COMPOSITIO MATHEMATICA

On fields of rationality for automorphic representations

Sug Woo Shin and Nicolas Templier

Compositio Math. **150** (2014), 2003–2053.

doi:10.1112/S0010437X14007428

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 150 (2014) 2003–2053
doi:10.1112/S0010437X14007428

On fields of rationality for automorphic representations

Sug Woo Shin and Nicolas Templier

Abstract

This paper proves two results on the field of rationality $\mathbb{Q}(\pi)$ for an automorphic representation π, which is the subfield of $\mathbb{C}$ fixed under the subgroup of $\mathrm{Aut}(\mathbb{C})$ stabilizing the isomorphism class of the finite part of π. For general linear groups and classical groups, our first main result is the finiteness of the set of discrete automorphic representations π such that π is unramified away from a fixed finite set of places, π_∞ has a fixed infinitesimal character, and $[\mathbb{Q}(\pi):\mathbb{Q}]$ is bounded. The second main result is that for classical groups, $[\mathbb{Q}(\pi):\mathbb{Q}]$ grows to infinity in a family of automorphic representations in level aspect whose infinite components are discrete series in a fixed L-packet under mild conditions.

1. Introduction

1.1 Modular form case

Let $S_k(N)$ be the space of cuspforms of weight $k \geqslant 2$ and level $\Gamma_0(N)$ with $N \geqslant 1$. Suppose that $f \in S_k(N)$ is an eigenform under the Hecke operator $\{T_p\}$ with eigenvalue $a_p(f) \in \mathbb{C}$ for each prime $p \nmid N$. It is well-known that $\{a_p(f)\}_{p\nmid N}$ are algebraic integers and that they generate a finite extension of $\mathbb{Q}$ (in $\mathbb{C}$), to be denoted $\mathbb{Q}(f)$. The field $\mathbb{Q}(f)$ encodes deep arithmetic information about f and is of our main concern here. To wit the significance of $\mathbb{Q}(f)$, the Eichler–Shimura construction associates to a weight 2 form f a GL_2-type abelian variety of dimension $[\mathbb{Q}(f):\mathbb{Q}]$ as a quotient of the Jacobian of the modular curve $X_0(N)$. Moreover the two-dimensional l-adic Galois representations associated to f are realized with coefficients in the completions of $\mathbb{Q}(f)$ at finite places.

We are interested in two aspects of $\mathbb{Q}(f)$. The first question is on the growth of $\mathbb{Q}(f)$ in a family of modular forms f with increasing level. Let $\mathcal{F}_k(N)$ be the set of normalized cuspidal eigenforms of weight $k \geqslant 2$. These are eigenforms for all T_p ($p \nmid N$) and define

$$\mathcal{F}_k(N)^{\leqslant A} := \{f \in \mathcal{F}_k(N) : [\mathbb{Q}(f):\mathbb{Q}] \leqslant A\}, \quad A \in \mathbb{Z}_{\geqslant 1}.$$

Serre has proved the following theorem, which serves as a prototype for one of our main results.

Theorem 1.1 [Ser97, Theorem 5]. *Fix $k \geqslant 2$ and a prime p. Then*

$$\lim_{\substack{N\to\infty,\\(N,p)=1}} |\mathcal{F}_k(N)^{\leqslant A}|/|\mathcal{F}_k(N)| = 0.$$

Received 16 September 2013, accepted in final form 29 April 2014, published online 11 September 2014.
2010 Mathematics Subject Classification 11F30, 11F70, 11F80, 11R39, 11S37 (primary).
Keywords: field of rationality, automorphic representations, Galois representations, finiteness.

Let us briefly recall Serre's argument. The key point is to show that

$$|\{a_p(f) : f \in \mathcal{F}_k(N)^{\leqslant A}\}| < \infty. \tag{1.1}$$

This follows from the fact that $a_p(f)$ is an algebraic integer which is the sum of a Weil p-number of weight $k-1$ and its complex conjugate. The condition $[\mathbb{Q}(f) : \mathbb{Q}] \leqslant A$ implies that $[\mathbb{Q}(a_p(f)) : \mathbb{Q}] \leqslant A$, so such a Weil number is a root of a monic polynomial in $\mathbb{Z}[x]$ whose degree and coefficients are bounded only in terms of p, k, and A. Clearly there are only finitely many such polynomials, hence (1.1). Finally, Theorem 1.1 is deduced from (1.1) by using a trace formula argument.

