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Semisimple rings of quotients

Julius M. Zelmanowitz

Necessary and sufficient conditions on an arbitrary Gabriel filter of left ideals of a ring R are determined in order that the ring of quotients of R with respect to the filter be semisimple artinian. Special instances include generalizations of earlier work on classical rings of quotients and maximal rings of quotients.

Introduction

There have been several interesting results which determine when a ring of quotients of a ring is semisimple artinian. For the classical ring of quotients Q with respect to the set of regular elements of a ring R, Levy proved in [5] that Q is semisimple artinian if and only if torsion-free divisible R-modules are injective. In [7], Sandomierski characterized rings with a semisimple artinian maximal quotient ring as being nonsingular and finite dimensional. More generally, for the ring of quotients with respect to a Gabriel topology, some progress has been made in [1] and [6] for the case of torsion-free rings. Hereditary rings with semisimple artinian rings of quotients are studied in [3].

The purpose of this article is the determination of when the ring of quotients with respect to an arbitrary Gabriel topology is semisimple artinian. This is accomplished in §2. In §3 and §4, respectively, the above-cited results of Sandomierski and Levy are shown to be special cases of the main theorem, and are extended to allow for the possibility of torsion. In §5, injective ideals of a ring of quotients are examined; this permits one to learn when the ring of quotients is simple. Some related observations are presented in §6.

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1.

Since the notation and language of torsion theories are not quite standardized, we will first indicate the usages of this article. Two basic references for the rudiments of the subject are [2] and [8].

By a module we mean a left R-module over some ring R. Actually, in all that follows R need not have an identity element, provided that one insists that all modules on which R acts trivially are torsion, and that one makes slight modifications in definitions and statements of theorems (such as replacing R by R^1). For simplicity however, we assume that R contains an identity element. For subsets N and N' of a module M, we set $(N : N') = \{r \in R \mid rN' \subseteq N\}$.

Throughout this paper F will denote a *Gabriel topology* of left ideals of R with $0 \notin F$. That is, F satisfies

(i) $I \in F$ and $a \in R$ implies $(I : a) \in F$, and

(ii) $(J : a) \in F$ for all $a \in I \in F$ implies that $J \in F$.

Such an F is necessarily a filter. We let

$$T(M) = T_F(M) = \{m \in M \mid (0 : m) \in F\} ,$$

the *F-torsion submodule* of a module M. A module is *F-torsion* if $T_F(M) = M$, and *F-torsion-free* if $T_F(M) = 0$. A submodule N of M is *F-dense* in M if M/N is F-torsion, and N is *F-closed* in M if M/N is F-torsion-free. We also set N^c equal to the submodule defined by $N^c/N = T(M/N)$; N^c is an F-closed submodule of M, called the *F-closure* of N. The class of F-torsion modules is closed under submodules, homomorphism, extension and direct sums; while the class of F-torsion-free modules is closed under isomorphism, submodules, extension, injective hulls and direct products. When no confusion can arise, we will delete the prefix "F-" from the preceding terms, and speak simply of torsion modules, dense submodules, and so on.

A module M is *F-injective* if every $f \in \hom_R(I, M)$ with $I \in F$ can be extended to an element of $\hom_R(R, M)$; equivalently, $T(\hat{M}/M) = 0$ where $\hat{M}$ denotes the R-injective hull of M. Every module has an

F*-injective hull* $\mathsf{F}(M)$ obtained as $\mathsf{F}(M)/M = T(\hat{M}/M)$; note that M is dense in $\mathsf{F}(M)$, and $\mathsf{F}(M)$ is closed in $\hat{M}$. One defines M_{F} , the *module of quotients* of M with respect to F , as $\mathsf{F}\big(M/T(M)\big)$; M_{F} is torsion-free and F-injective. $R_{\mathsf{F}} = \mathsf{F}\big(R/T(R)\big)$ forms a ring called the *ring of quotients* of R with respect to F ; and the multiplication on R_{F} extends its R-module structure. The assignment $M \to M_{\mathsf{F}}$ yields a left exact functor from R-modules to R_{F}-modules.

As it is our intention to make this article reasonably self-contained, we now list some elementary and well-known observations that are required in the sequel.

(1.1). *If* $N_1 \subseteq N_2 \subseteq N_3$ *is a trio of submodules and* N_i *is dense (respectively, closed) in* N_{i+1} *for* $i = 1, 2$ *, then* N_1 *is dense (respectively, closed) in* N_3 .

This is because the torsion (respectively, torsion-free) modules are closed under extension. //

(1.2). *A dense submodule* N *of a torsion-free module* M *is essential in* M .

For given $0 \neq m \in M$, $(N : m) \in \mathsf{F}$ and $0 \neq (N : m)m \subseteq Rm \cap N$. //

(1.3). *If* I *is a left ideal of* R *and* $I + T(R) \in \mathsf{F}$ *then* $I \in \mathsf{F}$.

