \mathcal{O}(T^*(\widetilde{V \times_{\mathcal{B}} V})_k).$$

Since p is a very good prime for $\mathbf{G}$, this complex was shown to be exact in the proof of [**8**, Proposition 3.4.1]. $\qquad\square$

COROLLARY. *$\Gamma(\mathcal{B}, \mathscr{D})$ is a faithfully flat right U-module.*

Proof. Since $\Gamma(\mathcal{B}, \mathscr{D}) \cong U \otimes_Z \widetilde{Z}$ by the Proposition, this follows from [**2**, Proposition 9.3], where it is shown that $\widetilde{Z}$ is free of rank $|\mathbf{W}|$ as a module over Z. $\qquad\square$

3.7. The J-adic associated graded ring. If J is a centrally generated ideal of a ring A, we denote the associated graded ring of A with respect to the J-adic filtration by $\text{gr}_J A$. Thus,

$$\text{gr}_J A := \bigoplus_{m \geqslant 0} \frac{J^m}{J^{m+1}}.$$

LEMMA. *Let A be a ring, and let Z be a central subring of A. Suppose that A is a flat Z-module and let J be an ideal of Z. Then*

$$\text{gr}_{JA} A \cong \text{gr}_J Z \underset{Z/J}{\otimes} \frac{A}{JA}.$$

Proof. Fix $m \in \mathbb{N}$. Since A is a flat Z-module by assumption and $0 \to J^m \to Z \to Z/J^m \to 0$ is exact, there is a short exact sequence

$$0 \to J^m \otimes_Z A \to Z \otimes_Z A \to (Z/J^m) \otimes_Z A \to 0$$

of A-modules. Therefore, $J^m \otimes_Z A \cong J^m A$. Applying flatness again to the short exact sequence $0 \to J^{m+1} \to J^m \to J^m/J^{m+1} \to 0$ produces the short exact sequence

$$0 \to J^{m+1} A \to J^m A \to (J^m/J^{m+1}) \otimes_Z A \to 0.$$

Since J^m/J^{m+1} is killed by J, we obtain isomorphisms

$$\frac{(JA)^m}{(JA)^{m+1}} = \frac{J^m A}{J^{m+1} A} \cong \frac{J^m}{J^{m+1}} \otimes_Z A \cong \frac{J^m}{J^{m+1}} \underset{Z/J}{\otimes} \frac{A}{JA}$$

for all $m \in \mathbb{N}$ and the result follows. $\qquad\square$

We will apply this Lemma in the following two cases:

PROPOSITION. *(a) $\mathscr{D}(V)$ is a flat $\widetilde{Z}$-module for any $V \in \mathcal{S}$.*
(b) $\mathscr{D}(\mathcal{B})$ is a flat $\widetilde{Z}$-module.

Proof. (a) Since $\widetilde{Z} \to \mathscr{D}(V)$ is a map of almost commutative affinoid K-algebras, by applying [**19**, Proposition 1.2] twice, it is enough to show that $\mathrm{Gr}(\mathscr{D}(V))$ is a flat $\mathrm{Gr}(\widetilde{Z})$-module. Now, $\mathrm{Gr}(\widetilde{Z}) \cong S(\mathfrak{t}_k)$ and $\mathrm{Gr}(\mathscr{D}(V)) \cong \mathcal{O}(\widetilde{T^*V_k})$ by [**2**, Proposition 5.10(a)]. Since V trivialises the torsor $\xi : \widetilde{\mathcal{B}} \to \mathcal{B}$, there is an isomorphism $\mathcal{O}(\widetilde{T^*V}) \cong \mathcal{O}(T^*V) \otimes_R S(\mathfrak{t})$, so $\mathcal{O}(\widetilde{T^*V_k}) \cong \mathcal{O}(T^*V_k) \otimes_k S(\mathfrak{t}_k)$ is a flat $S(\mathfrak{t}_k)$-module.

(b) Since $\mathscr{D}(\mathcal{B}) \cong U \otimes_Z \widetilde{Z}$ by Proposition 3.6, it is enough to show that U is a flat Z-module. Now, $\mathrm{Gr}(U) \cong S(\mathfrak{g}_k)$ and $\mathrm{Gr}(Z) \cong S(\mathfrak{g}_k)^{\mathbf{G}_k}$ by the proof of [**2**, Proposition 9.3], so again by [**19**, Proposition 1.2], it is enough to check that $S(\mathfrak{g}_k)$ is a flat $S(\mathfrak{g}_k)^{\mathbf{G}_k}$-module. But $\psi_k : S(\mathfrak{g}_k)^{\mathbf{G}_k} \to S(\mathfrak{t}_k)$ is an embedding with image $S(\mathfrak{t}_k)^{\mathbf{W}_k}$ by [**2**, Proposition 6.9] and $S(\mathfrak{t}_k)$ is a free graded $S(\mathfrak{t}_k)^{\mathbf{W}_k}$-module of rank $|\mathbf{W}|$ by [**10**, Théorème 2(c)], so $S(\mathfrak{g}_k)$ is actually a free graded $S(\mathfrak{g}_k)^{\mathbf{G}_k}$-module by [**6**, Proposition 3.1]. $\qquad\square$

3.8. The completion of $\mathscr{D}$ at a maximal ideal of the centre. Let $t_1, \ldots, t_l \in \mathfrak{h}$ be the simple coroots corresponding to the simple roots in $\mathfrak{t}_K^*$ given by the adjoint action of $\mathfrak{t}$ on $\mathfrak{n}^+$.