Serre then asked in [Ser97, § 6.1] whether the same type of result would be true without requiring some auxiliary prime p to be coprime to the level. (For instance, is the above result valid if the limit is taken along the sequence $N = 2, (2 \cdot 3)^2, (2 \cdot 3 \cdot 5)^3, \dots$?) In our paper we generalize Theorem 1.1 to higher rank classical groups and partially settles Serre's question in the generalized setting for a sequence of levels $N \to \infty$ such that there exists a prime whose order in N grows to infinity. Moreover, we improve on the rate of decay of the quotient as in Theorem 1.1 by a logarithmic order.

Another aspect of $\mathbb{Q}(f)$ is in relation to a finiteness result. Let us begin with recalling a deep theorem of Faltings, who also proved a stronger version in which 'up to isogeny' is replaced with 'up to isomorphism' (the Shafarevich conjecture).

Theorem 1.2 [Fal86, Theorem 5]. *Fix $n \in \mathbb{Z}_{\geqslant 1}$ and a finite set of primes S. Then there are only finitely many abelian varieties of dimension n having good reduction outside S up to isogeny.*

The Shimura–Taniyama conjecture, as confirmed by Wiles and Breuil–Conrad–Diamond–Taylor, translates the case $n = 1$ of the above theorem into a finiteness result about modular forms: namely there are only finitely many newforms f such that $[\mathbb{Q}(f) : \mathbb{Q}] = 1$ which are contained in $\mathcal{F}_2(N)$ for some level N whose prime divisors are all contained in S. With this motivation an automorphic analogue of the above finiteness theorem will be pursued in this paper.

To formulate and make progress toward the problems raised in this subsection we are going to introduce some definitions, concepts, and conjectures before stating the main results.

1.2 C-algebraic automorphic representations

Algebraicity of automorphic forms and representations has been studied by Shimura, Waldspurger, Harder, Harris, and many other mathematicians. Regarding automorphic representations of GL_n the definition of algebraicity was first formulated by Clozel [Clo90] and recently extended to arbitrary connected reductive groups by Buzzard and Gee [BG11]. In fact, one main point of their paper is to distinguish between the two possible definitions of algebraicity, namely C-algebraicity and L-algebraicity, the former generalizing Clozel's notion. In this article our attention is restricted to C-algebraic representations mainly because these are expected to be exactly those having number fields as their fields of rationality. (There is also W-algebraicity recently suggested by Patrikis [Pat12], but again C-algebraicity is believed to be the exact condition to ensure the finiteness of the field of rationality over $\mathbb{Q}$.)

To be precise let G be a connected reductive group over $\mathbb{Q}$. To avoid vacuous statements we assume throughout the paper that the rank of the groups under consideration is at least one. Let $\pi = \otimes_v \pi_v = \pi^\infty \otimes \pi_\infty$ be an automorphic representation of $G(\mathbb{A})$. Here π^∞ and π_∞ denote the finite and infinite components. We say that π is C-algebraic if, loosely speaking, the infinitesimal character of π_∞ is integral after a shift by the half sum of all positive roots

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

(for some thus for all choices of positivity on the set of roots). When σ is a field automorphism of $\mathbb{C}$, let $(\pi^\infty)^\sigma$ denote the $G(\mathbb{A}^\infty)$-representation on the underlying vector space of π^∞ twisted by a σ-linear automorphism. For any π define its field of rationality as the field of the definition of its isomorphism class, i.e.

$$\mathbb{Q}(\pi) := \{z \in \mathbb{C} : \sigma(z) = z,\ \forall \sigma \in \mathrm{Aut}(\mathbb{C}) \text{ s.t. } (\pi^\infty)^\sigma \simeq \pi^\infty\}. \tag{1.2}$$

The following was conjectured by Clozel (for $G = \mathrm{GL}_n$) and Buzzard and Gee.