To see this note that I is dense in $I + T(R)$ since

$$I + T(R)/I \cong T(R)/I \cap T(R) ;$$

and then apply (1.1). //

(1.4). *A closed submodule* N *of an* F*-injective module* M *is* F*-injective.*

$M = \mathsf{F}(M)$ is closed in $\hat{M}$, so N is closed in $\hat{M}$ by (1.1). Hence N is closed in $\hat{N} \subseteq \hat{M}$; that is, $N = \mathsf{F}(N)$. //

LEMMA 1.5. *Set* $\overline{\mathsf{F}} = \{\overline{I} \mid I \in \mathsf{F}\}$ *, a family of left ideals of* $\overline{R} = R/T_{\mathsf{F}}(R)$ *. Then*

(i) $\overline{I} \in \overline{\mathsf{F}}$ *if and only if* $\overline{I}$ *is* F*-dense in* $\overline{R}$ *;*

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(ii) $\overline{F}$ *is an idempotent filter of left ideals of* $\overline{R}$ *;*

(iii) $T_{\overline{F}}(\overline{R}) = \overline{0}$ *;*

(iv) $R_F = \overline{R}_{\overline{F}}$.

Proof. *(i)* follows from (1.3) and is in fact true for any factor ring of R .

(ii) is also true for every factor ring of R . The proof is routine.

(iii) If $\overline{a} \in T_{\overline{F}}(\overline{R})$, then $\left(T_F(R) : a\right) \in F$. But $T_F(R)$ is F-closed in R , so $\overline{a} = \overline{0}$.

(iv) We first note that R_F is an $\overline{R}$-module, and that R_F is $\overline{F}$-injective. To see the latter assertion, let $\overline{I} \in \overline{F}$ and $f \in \hom_{\overline{R}}\left(\overline{I}, R_F\right)$ be given; we must extend f to an element of $\hom_{\overline{R}}\left(\overline{R}, R_F\right)$. Let π denote the canonical epimorphism of R onto $\overline{R}$. Then $f \circ \pi \in \hom_R\left(I, R_F\right)$ with $I \in F$. Since R_F is F-torsion-free and F-injective, $f \circ \pi$ has a unique extension $g' \in \hom_R\left(R, R_F\right)$. Now $\ker g' \supseteq T_F(R)$, so g' induces a unique homomorphism $g \in \hom_R\left(\overline{R}, R_F\right)$ with $g \circ \pi = g'$. Thus g is an $\overline{R}$-homomorphism and $(g \circ \pi)|_I = g'|_I = f \circ \pi$, so g extends f .

Since $T_{\overline{F}}(\overline{R}) = \overline{0}$, it remains only to prove that $\overline{R}$ is $\overline{F}$-dense in R_F . But this is evident, and we are done. //

2.

We begin by presenting the main result, for which we require the following definition. A set S of left ideals of R will be called *cofinally finite* if given any $I \in S$ there exists a finitely generated left ideal $J \subseteq I$ with J F-dense in I .

THEOREM 2.1. *The following conditions are equivalent:*

(1) R_F *is semisimple artinian;*

(2) R *satisfies the ascending chain condition on closed left ideals, and torsion-free* F*-injective modules are injective;*

(3) F *is cofinally finite, and torsion-free* F*-injective modules are injective.*

Proof. (1) ⇒ (2). The fact that R satisfies the ascending chain condition on closed left ideals follows immediately from the observation that $\psi^{-1}\left(R_F\overline{I} \cap \overline{R}\right) = I$ for any closed left ideal I of R, where ψ denotes the canonical homomorphism of R onto $\overline{R} = R/T(R)$. Indeed, for the same conclusion, it would clearly suffice to have R_F noetherian (see also [2, p. 136]). To see this formula, in turn, note that $I \subseteq \psi^{-1}\left(R_F\overline{I} \cap \overline{R}\right)$, and that for any $x \in \psi^{-1}\left(R_F\overline{I} \cap \overline{R}\right)$, $(I : x) = (\overline{I} : \overline{x}) \in F$. Since I is closed we learn that $I = \psi^{-1}\left(R_F\overline{I} \cap \overline{R}\right)$.

Next, let M be a torsion-free F-injective R-module. Then $M = M_F$ is an R_F-module, as is the R-injective hull $\hat{M}$ of M. Since all R_F-modules are injective, M must be an R_F-direct summand of $\hat{M}$. But then necessarily $M = \hat{M}$.

(2) ⇒ (3). We prove that the set of all left ideals of R is cofinally finite. For let I be any nonzero left ideal of R. Choose $0 \neq a_1 \in I$. If Ra_1 is dense in I, we are done. If not, $(Ra_1)^c \not\supseteq I$, where $(Ra_1)^c$ denotes the closure of Ra_1 in R. Choose $0 \neq a_2 \in I\setminus(Ra_1)^c$; then $(Ra_1)^c \subsetneqq (Ra_1+Ra_2)^c$. If $Ra_1 + Ra_2$ is dense in I, we are done. If not, $(Ra_1+Ra_2)^c \not\supseteq I$, and we may continue this construction. Since R satisfies the ascending chain condition on closed left ideals, the construction must terminate. Thus for some integer n there exists $Ra_1 + \ldots + Ra_n$ dense in I.