DEFINITION. For any $\lambda \in \mathrm{Hom}_R(\pi^n \mathfrak{t}, R)$, let $\mathfrak{m}_\lambda$ be the ideal of $\widetilde{Z} = \widehat{U(\mathfrak{t})_{n,K}}$ generated by the elements $t_i - \lambda(t_i)$ for all $i = 1, \ldots, l$, and let

$$\widehat{\mathscr{D}} := \varprojlim \mathscr{D}/\mathfrak{m}_\lambda^n \mathscr{D}$$

be the $\mathfrak{m}_\lambda$-adic completion of $\mathscr{D}$.

PROPOSITION. *Let $V \in \mathcal{S}$.*
(a) $\widehat{\mathscr{D}}(V) \cong \widehat{\mathscr{D}(V)}$.
(b) $\widehat{\mathscr{D}}(V)$ is Noetherian.
(c) $\widehat{\mathscr{D}}(V')$ is a flat right $\widehat{\mathscr{D}}(V)$-module for all $V' \in \mathcal{S}$ contained in V.

Proof. (a) Since V is coherently $\mathscr{D}$-affine by [**2**, Theorem 5.13], $\Gamma(V, -)$ is exact on coherent $\mathscr{D}$-modules. Since $\widetilde{Z}$ is Noetherian,

$$(\mathscr{D}/\mathfrak{m}_\lambda^a \mathscr{D})(V) \cong \mathscr{D}(V)/\mathfrak{m}_\lambda^a \mathscr{D}(V)$$

for all $a \geqslant 1$ by [**2**, Lemma 5.2]. Hence, $\widehat{\mathscr{D}}(V)$ is the $\mathfrak{m}_\lambda$-adic completion of $\mathscr{D}(V)$.

(b) $\mathscr{D}(V)$ is Noetherian by [**2**, Proposition 5.10]. Hence, $\widehat{\mathscr{D}}(V) \cong \widehat{\mathscr{D}(V)}$ is also Noetherian by [**7**, Section 3.2.3(vi)].

(c) By part (a), the associated graded ring of $\widehat{\mathscr{D}}(V)$ with respect to the $\mathfrak{m}_\lambda$-adic filtration is isomorphic to $\mathrm{gr}\,\mathscr{D}(V)$. So by part (b) and [**19**, Proposition 1.2], it is enough to prove that $\mathrm{gr}\,\mathscr{D}(V')$ is a flat right $\mathrm{gr}\,\mathscr{D}(V)$-module. Since $\mathscr{D}(V)$ is flat as a $\widetilde{Z}$-module by Proposition 3.7(a), $\mathrm{gr}\,\widehat{\mathscr{D}}(V)$ is isomorphic to

$$\mathrm{gr}_{\mathfrak{m}_\lambda} \widetilde{Z} \underset{\widetilde{Z}/\mathfrak{m}_\lambda}{\otimes} \mathscr{D}(V)/\mathfrak{m}_\lambda \mathscr{D}(V)$$

by Lemma 3.7. Since $\widetilde{Z}/\mathfrak{m}_\lambda$ is a copy of the ground field K, it is enough to show that $\mathscr{D}(V')/\mathfrak{m}_\lambda \mathscr{D}(V')$ is a flat right $\mathscr{D}(V)/\mathfrak{m}_\lambda \mathscr{D}(V)$-module. But the proof of [**2**, Proposition 6.5(c)] shows that there is an isomorphism between $\mathscr{D}(V)/\mathfrak{m}_\lambda \mathscr{D}(V)$ and the algebra

$\widehat{\mathcal{D}(V)_{n,K}}$, which is compatible with the restriction maps to the corresponding algebras over $V' \subseteq V$. The flatness of $\widehat{\mathcal{D}(V')_{n,K}}$ as a right $\widehat{\mathcal{D}(V)_{n,K}}$-module in turn follows from the proof of [**2**, Proposition 5.7(d)]. $\square$

COROLLARY. $\widehat{\mathcal{D}}$ is coherent.

Proof. This follows from [**7**, Proposition 3.1.1] and the Proposition above. $\square$

3.9. Global sections of $\widehat{\mathcal{D}}$. Recall the central reduction $\widehat{\mathcal{D}^\lambda_{n,K}}$ of the sheaf $\mathcal{D} = \widehat{\mathcal{D}_{n,K}}$ defined in [**2**, Section 6.5].

PROPOSITION. $\widehat{\mathcal{D}}(\mathcal{B})$ is isomorphic to the $\mathfrak{m}_\lambda$-adic completion of $\mathcal{D}(\mathcal{B})$.