Conjecture 1.3. The automorphic representation π is C-algebraic if and only if $\mathbb{Q}(\pi)$ is finite over $\mathbb{Q}$.

It is worth noting that in the special but subtle case of Maass cusp forms for GL_2 over $\mathbb{Q}$, Sarnak [Sar02] classified the forms with integer coefficients, showing in particular that they are C-algebraic (i.e. Laplace eigenvalue being $\frac{1}{4}$), and made a remark on the transcendence of $\mathbb{Q}(\pi)$.

According to the conjecture, C-algebraic representations are the most suitable for studying questions on the growth of fields of rationality. To obtain unconditional results, we show that $\mathbb{Q}(\pi)$ is a number field for cohomological representations π, which form a large subset inside the set of C-algebraic representations, by adapting an argument of Clozel using arithmetic cohomology spaces. See § 2.2 below. Note that if G is semisimple then any π such that π_∞ is a discrete series is always cohomological.

1.3 Conjectures

Let us highlight two interesting conjectures that we were led to formulate during our investigation of fields of rationality for automorphic representations. Some partial results and remarks are found in the next subsection as well as in the main body of our paper.

The first conjecture, a small refinement of the well-known Fontaine–Mazur conjecture, is not directly concerned with field of rationality but rather with integrality of local parameters (e.g. Satake parameters or Frobenius eigenvalues of a Galois representation). The question arises naturally as a weak form of integrality is needed to answer a generalization of Theorem 1.1.

Conjecture 1.4. Let F be a number field and $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ a continuous irreducible representation unramified outside finitely many places. The following are equivalent:

- (i) ρ is de Rham at every place $v|l$ with nonnegative Hodge–Tate weights (adopting the convention that the cyclotomic character has Hodge–Tate weight -1);
- (ii) the Weil–Deligne representation associated with ρ at every finite place $v \nmid l$ is integral and pure of weight $w \in \mathbb{Z}$ which is independent of v;
- (iii) ρ appears as a subquotient of $H^i_{\text{ét}}(X \times_F \overline{F}, \overline{\mathbb{Q}}_l)$ for some proper smooth scheme X over F and some $i \in \mathbb{Z}_{\geqslant 0}$.

The motivation for the conjecture comes from our effort to obtain Theorem 1.7 below (which generalizes Theorem 1.1), where we need a version of the statement that $a_p(f)$ is an algebraic *integer*. We derive a partial result toward Conjecture 1.4 (Proposition 4.1) for the Galois representations arising from (conjugate) self-dual automorphic representations by exploiting the fact that they appear in the cohomology of Shimura varieties. This serves as a crucial ingredient in the proof of Theorem 1.7.

The second conjecture is on the finiteness of automorphic representations with bounded field of rationality. It is an automorphic analogue of (the isogeny version of) the Shafarevich conjecture

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

and its analogue for Galois representations formulated by Fontaine and Mazur [FM95, I.§ 3]. Theorem 1.6 below partially confirms the conjecture.

Conjecture 1.5. Fix $A \in \mathbb{Z}_{\geqslant 1}$, S a finite set of places of F containing all infinite places, and an infinitesimal character χ_∞ for $G(F \otimes_{\mathbb{Q}} \mathbb{R})$. Then there are only finitely many discrete automorphic representations π of $G(\mathbb{A}_F)$ with infinitesimal character χ_∞ such that π^S is unramified and $[\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A$.