(3) ⇒ (1). We first show that every finitely generated left ideal of R_F is injective. For any $x \in R_F$, we have the exact sequence

$$0 \to (0 : x)_{R_F} \rightarrowtail R_F \to R_Fx \to 0 ,$$

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where $(0 : x)_{R_F} = \{r \in R_F \mid rx = 0\}$. Now R_F is F-injective, and $(0 : x)_{R_F}$ is closed since $R_F x$ is torsion-free, so by (1.4), $(0 : x)_{R_F}$ is F-injective. By our hypothesis then, $(0 : x)_{R_F}$ is injective. It follows that $R_F x$ is injective for any $x \in R_F$.

Next, consider the exact sequence

$$0 \to \ker \Sigma \to \bigoplus_{i=1}^{t} R_F x_i \xrightarrow{\Sigma} \sum_{i=1}^{t} R_F x_i \to 0 ,$$

where Σ is the canonical epimorphism. Then as in the previous paragraph, $\ker \Sigma$ is closed. Since $\bigoplus_{i=1}^{t} R_F x_i$ is injective, it follows that $\ker \Sigma$ is torsion-free and F-injective, hence injective. Thus the sequence splits, and $\sum_{i=1}^{t} R_F x_i$ is injective.

Now let $I \in F$ be given. Since F is cofinally finite, there exists $\sum_{i=1}^{t} Ra_i \in F$ with $\sum_{i=1}^{t} Ra_i \subseteq I$. Then $\sum_{i=1}^{t} R_F \bar{a}_i \subseteq R_F \bar{I} \subseteq R_F$ with each a dense submodule of its successor. Hence $\sum_{i=1}^{t} R_F \bar{a}_i$ is an injective R-module which is dense in R_F. By (1.2), $\sum_{i=1}^{t} R_F \bar{a}_i = R_F \bar{I} = R_F$. (Thus F is a *perfect* topology, in the sense of [8, p. 231].)

It follows that every R_F-module is torsion-free (for if x is an element of an R_F-module and $Ix = 0$ with $I \in F$, then $R_F x = R_F I x = 0$). In particular, every left ideal of R_F is closed, hence is torsion-free and F-injective. By hypothesis, then, every left ideal of R_F is R-injective. Since R-injective implies R_F-injective for torsion-free R_F-modules, this completes the proof. //

We remark that, in the preceding theorem, we could replace the requirement that F is cofinally finite by the stronger hypothesis that F

be perfect (that is, $R_F I = R_F$ for all $I \in F$). This was noted in the course of proving the implication (3) ⇒ (1).

It is now our intention to examine for a moment the hypotheses of Theorem 2.1 in order to determine their relationship with other familiar conditions. For instance, F being cofinally finite is a weaker condition than the ascending chain condition on closed left ideals. Their precise correlation is now given. We let $C_F(R)$ denote the lattice of closed left ideals of R.

PROPOSITION 2.2. *The following conditions are equivalent:*

(i) R *satisfies the ascending chain condition on closed left ideals;*

(ii) F *and* $C_F(R)$ *are cofinally finite;*

(iii) the set of all left ideals of R *is cofinally finite.*

Proof. The equivalence of *(i)* and *(iii)* is noted as part of Proposition XIII.2.4 in [8], and we have proved that *(i)* ⇒ *(iii)* in (2) ⇒ (3) of the preceding theorem. For the sake of completeness, then, we will demonstrate that *(ii)* ⇒ *(i)*.

Let $J_1 \subseteq J_2 \subseteq \ldots$ be a sequence of closed left ideals. Set $J = \bigcup_{i=1}^{\infty} J_i$; we first show that J is closed.

Suppose that J is dense in $I \subseteq R$. Let $a \in I$ be arbitrary. Then $(J : a) \in F$, so by hypothesis there exists $I_0 = Ra_1 + \ldots + Ra_t$ dense in $(J : a)$. For each $k = 1, \ldots, t$, $a_k a \in J$, so there exists an integer n with $I_0 a \subseteq J_n$. Since $I_0 \in F$ and J_n is closed, $a \in J_n$. Thus $J = I$ and J is closed.

Next, we know by hypothesis that there exists $Rb_1 + \ldots + Rb_s$ dense in J . But then there is an integer m with $Rb_1 + \ldots + Rb_s \subseteq J_m$. This implies that J_m is dense in J . Since J_m is closed, $J_m = J$, and the sequence is finite. //

As we have seen in (1.2), a dense submodule of a torsion-free module is an essential submodule. The converse statement (that is that essential submodules of torsion-free modules be dense) can be seen to be equivalent to torsion-free F-injectives being injective. The proof is a simple variant of the one given for Proposition 2.4 in [4], and will therefore not be presented here.