Proof. Let $\{V_1, \ldots, V_m\}$ be an $\mathcal{S}$-cover of $\mathcal{B}$, and let $V := \coprod V_i$. The sequence

$$0 \to \mathcal{D}(\mathcal{B}) \to \mathcal{D}(V) \longrightarrow \mathcal{D}(V \times_\mathcal{B} V)$$

is exact, and because $\widehat{\mathcal{D}}(V) \cong \widehat{\mathcal{D}(V)}$ by Lemma 3.8(a) it will be enough to prove that the associated graded of this sequence with respect to the $\mathfrak{m}_\lambda$-adic filtration is exact. Now $\mathcal{D}(\mathcal{B}) \cong U \otimes_Z \widetilde{Z}$ by Proposition 3.6, so each term in this sequence is flat as a $\widetilde{Z}$-module by Proposition 3.7. So by Lemma 3.7, this associated graded is isomorphic to the tensor product of $\mathrm{gr}_{\mathfrak{m}_\lambda} \widetilde{Z}$ over $\widetilde{Z}/\mathfrak{m}_\lambda$ with the complex

$$0 \to \frac{\mathcal{D}(\mathcal{B})}{\mathfrak{m}_\lambda \mathcal{D}(\mathcal{B})} \to \frac{\mathcal{D}(V)}{\mathfrak{m}_\lambda \mathcal{D}(V)} \to \frac{\mathcal{D}(V \times_\mathcal{B} V)}{\mathfrak{m}_\lambda \mathcal{D}(V \times_\mathcal{B} V)}.$$

Since $\widetilde{Z}/\mathfrak{m}_\lambda$ is a copy of the field K, it is enough to prove that this complex is exact. Now, [**2**, Theorem 6.10(a) and (b)] tell us that

$$\frac{\mathcal{D}(\mathcal{B})}{\mathfrak{m}_\lambda \mathcal{D}(\mathcal{B})} \cong (U \otimes_Z \widetilde{Z}) \otimes_{\widetilde{Z}} (\widetilde{Z}/\mathfrak{m}_\lambda) \cong U \otimes_Z (\widetilde{Z}/\mathfrak{m}_\lambda) \cong \widehat{\mathcal{D}^\lambda_{n,K}}(\mathcal{B}),$$

and [**2**, Proposition 6.5(c)] tells us that

$$\mathcal{D}(V') \otimes_{\widetilde{Z}} (\widetilde{Z}/\mathfrak{m}_\lambda) \cong \widehat{\mathcal{D}^\lambda_{n,K}}(V') \quad \text{for any} \quad V' \in \mathcal{S}.$$

This complex can thus be identified with

$$0 \to \widehat{\mathcal{D}^\lambda_{n,K}}(\mathcal{B}) \to \widehat{\mathcal{D}^\lambda_{n,K}}(V) \to \widehat{\mathcal{D}^\lambda_{n,K}}(V \times_\mathcal{B} V),$$

and is therefore exact. $\square$

COROLLARY. $\widehat{\mathcal{D}}(\mathcal{B})$ is Noetherian, and a flat right $\mathcal{D}(\mathcal{B})$-module.

Proof. The algebra $\mathcal{D}(\mathcal{B})$ is isomorphic to $U \otimes_Z \widetilde{Z}$ by Proposition 3.6, which is an almost commutative affinoid K-algebra by Lemma 3.4. It is therefore Noetherian. Now, apply the Proposition together with [**7**, Section 3.2.3 (iv) and (vi)]. $\square$

3.10. The Beilinson–Bernstein theorem for $\widehat{\mathcal{D}}$. We assume from now on that K has characteristic zero. Let $\omega_1, \ldots, \omega_l \in \mathfrak{h}_K^*$ be the system of fundamental weights corresponding to the coroots $\{t_1, \ldots, t_l\}$, and let $\rho = \omega_1 + \ldots + \omega_l$. Following [**5**], we say that a weight $\mu \in \mathfrak{h}_K^*$ is *dominant* if $\mu(h) \notin \{-1, -2, -3, \cdots\}$ for any positive coroot

$h \in \mathfrak{h}$, and we say that μ is *regular* if its stabiliser under the action of $\mathbf{W}$ is trivial. Finally, we will say that λ is *ρ-dominant* if $\lambda + \rho$ is dominant, and λ is *ρ-regular* if $\lambda + \rho$ is regular.

Recall [**2**, Section 5.1] that if $\mathcal{A}$ is a sheaf of rings on $\mathcal{B}$, then we say that $\mathcal{S}$ is *coherently $\mathcal{A}$-acyclic*, respectively *coherently $\mathcal{A}$-affine*, if for all $U \in \mathcal{S}$, U is coherently $\mathcal{A}_{|U}$-acyclic, respectively coherently $\mathcal{A}_{|U}$-affine.

THEOREM. *(a) $\mathcal{S}$ is coherently $\mathscr{D}$-affine.*
 (b) $\mathcal{S}$ is coherently $\widehat{\mathscr{D}}$-affine.
 (c) If λ is ρ-dominant, then $\mathcal{B}$ is coherently $\widehat{\mathscr{D}}$-acyclic.
 (d) If λ is ρ-dominant and ρ-regular, then $\mathcal{B}$ is coherently $\widehat{\mathscr{D}}$-affine.

Proof. (a) This is [**2**, Theorem 5.13].

(b) $\mathscr{D}(V)$ is Noetherian for all $V \in \mathcal{S}$ by [**2**, Proposition 5.10(a)], $\mathcal{S}$ is coherently $\mathscr{D}$-affine by part (a), and $\widehat{\mathscr{D}}$ is coherent by Corollary 3.8. Therefore, $\mathcal{S}$ is coherently $\widehat{\mathscr{D}}$-affine by [**2**, Theorem 5.5].