1.4 Main results

Let us make it clear at the outset that our results concerning quasi-split classical (i.e. symplectic, orthogonal,[1] or unitary) groups rely on Arthur's endoscopic classification [Art13] and its analogue for unitary groups due to Mok [Mok12]. (However, our finiteness theorem for general linear groups, cf. Theorem 1.6 below, is unconditional.) The classification is based on some unproven assertions on the stabilization of the twisted trace formula for GL_n and a little more, which are hoped to be proved in the near future. So we are making the same hypotheses as Arthur does in his work. (Also see [BMM11, 1.18] and the footnote around Hypothesis 4.8 for a discussion of the hypotheses.) We only deal with quasi-split groups mainly because the analogous theorems for inner forms are not complete (see the last chapter of [Art13] for a sketch), but our argument should apply equally well to the inner forms. With this in mind we have written the argument in such a way that our main theorems remain true for non-quasi-split classical groups with little change in the proof once the necessary classification becomes available. As a matter of fact, Theorem 1.7 in case (i) is almost an unconditional theorem for (not necessarily quasi-split) unitary groups thanks to the base change results for cohomological representations in [Lab11]. (Unlike Arthur's work, the latter are not conditional on the full stabilization of the twisted trace formula or any other hypotheses.)

Our first main result is a finiteness theorem for automorphic representations with bounded field of rationality. It is worth emphasizing that we allow arbitrary infinitesimal characters (e.g. those corresponding to C-algebraic Maass forms in the case of GL_2 over $\mathbb{Q}$) even including transcendental ones (in which case the set of π is expected to be empty by Conjecture 1.3).

Theorem 1.6 (Theorems 5.18 and 5.19). *Conjecture 1.5 is true for general linear groups and quasi-split classical groups.*

Our second main result is on the growth of the field of rationality in a family of automorphic representations. We work with a quasi-split classical group G over $\mathbb{Q}$ for simplicity (in the main body G is over any totally field) and introduce a family in level aspect with prescribed local conditions as in [ST12]. Let $n_x \in \mathbb{Z}_{\geqslant 1}$, ξ be an irreducible algebraic representation of G over $\mathbb{C}$ whose highest weight is regular, S_0 be a finite set of finite primes (which could be empty so that no local condition may be imposed), and $\widehat{f}_{S_0}$ be a well-behaved function on the unitary dual of $G(\mathbb{Q}_{S_0})$. The family in question is a sequence

$$\mathcal{F}_x = \mathcal{F}(n_x, \widehat{f}_{S_0}, \xi),\ x \in \mathbb{Z}_{\geqslant 1} \quad \text{such that } n_x \to \infty \text{ as } x \to \infty,$$

where each $\mathcal{F}_x$ consists of discrete automorphic representations π of G which, loosely speaking, has level n_x, weight ξ, and prescribed local conditions at S_0 by $\widehat{f}_{S_0}$. Then each $\mathcal{F}_x$ is a finite set whose cardinality $|\mathcal{F}_x|$ tends to infinity as $x \to \infty$. Actually in our formulation $\mathcal{F}_x$ is a multi-set in that each π is weighted by the dimension of the fixed vectors of π^∞ under the principal congruence subgroup of level n_x. (See § 6.1 for the precise definition of $\mathcal{F}_x$ and $|\mathcal{F}_x|$.)

[1] As we will never deal with the usual (disconnected) orthogonal groups, special orthogonal groups will be called orthogonal groups in favor of simpler terminology. We will be precise where we have to be.

For $A \in \mathbb{Z}_{\geqslant 1}$ define
$$\mathcal{F}_x^{\leqslant A} := \{\pi \in \mathcal{F}_x : [\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A\}.$$
Note that we have $[\mathbb{Q}(\pi) : \mathbb{Q}] < \infty$ for every $\pi \in \mathcal{F}_x$ since π is cohomological in that $\pi_\infty \otimes \xi$ has non-vanishing Lie algebra cohomology. We prove a theorem roughly saying that the field of rationality grows generically in the family $\{\mathcal{F}_x\}_{x \geqslant 1}$ in the case (i) or (ii) below. Note that case (ii) includes the level sequence $2, (2 \cdot 3)^2, (2 \cdot 3 \cdot 5)^3, (2 \cdot 3 \cdot 5 \cdot 7)^4, \ldots$ for instance. Unfortunately neither case (i) nor case (ii) includes the sequence $2, 2 \cdot 3, 2 \cdot 3 \cdot 5, \ldots$.