(2.3). *Torsion-free F-injective modules are injective if and only if essential submodules of torsion-free modules are dense.*

3.

Throughout this section we let G denote the Goldie topology,

$$G = \{{}_R I \mid {}_R I \subseteq {}_R J \text{ where } J \text{ and } (I : a) \text{ are essential left ideals of } R \text{ for every } a \in J\} ;$$

that is, G is the smallest Gabriel topology which contains the set of essential left ideals of R (see [8, p. 148]). G-injective modules are of course injective, and so Theorem 2.1 specializes as follows.

THEOREM 3.1. *For F any Gabriel topology containing G, the following conditions are equivalent:*

(1) *R_F is semisimple artinian;*

(2) *$C_F(R)$ satisfies the ascending chain condition;*

(3) *F is cofinally finite;*

(4) *there is no infinite independent family of F-torsion-free left ideals of R.*

Since $F \supseteq G$ is equivalent to F-injectives being injective, R_F is semisimple artinian with $F \supseteq G$ if and only if F is cofinally finite and F-injectives are injective.

(1), (2), and (3) are equivalent by Theorem 2.1. The proof that (3) ⇒ (4) ⇒ (2) involves a rather standard argument which appears also in Proposition XIII.3.1 of [8], where additional equivalent conditions are listed. We will therefore omit the proof.

LEMMA 3.2. *Assume that torsion-free F-injective modules are injective. Then $\overline{F}$ = {essential left ideals of $\overline{R}$}, the left singular*

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ideal of $\overline{R} = R/T(R)$ *is zero, and* R_F *is the maximal left quotient ring of* $\overline{R}$.

Proof. Let $\overline{I}$ be an essential left ideal of $\overline{R}$. By (2.3) and Lemma 1.5 *(i)*, $\overline{I} \in \overline{F}$. Conversely, $\overline{I} \in \overline{F}$ implies that $\overline{I}$ is essential in $\overline{R}$ by (1.2). Since $T_{\overline{F}}(\overline{R}) = \overline{0}$ by Lemma 1.5 *(iii)*, the right annihilator of $\overline{I}$ in $\overline{R}$ equals zero. This proves that the left singular ideal of $\overline{R}$ is zero.

As is well-known, the maximal left quotient ring of a nonsingular ring is just the injective hull [8, p. 149]. Now because of the hypothesis and Lemma 1.5 *(iv)*, R_F must be the maximal left quotient ring of ${}_{\overline{R}}\overline{R}$. //

Combining these results we obtain the following consequence.

COROLLARY 3.3. *If* R *is a finite dimensional ring and* $F \supseteq G$, *then* R_F *is semisimple artinian and is the maximal left quotient ring of* $R/T_F(R)$.

4.

Another important special case of the main theorem occurs when F is *cofinitely principal*; that is, when each left ideal in F contains a principal left ideal in F .

Recall that when S is a multiplicatively closed subset of a ring R , a *classical left quotient ring* of R with respect to S is defined to be a ring R_S together with a ring homomorphism $\varphi : R \to R_S$ such that

(i) $\varphi(s)$ is a unit in R_S for each $s \in S$;

(ii) every element of R_S has the form $\varphi(s)^{-1}\varphi(a)$ with $s \in S$, $a \in R$; and

(iii) $\varphi(a) = 0$ if and only if $sa = 0$ for some $s \in S$.

It can be established [8, p. 51] that the classical left quotient ring of R with respect to S exists if and only if S satisfies:

(a) if $s \in S$ and $a \in R$, then there exists $t \in S$ and $b \in R$ with $ta = bs$; and

(b) if $as = 0$ with $a \in R$, $s \in S$, then $ta = 0$ for some $t \in S$.

Such a multiplicatively closed set S will be called a *left denominator set*. For the usual reasons, when R_S exists, it is unique up to isomorphism over R; furthermore, $F = \{{}_R I \mid I \cap S \neq \emptyset\}$ is then a cofinitely finite Gabriel topology and $R_F = R_S$ [8, p. 238].

PROPOSITION 4.1. *Assume that* F *is cofinitely principal and that* F*-torsion-free* F*-injectives are injective. Then* R_F *is semisimple artinian and is the classical left quotient ring of* R *with respect to* $S = \{s \in R \mid Rs \in F\}$.