(c),(d) By [**2**, Proposition 6.12], $\mathcal{B}$ is coherently $\mathscr{D}$-acyclic whenever λ is ρ-dominant, and it is coherently $\mathscr{D}$-affine if λ is in addition ρ-regular. Since $\widehat{\mathscr{D}}$ is coherent by Corollary 3.8 and $\widehat{\mathscr{D}}(\mathcal{B})$ is Noetherian by Corollary 3.9, both parts follow from [**2**, Theorem 5.5] applied to the topological space $\mathcal{B}$ equipped with the base $\mathcal{S} \cup \{\mathcal{B}\}$. $\qquad\square$

3.11. Base change.   Let K'/K be a finite extension with rings of integers R'/R and ramification index e, and let $\mathcal{B}' := \mathcal{B} \times_R R'$, $\mathbf{H}' := \mathbf{H} \times_R R'$, $\xi' := \xi \times_R R'$ and $\mathfrak{h}' := \mathfrak{h} \otimes_R R'$ be the corresponding base-changed objects.

Let $\mathscr{D}' := \widehat{\mathcal{D}_{ne,K'}}$ be the sheaf of K'-algebras on $\mathcal{B}'$ obtained as in Section 3.6 using the $\mathbf{H}'$-torsor ξ' and the deformation parameter ne, and let $\lambda : \pi^n \mathfrak{h}' \to R'$ be a character. We let $\widehat{\mathscr{D}'}$ denote the completion of $\mathscr{D}'$ at the maximal ideal $\mathfrak{m}_\lambda$ of $\widehat{Z}' := \widehat{U(\mathfrak{h}')_{ne,K}}$ defined in Section 3.8.

LEMMA. *Let $\tau : \mathcal{B}' \to \mathcal{B}$ denote the natural projection.*
 (a) $\Gamma(\mathcal{B}, \tau_* \widehat{\mathscr{D}'})$ is isomorphic to the $\mathfrak{m}_\lambda$-adic completion of $K' \otimes_K \Gamma(\mathcal{B}, \mathscr{D})$.
 (b) The sheaf of rings $\tau_* \widehat{\mathscr{D}'}$ is coherent.
 (c) $\mathcal{B}$ is coherently $\tau_* \widehat{\mathscr{D}'}$-acyclic, whenever λ is ρ-dominant.

Proof. (a) Let $U' := \widehat{U(\mathfrak{g}')_{ne,K}}$, $Z' := \widehat{U(\mathfrak{g}')^{\mathbf{G}'}_{ne,K}}$ and $\widetilde{Z}' := \widehat{U(\mathfrak{t}')_{ne,K}}$ be the corresponding base-changed objects. Then [**2**, Lemma 3.9(c) and Lemma 9.5] tell us that $U' \cong K' \otimes_K U$, $Z' \cong K' \otimes_K Z$ and $\widetilde{Z}' \cong K' \otimes_K \widetilde{Z}$, so

$$\mathscr{D}'(\mathcal{B}') \cong U' \otimes_{Z'} \widetilde{Z}' \cong K' \otimes_K (U \otimes_Z \widetilde{Z}) \cong K' \otimes_K \mathscr{D}(\mathcal{B})$$

by applying Proposition 3.6 twice. Therefore, $\Gamma(\mathcal{B}, \tau_* \widehat{\mathscr{D}'}) = \Gamma(\mathcal{B}', \widehat{\mathscr{D}'})$ is the $\mathfrak{m}_\lambda$-adic completion of $\mathscr{D}'(\mathcal{B}') \cong K' \otimes_K \mathscr{D}(\mathcal{B})$ by Proposition 3.9.

(b) Let $\mathcal{S}'$ be the base of open subschemes of $\mathcal{B}'$ that trivialise ξ', and note that $\tau^{-1}(V) = V \times_R R'$ is in $\mathcal{S}'$ whenever $V \in \mathcal{S}$. Now, $\Gamma(V, \tau_* \widehat{\mathscr{D}'}) = \Gamma(\tau^{-1}(V), \mathscr{D}')$ is left Noetherian by Proposition 3.8(b), and for any open $V' \in \mathcal{S}$ contained in V, $\Gamma(V', \tau_* \widehat{\mathscr{D}'}) = \Gamma(\tau^{-1}(V'), \mathscr{D}')$ is a flat right $\Gamma(\tau^{-1}(V), \mathscr{D}')$-module by Proposition 3.8(c), so $\tau_* \widehat{\mathscr{D}'}$ is coherent by [**7**, Proposition 3.1.1].

(c) Let $\mathcal{N}$ be a coherent $\widehat{\mathscr{D}'}$-module. Then $H^i(\tau^{-1}(V), \mathcal{N}) = 0$ for all $i > 0$ and all $V \in \mathcal{S}$ by Theorem 3.10(b), so $R^i \tau_* \mathcal{N} = 0$ for all $i > 0$ by [**13**, Proposition III.8.1]. Hence, τ_* is exact on coherent $\widehat{\mathscr{D}'}$-modules.