Theorem 1.7 (Theorems 6.1 and 6.6). *Let $G \neq \{1\}$ be a quasi-split classical group, or a non-quasi-split unitary group. Suppose there exists a prime $p \notin S_0$, at which G is unramified, such that either:*

(i) $(n_x, p) = 1$ *for all but finitely many x; or*

(ii) $\mathrm{ord}_p(n_x) \to \infty$ *as $x \to \infty$.*

Then for every $A \in \mathbb{Z}_{\geqslant 1}$, $\lim_{x \to \infty} (|\mathcal{F}_x^{\leqslant A}|/|\mathcal{F}_x|) = 0$. Moreover, let S_{unr} be the number of primes p satisfying case (i) (which could be infinite) and such that G is unramified at p. Put $R_{\mathrm{unr}} := \sum_{p \in S_{\mathrm{unr}}} \mathrm{rank}\, G_{\mathbb{Q}_p}$. Then
$$|\mathcal{F}_x^{\leqslant A}| = O(|\mathcal{F}_x|/(\log |\mathcal{F}_x|)^R) \quad \text{for all } R \leqslant R_{\mathrm{unr}}.$$

Especially pleasing features of the theorem are that some arbitrarily high ramification can be treated as seen in case (ii) and that the upper bound has a logarithmic power-saving. The case (ii) seems to be new already in the case of modular forms while the logarithmic saving generalizes [Roy00, GJS99]. It would be nice to prove (or disprove) the theorem without cases (i) and (ii). We can do it under some restrictive hypotheses (which are too special to be discussed here) but do not know of any general type of result.

It is natural to ask whether $|\mathcal{F}_x^{\leqslant A}| = O(|\mathcal{F}_x|^\delta)$ for some $\delta < 1$ for a level aspect family $\mathcal{F}_x$ (whose level $n_x \to \infty$) for an arbitrary reductive group G, cf. Question 6.5 below. This is already challenging for $G = \mathrm{GL}(2)$ (see [Ser97, p. 89]). The above theorem does not achieve this. However, we do provide a nearly optimal answer under a hypothesis on $\{n_x\}_{x \geqslant 1}$ (Corollary 6.8). Let G be a group as in Theorem 1.6 and suppose that $\{n_x\}_{x \geqslant 1}$ is supported on a finite set S of finite primes in the sense that for all but finitely many x, every prime divisor of n_x is in S. Then $|\mathcal{F}_x^{\leqslant A}| = O(1)$. This is actually an easy corollary of Theorem 1.6. Again no condition on infinitesimal characters at ∞ is needed (so it applies to C-algebraic Maass forms when $G = \mathrm{GL}_2$ for instance).

In the following we sketch the proof of Theorems 1.6 and 1.7. Both theorems take local finiteness results as key inputs. The former theorem in the case of GL_n uses the following result.

Proposition 1.8. *Fix $A \geqslant 1$ and a prime p and an integer $n \geqslant 2$. There exists a constant $C = C(A, p, n)$ such that every irreducible smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ with $[\mathbb{Q}(\pi_p) : \mathbb{Q}] \leqslant A$ has conductor $\leqslant C$. (Here $\mathbb{Q}(\pi_p)$ is the field of rationality for π_p defined as in (1.2).)*

For the proof of the proposition we pass to the Galois side via the local Langlands correspondence and examine the representation of the inertia group. Note that a suitable normalization of the local Langlands correspondence preserves the field of rationality. Since the inertia representation must have finite image, it is possible to conclude with some elementary representation theory and ramification theory for local fields. Once the proposition is in place, Theorem 1.6 is an easy consequence of Harish-Chandra's finiteness theorem for automorphic forms.

Theorem 1.7 requires a more arithmetic kind of local finiteness theorem. When G is a quasi-split classical group, we show the following far-reaching generalization of the finiteness of Weil numbers (§ 1.1) to the case for higher rank groups allowing arbitrary ramification at p.