Proof. It is well-known and easy to prove that for a cofinitely principal Gabriel topology F, $S = \{s \in R \mid Rs \in F\}$ is a multiplicatively closed subset of R satisfying (a) [8, p. 237]. To see that (b) holds, suppose that $as = 0$ with $a \in R$, $s \in S$. From §2, we know that F is a perfect topology, so $R_F\bar{s} = R_F$ where $\bar{s} = s + T(R) \in R/T(R)$. Since R_F is artinian, $\bar{s}$ is a unit of R_F. Now $as = 0$ implies that $\bar{a}\bar{s} = \bar{0}$, whence $a \in T_F(R)$. It follows that there exists $t \in S$ with $ta = 0$. Thus we know that R has a classical left quotient ring with respect to S. //

The preceding proposition extends Theorem 1.7 of [7].

A result due to Levy states that if a ring R has a classical left quotient ring Q, then Q is semisimple artinian if and only if torsion-free divisible R-modules are injective [5, Theorem 3.3]. In Theorem 4.3 we provide a generalization of this to classical left quotient rings with respect to left denominator sets.

Given any $S \subseteq R$ and a module ${}_R M$, we call M *S-torsion-free* if whenever $sm = 0$ with $s \in S$ and $m \in M$, then $m = 0$; and M is called *S-divisible* if $sM = M$ for each $s \in S$.

LEMMA 4.2. *If* S *is a left denominator set and* I *is a left ideal of* R *then there is a natural isomorphism* $R_S \otimes_R \varphi(I) \cong R_S\varphi(I)$, *where* φ *denotes the canonical homomorphism of* R *into* R_S.

Proof. The isomorphisms $R_S \otimes_R \varphi(I) \underset{\nu}{\overset{\mu}{\rightleftarrows}} R_S\varphi(I)$ are defined by

$$\mu\left(\sum_{i=1}^{n} x_i \otimes \varphi(a_i)\right) = \sum_{i=1}^{n} x_i\varphi(a_i) \quad \text{and} \quad \nu\left(\sum_{i=1}^{n} x_i\varphi(a_i)\right) = \sum_{i=1}^{n} x_i \otimes \varphi(a_i) ,$$

where $x_i \in R_S$, $a_i \in I$. μ is well-defined by the universal mapping property of tensor products, and the proof is completed by showing that ν is well-defined.

To see this, let $\sum_{i=1}^{n} x_i\varphi(a_i) = 0$ with $x_i \in R_S$, $a_i \in I$. We may choose a common denominator $s \in S$ and $b_1, \ldots, b_n \in R$ so that each $x_i = \varphi(s)^{-1}\varphi(b_i)$. Then $\sum_{i=1}^{n} \varphi(b_i)\varphi(a_i) = 0$, and so

$$\varphi(s)\left(\sum_{i=1}^{n} x_i \otimes \varphi(a_i)\right) = \sum_{i=1}^{n} \varphi(b_i) \otimes \varphi(a_i)$$

$$= \sum_{i=1}^{n} \varphi(1) \otimes \varphi(b_i)\varphi(a_i) = \varphi(1) \otimes \sum_{i=1}^{n} \varphi(b_i)\varphi(a_i) = 0 .$$

Since $\varphi(s)$ is a unit, $\sum_{i=1}^{n} x_i \otimes \varphi(a_i) = 0$, and ν is well-defined. //

THEOREM 4.3. *Let S be a left denominator set in R . Then R_S is semisimple artinian if and only if S-torsion-free S-divisible R-modules are injective.*

Proof. Set $F = \{{}_RI \mid I \cap S \neq \emptyset\}$. It is straightforward to check that a module is F-torsion-free if and only if it is S-torsion-free. We begin by showing that the F-torsion-free F-injectives are precisely the S-torsion-free S-divisibles.

Suppose that M is F-torsion-free and F-injective and let $s \in S$, $m \in M$ be given. Define $f \in \hom_R(Rs, M)$ by $f(as) = am$, $a \in R$. f is well-defined; for $as = 0$ implies that there exists $t \in S$ with $ta = 0$, and so $tam = 0$, whence $am = 0$, since M is F-torsion-free. Since M is F-injective there exists $g \in \hom_R(R, M)$ with $g|_{Rs} = f$. Now $m = f(s) = g(s) = sg(1)$, and this proves that M is S-divisible.

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Conversely, assume that M is S-torsion-free and S-divisible, and let $f \in \hom_R(I, M)$ be given with $I \in F$. Choose $s \in I \cap S$; $f(s) = m$ for some $m \in M$. Since M is S-divisible, we may choose $n \in M$ with $m = sn$. Now define $g \in \hom_R(R, M)$ by $g(r) = rn$. We claim that $g|_I = f$. For given $a \in I$, there exists $t \in S$, $b \in R$ with $ta = bs$. So

$$t\big(f(a)-g(a)\big) = f(ta) - g(ta) = f(bs) - g(bs)$$
$$= bf(s) - bg(s) = bm - bsn = 0 .$$

Since M is S-torsion-free, $f(a) = g(a)$, and so M is F-injective.

Now assume that S-torsion-free S-divisible modules are injective. By Proposition 4.1, $R_S = R_F$ is semisimple artinian.