Now let $\mathcal{M}$ be a coherent $\mathcal{A} := \tau_*\widehat{\mathscr{D}'}$-module and let $\mathcal{N} := \widehat{\mathscr{D}'} \otimes_{\tau^{-1}\mathcal{A}} \tau^{-1}\mathcal{M}$. We will show that $\mathcal{N}$ is a coherent $\widehat{\mathscr{D}'}$-module, and that the natural map $\eta_{\mathcal{M}} : \mathcal{M} \to \tau_*\mathcal{N}$ is an isomorphism. Since these are local properties, we may assume that $\mathcal{M}$ has a finite presentation $\mathcal{A}^r \to \mathcal{A}^s \to \mathcal{M} \to 0$. Then $\widehat{\mathscr{D}'}^r \to \widehat{\mathscr{D}'}^s \to \mathcal{N} \to 0$ is a presentation for $\mathcal{N}$; hence, $\mathcal{N}$ is coherent because $\widehat{\mathscr{D}'}$ is coherent by Corollary 3.8. As τ_* is exact on coherent $\widehat{\mathscr{D}'}$-modules by the first paragraph, $\mathcal{A}^r \to \mathcal{A}^s \to \tau_*\mathcal{N} \to 0$ is exact. Hence, $\eta_{\mathcal{M}} : \mathcal{M} \to \tau_*\mathcal{N}$ is an isomorphism as claimed, so we may invoke [**13**, Exercise III.8.1] to deduce that

$$H^i(\mathcal{B}, \mathcal{M}) = H^i(\mathcal{B}, \tau_*\mathcal{N}) = H^i(\mathcal{B}', \mathcal{N}) \quad \text{for all} \quad i \geqslant 0.$$

The result now follows from Theorem 3.10(c) applied to the sheaf $\widehat{\mathscr{D}'}$ on $\mathcal{B}'$. $\qquad\square$

3.12. Lemma. Let M be a simple left U-module, and suppose that $n > 0$. Then there exists a finite field extension K'/K and a ρ-dominant character $\lambda : \pi^n\mathfrak{t} \to R'$ such that if $N := K' \otimes_K \widetilde{Z} \otimes_Z M$, then $\mathfrak{m}_\lambda \cdot N < N$.

Proof. Let M be a simple U-module and let $P = \mathrm{Ann}_Z(M)$. Since $n > 0$ by assumption, the affinoid Quillen Lemma [**2**, Theorem 9.4] implies that Z/P is finite dimensional over K. Since $\widetilde{Z}$ is a finitely generated Z-module via $\widehat{\phi}$, the algebra $\widetilde{Z} \otimes_Z Z/P$ is finite dimensional over K. Using the notation of Section 3.11, choose a finite field extension K'/K large enough so that every maximal ideal of

$$\widetilde{Z}'/\widetilde{Z}' \cdot P \cong K' \otimes_K \widetilde{Z} \otimes_Z Z/P$$

is of the form $\mathfrak{m}_\lambda/\widetilde{Z}' \cdot P$ for some $\lambda : \pi^n\mathfrak{t} \to R'$, and let $\Lambda \subset \pi^n\mathfrak{t}'^*$ be the finite set of characters obtained in this way. Since $\widehat{\phi}(Z)$ consists of $\mathbf{W}$-invariant elements of $\widetilde{Z}$ under the dot action, Λ is a union of $\mathbf{W}$-orbits.

Suppose for a contradiction that $\mathfrak{m}_\lambda \cdot N = N$ for all ρ-dominant $\lambda \in \Lambda$. Using [**2**, Lemma 9.6], we see that $\mathfrak{m}_\lambda \cdot N = N$ for *all* $\lambda \in \Lambda$, and hence $\mathfrak{m}_\lambda^t \cdot N = N$ for all $\lambda \in \Lambda$ and all integers $t \geqslant 1$. Since $\widetilde{Z}'/\widetilde{Z}' \cdot P$ is finite dimensional, we can find some $t \geqslant 1$ such that $\prod_{\lambda \in \Lambda} \mathfrak{m}_\lambda^t \subseteq \widetilde{Z} \cdot P$, and therefore $P \cdot N = N$. But $P \cdot N = 0$ by construction and hence $N = 0$. On the other hand, $\widetilde{Z}$ is a finitely generated free Z-module by [**2**, Proposition 9.3], so N is a direct sum of finitely many copies of M – a contradiction. $\square$

We can now state and prove the main result of this section.

3.13. Theorem. Let $\{V_1, \ldots, V_m\}$ be an open $\mathcal{S}$-cover of $\mathcal{B}$, let $\mathscr{D} := \widehat{\mathcal{D}_{n,K}}$ and let $U = \widehat{U(\mathfrak{g})_{n,K}}$. If $n > 0$, then $\mathcal{K}(U) \leqslant \mathcal{K}(\oplus_{i=1}^m \mathscr{D}(V_i))$.

Proof. We will apply Theorem 2.3 to the sheaf $\mathscr{D}$ on $\mathcal{B}$, which is coherent by [**2**, Proposition 5.10(c)]. By Proposition 3.6, $D := \Gamma(\mathcal{B}, \mathscr{D})$ is isomorphic to $U \otimes_Z \widetilde{Z}$ and therefore contains U. We will now verify the hypotheses of Theorem 2.3.

(1) By Lemma 3.4, D is an almost commutative affinoid K-algebra, so it is automatically Noetherian – see [**2**, Section 3.8]. It is a faithfully flat right U-module by Corollary 3.6.

(2) $\mathcal{S}$ is coherently $\mathscr{D}$-affine by Theorem 3.10(a).

(3) Let M be a simple left U-module. By Lemma 3.12, we can find a finite field extension K'/K and a ρ-dominant $\lambda : \pi^n\mathfrak{t} \to R'$ such that $\mathfrak{m}_\lambda \cdot N < N$, where

$N = K' \otimes_K \widetilde{Z} \otimes_Z M$. Let $\widehat{\mathscr{D}'}$ be the completion of $\mathscr{D}'$ considered in Section 3.11, and set $\mathscr{D}' := \tau_* \widehat{\mathscr{D}'}$.

(a) Since λ is ρ-dominant, $\mathcal{B}$ is coherently $\mathscr{D}'$-acyclic by Lemma 3.11(c).

(b) By Lemma 3.11(a), $D' := \Gamma(\mathcal{B}, \mathscr{D}')$ is the $\mathfrak{m}_\lambda$-adic completion of $K' \otimes_K D$. This algebra is left Noetherian and flat over $K' \otimes_K D$ (and hence also over D) by [**7**, Section 3.2.3(vi) and (iv)].

(c) It follows from Proposition 3.6 that $N = K' \otimes_K \widetilde{Z} \otimes_Z M \cong K' \otimes_K D \otimes_U M$. Now, $D' \otimes_U M \cong D' \otimes_{K' \otimes_K D} N$ is the $\mathfrak{m}_\lambda$-adic completion of N by [**7**, Section 3.2.3(iii)], and it is non-zero because $\mathfrak{m}_\lambda \cdot N < N$ by our choice of λ. $\qquad \square$

4. Krull dimension of Extended Tate–Weyl algebras.

4.1. The injective dimension of almost commutative affinoid K-algebras. In this subsection, K can have arbitrary characteristic. Let A be an almost commutative affinoid K-algebra, and let M be a finitely generated A-module. The *characteristic variety* of M was defined in [**2**, Section 3.3] to be the support

$$\mathrm{Ch}(M) = \mathrm{Supp}(\mathrm{Gr}(M)) \subseteq \mathrm{Spec}(\mathrm{Gr}(A))$$

of the associated double graded module $\mathrm{Gr}(M)$ of M with respect to a good double filtration on M. By definition, the ambient space $\mathrm{Spec}(\mathrm{Gr}(A))$ containing these characteristic varieties is an affine variety of finite type over k.

LEMMA. *If $\mathrm{Spec}(\mathrm{Gr}(A))$ is smooth, then the injective dimension of A is determined by the characteristic varieties of simple A-modules. More precisely, we have*

$$\mathrm{inj.dim}(A) = \dim \mathrm{Gr}(A) - \min_M \dim \mathrm{Ch}(M),$$

where the minimum is taken over all simple A-modules M.

Proof. It is explained in [**2**, Theorem 3.3] that the *grade number*

$$j_A(M) := \min\{j : \mathrm{Ext}_A^j(M, A) \neq 0\}$$

of any finitely generated A-module M can be computed using the characteristic variety using the formula

$$j_A(M) = \dim \mathrm{Gr}(A) - \dim \mathrm{Ch}(M).$$

It is well known that $\mathrm{inj.dim}(A) = \max_M j_A(M)$, where the maximum is taken over all non-zero finitely generated A-modules M, and therefore

$$\mathrm{inj.dim}(A) = \dim \mathrm{Gr}(A) - \min_M \dim \mathrm{Ch}(M).$$

Since $\dim \mathrm{Ch}(N) \leqslant \dim \mathrm{Ch}(M)$ for any quotient N of M, we may as well take the minimum over all *simple A-modules*. $\qquad \square$

4.2. Bernstein's Inequality and Quillen's Lemma. We return to assuming that K has characteristic zero. Recall from Section 3.8 that $l = \dim \mathbf{T}$ denotes the rank of $\mathbf{G}$.

THEOREM. *Let $D = \mathscr{D}(V)$ for some $V \in \mathcal{S}$, suppose that $n > 0$ and that $V \cong \mathbb{A}_R^m$, where $m = \dim \mathcal{B}$. Then*

$$\text{inj.dim}(D) \leqslant m + l.$$

Proof. The double associated graded of D was computed in [2, Proposition 5.10(a)] as follows:

$$\text{Gr}(D) = \text{Gr}(\widehat{\mathcal{D}_{n,K}}(V)) \cong \mathcal{O}(\widetilde{T^*V_k}).$$

Since V trivialises ξ by assumption, it follows from [2, Lemma 4.4] that $\widetilde{T^*V} \cong T^*V \times \mathfrak{h}^*$, so $\widetilde{T^*V_k}$ is smooth. Since we are assuming that $V \cong \mathbb{A}_R^m$, we see that

$$\dim \text{Gr}(D) = \dim \widetilde{T^*V_k} = 2 \dim V + \dim \mathfrak{h} = 2m + l.$$

By Lemma 4.1, it is therefore enough to show that $\dim \text{Ch}(M) \geqslant m$ for any simple D-module M.

Now $\widetilde{Z} = \widehat{U(\mathfrak{t})_{n,K}}$ is a central subalgebra of D. Let $P = \text{Ann}_{\widetilde{Z}}(M)$. By the affinoid Quillen Lemma [2, Corollary 8.6], P has finite codimension in $\widetilde{Z}$. Suppose first that $\widetilde{Z}/P$ is a copy of K. Then $\mathfrak{m}_\lambda$ kills M for some character $\lambda \in \pi^{-n}\mathfrak{t}^*$, so M is a module over $D/\mathfrak{m}_\lambda D$. It follows from [2, Proposition 6.5(a)] that this algebra is isomorphic to a Tate–Weyl algebra $\mathcal{D}(\widehat{\mathbb{A}^m})_{n,K}$. Since K has characteristic zero by assumption, we may apply the affinoid Bernstein Inequality, [2, Corollary 7.4].