Conversely, assume that R_S is semisimple artinian, and let ${}_RM$ be S-torsion-free and S-divisible. Then M is an R_S-module. For given $m \in M$ and $x = \varphi(s)^{-1}\varphi(a) \in R_S$ with $s \in S$, $a \in R$, there exists a unique element $n \in M$ with $am = sn$; and defining $x \cdot m = n$ determines the R_S-module structure of M, as can be readily verified.

Now let $f \in \hom_R(I, M)$ be given, with I a left ideal of R. Then $J = \ker \varphi \cap I = \{a \in I \mid sa = 0 \text{ for some } s \in S\}$ is a left ideal, and $f(J) = 0$ because M is S-torsion-free. Hence f induces $f' \in \hom_R\big(\varphi(I), M\big)$ with $f' \circ \varphi = f$. By Lemma 4.2, $R_S \otimes_R \varphi(I) \cong R_S\varphi(I)$, and therefore f' can be extended to $g' \in \hom_R\big(R_S\varphi(I), M\big)$. Since $R_S\varphi(I)$ is a left ideal of the semisimple artinian ring R_S, g' can be extended to $g \in \hom_R\big(R_S, M\big)$. But then $g \circ \varphi \in \hom_R(R, M)$ extends f. So ${}_RM$ is injective, and the proof is complete. //

5.

The next objective is to examine ideals of R_F in the case when torsion-free F-injectives are injective. Some observations can be made in a more general setting.

Assume that F and G are Gabriel topologies with $F \subseteq G$. Then $T_F(R) \subseteq T_G(R)$; and we let p denote the canonical homomorphism from $R/T_F(R)$ onto $R/T_G(R)$. Consider the diagram

$$\begin{tikzcd}
R_G & \\
R/T_G(R) \arrow[u, phantom, sloped, "\subseteq"] & \\
R/T_F(R) \arrow[u, "p"] \arrow[r, phantom, "\subseteq"] & R_F \arrow[uul, dashed, "q"']
\end{tikzcd}$$

Since R_G is F-torsion-free and F-injective there exists a unique R-homomorphism $q : R_F \to R_G$ extending p.

We claim that q is in fact a ring homomorphism. To see this, let $\bar{r}$ denote the coset of $r \in R$ in $R/T_F(R)$, and observe that the R-module structures of R_F and R_G are defined by $r\cdot\alpha = \bar{r}\alpha$, $r\cdot\gamma = p(\bar{r})\gamma$ for any $r \in R$, $\alpha \in R_F$, $\gamma \in R_G$. Now let α, β be arbitrary elements of R_F and set $I = \big(R/T_F(R) : \alpha\big) \in F$. Then for any $r \in I$,

$$\begin{aligned} r\big(q(\alpha\beta)-q(\alpha)q(\beta)\big) &= q(r\cdot\alpha\beta) - q(r\cdot\alpha)q(\beta) \\ &= q(r\alpha\cdot\beta) - p(r\alpha)q(\beta) = p(r\alpha)q(\beta) - p(r\alpha)q(\beta) = 0 \end{aligned}$$

since $r\alpha \in R/T_F(R)$ and q extends p. Since $r \in I$ was arbitrary and R_G is F-torsion-free, it follows that $q(\alpha\beta) = q(\alpha)q(\beta)$. We summarize this.

(5.1). *If $F \subseteq G$ are Gabriel topologies, then there is a unique R-algebra homomorphism $q : R_F \to R_G$ which extends the canonical epimorphism $p : R/T_F(R) \to R/T_G(R)$.* (See [8, p. 210, Exercise 1].)

PROPOSITION 5.2. *Suppose that F-injective ideals of R_F are R-injective, and let G be a Gabriel topology with $F \subseteq G$. Then R_G is isomorphic to a direct summand of R_F under a splitting of the homomorphism q of* (5.1).

Proof. Let $q : R_F \to R_G$ be the algebra homomorphism described by (5.1). Since R_G is F-torsion-free, kernel q is F-injective, and so by hypothesis R_F and kernel q are injective R-modules. Hence $R_F \cong \text{kernel } q \oplus q(R_F)$, from which we have that $q(R_F)$ is injective. But $R/T_G(R) \subseteq q(R_F)$, so necessarily $q(R_F) = R_G$. //

A converse is true as well.

PROPOSITION 5.3. *If $\underline{\underline{A}}$ is an ideal direct summand of R_F which is injective as a left R-module, then $\underline{\underline{A}}$ is a quotient ring of R with respect to some Gabriel topology $G \supseteq F$.*

Proof. Choose an ideal $\underline{\underline{B}}$ with $R_F = \underline{\underline{A}} \oplus \underline{\underline{B}}$. Set $T = \{{}_RM \mid \hom_R(M, \underline{\underline{A}}) = 0\}$; T is a hereditary torsion class containing the F-torsion modules because $\underline{\underline{A}}$ is an injective F-torsion-free R-module. Let G be the Gabriel topology of left ideals associated to T; then $G \supseteq F$ and $T_G(R) \supseteq T_F(R)$. We will show that $R_G \cong \underline{\underline{A}}$.