In the general case, pass to a finite field extension using [2, Proposition 3.9]. $\quad\square$

COROLLARY. *The Krull dimension of D is at most $m + l$.*

Proof. This follows from the inequality

$$\mathcal{K}(D) \leqslant \text{inj.dim}(D)$$

which is apparently originally due to Roos (see [1, Corollary 1.3]). $\quad\square$

4.3. Levasseur's Theorem. We can finally state and prove the main result of this paper.

THEOREM. *Let $U = \widehat{U(\mathfrak{g})_{n,K}}$ and suppose that $n > 0$. Then $\mathcal{K}(U) \leqslant \dim \mathbf{B}$.*

Proof. Let $\{V_1, \ldots, V_m\}$ be the $\mathbf{W}$-translates of a big cell in $\mathcal{B}$. Then each $\Gamma(V_i, \mathscr{D})$ is a copy of D, so

$$\mathcal{K}(U) \leqslant \oplus_{i=1}^m \mathcal{K}(\mathscr{D}(V_i)) \leqslant \text{inj.dim}(\mathscr{D}(D)) \leqslant m + l$$

by Theorem 3.13 and Corollary 4.2. $\quad\square$

We remark that the reverse inequality $\mathcal{K}(U) \geqslant \dim B$ in Theorem 4.3 can be established along classical lines, and the restriction $n > 0$ in the affinoid Quillen Lemma is not really necessary, and will be removed in a future paper. Levasseur's original result immediately follows as a consequence.

COROLLARY. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{b}$ be a Borel subalgebra. Then $\mathcal{K}(U(\mathfrak{g})) \leqslant \dim \mathfrak{b}$.*

Proof. We regard $K = \mathbb{C}$ as being complete with respect to the *trivial* discrete valuation obtained by setting $\pi = 0$. Let the deformation parameter n be equal to zero. Then $U(\mathfrak{g})_0$ is just the enveloping algebra $U(\mathfrak{g})$ and the π-adic filtration on this algebra is trivial, so $U(\mathfrak{g})$ is isomorphic to $\widehat{U(\mathfrak{g})_{0,K}}$. Thus, the result would follow from Theorem 4.3, had the restriction $n > 0$ not been present. However, this restriction is only needed in the proof to invoke the affinoid Quillen Lemma, which reduces to the classical Quillen Lemma [**11**, Proposition 2.6.8] in this case.   □

Levasseur's Theorem can also be deduced directly from Theorem 2.3 as follows. Take $\mathcal{D}$ to be $\widetilde{\mathcal{D}}$ which is a quasi-coherent sheaf of $\mathcal{O}$-modules on the flag variety; then (2) is immediate. It is coherent since its associated graded sheaf is Noetherian, and (1) holds because $\Gamma(\widetilde{\mathcal{D}})$ is a finitely generated free $U(\mathfrak{g})$-module of rank $|W|$. Finally for (3), every simple $U(\mathfrak{g})$-module has a central character by the classical Quillen Lemma. Choose a ρ-dominant weight λ that lifts that central character, and take $\mathcal{D}'$ to be the $\mathfrak{m}_\lambda$-adic completion of $\widetilde{\mathcal{D}}$. Beilinson–Bernstein [**5**] proved that the flag variety is coherently $\widetilde{\mathcal{D}}/\mathfrak{m}_\lambda^n\widetilde{\mathcal{D}}$-acyclic for all $n \geqslant 1$, and a straightforward Mittag–Leffler argument gives the remaining conditions of (3).

4.4. Enhanced localisation is not flat. We conclude by giving an example which partially justifies the somewhat long argument presented in Theorem 2.3. Geometrically, this example is plausible because the Grothendieck–Springer resolution $\widetilde{T^*\mathcal{B}} \to \mathfrak{g}^*$ is not flat.

EXAMPLE. Let $\mathbf{G} = SL_2$ and let $V = \mathrm{Spec}(R[z])$ be a big cell in the corresponding flag variety $\mathbb{P}^1$. Then $\widetilde{\mathcal{D}}(V)$ is not a flat right $U(\mathfrak{g})$-module.

Proof. Let f, h, e be the standard basis for $\mathfrak{g}$ and identify $\widetilde{\mathcal{D}}(V)$ with the polynomial algebra $A_1[t]$ over the first Weyl algebra $A_1 = R[z; \partial]$. The algebra homomorphism $U := U(\mathfrak{g}) \to A_1[t]$ is given on generators by

$$f \mapsto -\partial, \quad h \mapsto 2z\partial - t, \quad \text{and} \quad e \mapsto z^2\partial - zt.$$

Consider the trivial left U-module R. We compute $\mathrm{Tor}_1^U(A_1[t], R)$ using the standard Chevalley complex [**22**, Section 7.7]: this Tor group is equal to the middle homology of the complex

$$A_1[t] \otimes \Lambda^2\mathfrak{g} \xrightarrow{d_2} A_1[t] \otimes \mathfrak{g} \xrightarrow{d_1} A_1[t],$$

where the maps are given explicitly by

$$
\begin{aligned}
d_2(u \otimes f \wedge h) &= uf \otimes h - uh \otimes f - u \otimes 2f \\
d_2(v \otimes f \wedge e) &= vf \otimes e - ve \otimes f + v \otimes h \\
d_2(w \otimes h \wedge e) &= wh \otimes e - we \otimes h - w \otimes 2e \\
d_1(u \otimes x) &= ux
\end{aligned}
$$

for all $u, v, w \in A_1[t]$ and $x \in \mathfrak{g}$. So elements in the image of d_2 are of the form

$$(-uh - ve - 2u) \otimes f + (uf + v - we) \otimes h + (vf + wh - 2w) \otimes e$$

for some $u, v, w \in A_1[t]$. Now,

$$d_1(z^2 \otimes f + z \otimes h - 1 \otimes e) = z^2(-\partial) + z(2z\partial - t) - (z^2\partial - zt) = 0;$$

suppose for a contradiction that $z^2 \otimes f + z \otimes h - 1 \otimes e$ is in the image of d_2. Equating the coefficient of e gives elements $v, w \in A_1[t]$ such that

$$-1 = vf + w(h - 2) = -v\partial + w(2z\partial - t - 2) = (2wz - v)\partial - w(t + 2).$$

Setting $t = -2$ now implies that -1 lies in the left ideal $A_1 \cdot \partial$ of the first Weyl algebra, a contradiction. Therefore, $\mathrm{Tor}_1^U(A_1[t], R)$ is non-zero. $\qquad\square$

REFERENCES

1. K. Ajitabh, S. P. Smith and J. J. Zhang, Auslander–Gorenstein rings, *Comm. Algebra* **26**(7) (1998), 2159–2180.

2. K. Ardakov and S. J. Wadsley, On irreducible representations of compact p-adic analytic groups, preprint, *Ann. Math.* **178**(2) (2013), 453–557.

3. K. Ardakov, Krull dimension of Iwasawa algebras, *J. Algebra* **280**(1) (2004), 190–206.

4. A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, *C. R. Acad. Sci. Paris. Sér.I Math.* **292**(1) (1991), 15–18.

5. A. Beĭlinson and J. Bernstein, *A proof of Jantzen conjectures*. In *I. M. Gel'fand Seminar*,Advances in Soviet Mathematics, vol. 16 (American Mathematical Society, Providence, RI, 1993), 1–50.

6. J. Bernstein and V. Lunts, A simple proof of Kostant's theorem that $U(\mathfrak{g})$ is free over its center, *Amer. J. Math.* **118**(5) (1996), 979–987.

7. P. Berthelot, $\mathscr{D}$-modules arithmétiques I. opèrateurs diff'erentiels de niveau fini, *Ann. Sci. École Norm. Sup. (4)* **29**(2) (1996), 185–272.

8. R. Bezrukanvikov, I. Mirkovic and D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, *Ann. Math.* **167**(3) (2008), 945–991.

9. R. Bezrukavnikov, A. Braverman and L. Positselskii, Gluing of abelian categories and differential operators on the basic affine space, *J. Inst. Math. Jussieu* **1**(4) (2002), 543–557.

10. M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, *Invent. Math.* **21** (1973), 287–301.

11. J. Dixmier, *Enveloping algebras*, Graduate Studies in Mathematics, vol. 11 (American Mathematical Society, Providence, RI, English edition, 1996).

12. A. Grothendieck, Éléments de géométrie algébrique. I. Le langage des schémas, *Inst. Hautes Études Sci. Publ. Math.* **4** (1960), 228.

13. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer-Verlag, New York, 1997).

14. T. Levasseur, Sur la dimension de Krull de l'algèbre enveloppante d'une algèbre de Lie semi-simple. In *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 34th Year (Paris, 1981)*, Lecture Notes in Mathematics, vol. 924 (Springer, Berlin, 1982), 173–183.

15. T. Levasseur, Krull dimension of the enveloping algebra of a semisimple Lie algebra, *Proc. Amer. Math. Soc.* **130**(12) (2002), 3519–3523 (electronic).

16. T. Levasseur and J. T. Stafford. *Differential operators and cohomology groups on the basic affine space, Studies in Lie theory*,Progress in Mathematics, vol. 243 (Birkhäuser/Boston, MA, 2006), 377–403.

17. H. Li and F. Van Oystaeyen, *Zariskian filtrations* (Kluwer Academic Publishers, Dordrecht, the Netherlands, 1996).

18. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*,Graduate Studies in Mathematics, vol. 30 (American Mathematical Society, Providence, RI, revised edition, 2001). With the cooperation of L. W. Small.

19. P. Schneider and J. Teitelbaum, Algebras of p-adic distributions and admissible representations, *Invent. Math.* **153**(1) (2003), 145–196.

20. S. P. Smith, Krull dimension of the enveloping algebra of sl(2, **C**), *J. Algebra* **71**(1) (1981), 189–194.

21. S. P. Smith, Krull dimension of factor rings of the enveloping algebra of a semisimple Lie algebra, *Math. Proc. Cambridge Philos. Soc.* **93**(3) (1983), 459–466.

22. C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38 (Cambridge University Press, Cambridge, 1994).