Let $q : R_F \to R_G$ be the ring homomorphism of (4.1). Since $\hom_{R_F}(\underline{\underline{B}}, \underline{\underline{A}}) = 0$, $\hom_R(\underline{\underline{B}}, \underline{\underline{A}}) = 0$. So $\underline{\underline{B}} \in T$, and hence $\underline{\underline{B}} \subseteq \text{kernel } q$. If $\underline{\underline{A}} \cap \ker q \neq 0$, then $A = \underline{\underline{A}} \cap \ker q \cap R/T_F(R) \neq 0$, and consequently $Ap = Aq = 0$. Hence $A \subseteq \text{kernel } p = T_G(R)/T_F(R) \in T$. But then $\hom_R(A, \underline{\underline{A}}) = 0$, which forces $A = 0$, a contradiction. Thus it must be the case that $\underline{\underline{A}} \cap \ker q = 0$, and so $q|_{\underline{\underline{A}}}$ is a monomorphism. Next $q(R_F) = q(\underline{\underline{A}})$ is an injective R-submodule of R_G which contains $R/T_G(R)$, and we conclude from this that $q(\underline{\underline{A}}) = R_G$. So $\underline{\underline{A}} \cong R_G$ under the homomorphism q. //

An ideal T of a ring R is a *torsion ideal* if there exists a proper Gabriel topology G with $T_G(R) = T$. We can now apply the information above to learn when a semisimple artinian ring of quotients is simple.

THEOREM 5.4. *Assume that R_F is semisimple artinian. Then the* *following conditions are equivalent:*

(i) R_F *is simple;*

(ii) $R_F = R_G$ *for every proper Gabriel topology* $G \supseteq F$ *;*

(iii) $T_F(R)$ *is a maximal torsion ideal.*

Proof. *(i) ⇒ (iii).* Suppose that R_F is simple, and say $T_F(R) \subsetneqq T_G(R)$ for some Gabriel topology G . Then $\overline{T} = \overline{T_G(R)}$ is a non-zero G-torsion submodule of $\overline{R} = R/T_F(R)$. Since R_F is simple, $R_F\overline{T}R_F = R_F$. Write $1 = \sum_{i=1}^{t} \alpha_i\overline{t}_i\beta_i$ with $\alpha_i, \beta_i \in R_F$, $\overline{t}_i \in \overline{T}$. Set $J = \bigcap_{i=1}^{t} (\overline{R} : \alpha_i) \in F$. Then for any $a \in J$,

$$\overline{a} = \overline{a} \sum_{i=1}^{t} \alpha_i\overline{t}_i\beta_i = \sum_{i=1}^{t} (\overline{a}\alpha_i)\overline{t}_i\beta_i \subseteq \overline{R} \cap \overline{T}R_F \subseteq T_G(\overline{R}) .$$

It follows that $\overline{J} \subseteq T_G(\overline{R})$, and hence by (1.2), $T_G(\overline{R})$ is an essential left ideal of $\overline{R}$. But $T_G(\overline{R}) \subseteq T_G(R_F)$, so $T_G(R_F)$ must be an essential ideal of R_F . Hence $T_G(R_F) = R_F$. By considering again the identity element of R_F , this forces $T_G(R) \in G$, which is impossible unless $0 \in G$.

(iii) ⇒ (ii). This is clear from Proposition 5.2.

(ii) ⇒ (i). If R_F is not simple, let $\underline{A}$ be any proper ideal of R_F . By Proposition 5.3, there exists a Gabriel topology $G \supseteq F$ with $R_G = \underline{A} \subsetneqq R_F$. //

We conclude this section with the following result, which is actually a consequence of the material in §2.

THEOREM 5.5. R_F *is a division ring if and only if* $T_F(R)$ *is the only proper closed left ideal of* R *.*

Proof. Suppose that R_F is a division ring, and that I is a closed left ideal of R. If $I \neq T(R)$, then $R_F\overline{I}$ is a nonzero left ideal of R_F, so $R_F\overline{I} = R_F$. But then I is clearly dense in R, so, necessarily, $I = R$.

Conversely, assume that $T(R)$ is the only closed left ideal of R, other than R. We first show that a torsion-free F-injective module M is injective. For let $f \in \hom_R(I, M)$ with I a left ideal of R. By our hypothesis either $I \subseteq T(R)$, in which case $f = 0$, or else $I \in F$. In any event, f can be extended to an element of $\hom_R(R, M)$, and so $_RM$ is injective.

By Theorem 2.1 (2), R_F is semisimple artinian; and from the discussion immediately following that theorem, we know that F is perfect. Let K now be any left ideal of R_F. Since F is perfect, K is closed in R_F. It follows that $I = \psi^{-1}(K \cap \overline{R})$ is closed in R, where $\psi : R \to R_F$ is the canonical homomorphism. By hypothesis, $I = T(R)$ or $I = R$, and from this we can conclude that $K = 0$ or $K = R_F$. Thus R_F is a division ring. //

6.

In this final section we treat some related facts, which extend results known for classical rings of quotients. For instance, a ring with a simple (respectively, semisimple) artinian classical ring of quotients is prime (respectively, semiprime). More generally, we have the following:

PROPOSITION 6.1. *Suppose that R_F is a simple ring (respectively, a finite direct sum of simple rings), and that every left ideal in $\overline{F}$ is a faithful $\overline{R}$-module. Then $T_F(R)$ is a prime (respectively, semiprime) ideal of R.*

Proof. First assume that R_F is simple, and let $AB \subseteq T_F(R)$ with A and B left ideals of R and $B \nsubseteq T_F(R)$. Then $\overline{B} \neq \overline{0}$ in $\overline{R} = R/T_F(R)$, so $R_F\overline{B}R_F = R_F$ by the hypothesis on R_F. Write $1 = \sum_{i=1}^{t} p_i\overline{b}_iq_i$ with

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$p_i, q_i \in R_F$, $b_i \in B$. Let $I = \bigcap_{i=1}^{t} (\bar{R} : p_i) \in F$. Now

$\bar{I} = \bar{I}1 \subseteq \sum_{i=1}^{t} (Ip_i)\bar{b}_i q_i \subseteq \bar{R}\bar{B}R_F \subseteq \bar{B}R_F$. Hence $\bar{A}\bar{I} \subseteq \bar{A}\bar{B}R_F = \bar{0}$. Since ${}_{\bar{R}}\bar{I}$ is faithful, we conclude that $\bar{A} = \bar{0}$, proving that $T_F(R)$ is a prime ideal.

For the semisimple case, let B be a left ideal of R with $B^n \subseteq T_F(R)$, $n \geq 1$. We may write $R_F\bar{B}R_F = R_F e$ with e a central idempotent of R . As above, write $e = \sum_{i=1}^{t} p_i\bar{b}_i q_i$ and let $I = \bigcap_{i=1}^{t} (\bar{R} : p_i) \in F$. Then $e\bar{I} = \bar{I}e \subseteq \bar{B}R_F$, so that $\bar{B}^{n-1}e\bar{I} \subseteq \bar{B}^n = \bar{0}$. Since $\bar{I}$ is faithful, $\bar{B}^{n-1}e = \bar{0}$. But e is an identity element on $\bar{B}$, so $\bar{B}^{n-1} = \bar{0}$. Continuing in this manner, we eventually learn that $\bar{B} = \bar{0}$, and this proves that $T_F(R)$ is a semiprime ideal. //

COROLLARY 6.2. *If* S *is a left denominator set such that* R_S *is a simple ring (respectively, a direct sum of simple rings), then* $T(R) = \{a \in R \mid sa = 0$ *for some* $s \in S\}$ *is a prime (respectively, semi-prime) ideal.*

Let us call a topology F *hereditary* if every left ideal in F is projective. Cofinally finite hereditary topologies (which are not necessarily Gabriel topologies) often consist of finitely generated left ideals, as we now see.

PROPOSITION 6.3. *Assume that* $T_F(R) = 0$ *and that* F *is a cofinally finite hereditary topology. Then every* $I \in F$ *is finitely generated.*

Proof. Let $I \in F$ be given. Choose $J = \sum_{i=1}^{t} Rx_i \subseteq I$ with $J \in F$. By hypothesis I is projective, so we may choose $f_\alpha \in \hom_R(I, R)$, $y_\alpha \in I$ with $y = \sum_{\alpha\in A} (yf_\alpha)y_\alpha$ for each $y \in I$. Since J is finitely

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generated, $f_\alpha|_J = 0$ for all but finitely many $\alpha \in A$. For simplicity, let us assume that $f_\alpha|_J = 0$ for $\alpha \notin \{1, 2, \ldots, k\}$. Now for each $\alpha \notin \{1, 2, \ldots, k\}$, f_α induces $f'_\alpha : I/J \to R$ and $(I/J)f'_\alpha$ is F-torsion. Since $T_F(R) = 0$, $f'_\alpha = 0$; that is, $f_\alpha = 0$ for each $\alpha \notin \{1, 2, \ldots, k\}$, and it follows that $y_1, y_2, \ldots, y_k$ generate I. //

COROLLARY 6.4 [8, p. 260]. *A finite dimensional hereditary ring is noetherian.*

Proof. A hereditary ring is nonsingular. So F = {essential left ideals} forms a Gabriel topology satisfying the previous proposition. It follows that every left ideal, being a summand of an essential left ideal, is finitely generated. //

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Department of Mathematics,
University of California, Santa Barbara,
Santa Barbara,
California,
USA.

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