Proposition 1.9 (Corollary 5.7). *Fix $A \geqslant 1$, a prime p, and an irreducible algebraic representation ξ of G. Then the set of irreducible tempered representations π_p of $G(\mathbb{Q}_p)$ with $[\mathbb{Q}(\pi_p) : \mathbb{Q}] \leqslant A$ which may be realized as the p-components of discrete ξ-cohomological automorphic representations π of $G(\mathbb{A})$ is finite.*

A crucial input in the proof is the properties of the Galois representations associated with π concerning weight and integrality, which we justify along the way. The integrality here is the same kind as in Conjecture 1.4(ii). In fact, this consideration led us to formulate the conjecture. To associate Galois representations, the work of Arthur and Mok is applied to transfer π to a suitable general linear group, and the field $\mathbb{Q}(\pi)$ has to be kept track of during the transfer. To this end we check the nontrivial fact that the transfer from G to the general linear group is rational in the sense that it commutes with the $\mathrm{Aut}(\mathbb{C})$-action on the coefficients. It would be of independent interest that a similar argument would show that many other endoscopic transfers are rational (sometimes with respect to the $\mathrm{Aut}(\mathbb{C}/F)$-action for a number field F).

Both cases (i) and (ii) of Theorem 1.7 are deduced from Proposition 1.9 via the theorem (proved earlier by us in [Shi12] and [ST12]) that π_p are equidistributed with respect to the Plancherel measure for $G(\mathbb{Q}_p)$. The equidistribution reduces the proof to showing that the set in Proposition 1.9 has negligible Plancherel measure in the subset of the unitary dual of $G(\mathbb{Q}_p)$ consisting of representations whose levels are at most (the p-part of) n_x. Part (i) results from the fact that the Plancherel measure is atomless when restricted to the unramified unitary dual. The saving by $(\log|\mathcal{F}_x|)^R$ in the denominator comes from the quantitative Plancherel equidistribution theorem [ST12] and a uniform approximation of characteristic functions in the unramified unitary dual by Hecke functions of bounded degree. For part (ii) observe that the condition there implies that the mass of the set in Proposition 1.9, which may *not* be zero since some points may correspond to discrete series, becomes negligible relative to the mass of the level $\leqslant n_x$ part of the unitary dual as $\mathrm{ord}(n_x) \to \infty$.

1.5 Organization

Section 2 introduces basic notions such as C-algebraic, C-arithmetic, and cohomological automorphic representations as well as the field of rationality for local and global representations, and then builds background materials. The key result is that cohomological representations are C-algebraic and strongly C-arithmetic, indicating that a good playground to study field of rationality is the world of cohomological representations. We included many supplementary results which do not play roles in proving main theorems but are interesting in their own right. Section 3 is mainly local and Galois-theoretic. We prove the fundamental proposition that a Weil–Deligne representation with bounded field of rationality has bounded ramification and transfer the result to the automorphic side via the local Langlands correspondence for GL_n. Section 4 is global in nature and draws deep facts from both Galois and automorphic sides. It is shown that the Galois representations associated with (conjugate) self-dual automorphic representations of GL_n are pure and integral. The remainder of § 4 is concerned with the twisted endoscopic transfer and classification theorems for quasi-split classical groups relative to GL_n. This is where Arthur's work is invoked. Section 5 proves key local finiteness results to be used in the proof for main theorems. The basic strategy is to prove something for GL_n and transfer the result to classical groups or vice versa. To play this game the rationality of endoscopic transfer as proved in § 5.2 is essential. The culmination of § 5 is the finiteness theorems in § 5.5. In the last § 6 we prove several results on the field of rationality for families of automorphic representations in level aspect and conclude with remarks on counting elliptic curves and some outlook.

1.6 Notation and convention

We use the following notation and convention:

- $\overline{k}$ denotes an algebraic closure of k for any field k;
- $\mathrm{Res}_{k'/k}$ denotes the Weil restriction of scalars from a finite extension field k' to k;
- Ind and n-ind denote the unnormalized and normalized inductions from parabolic subgroups, respectively;
- F is a number field, $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$, and W_F is the Weil group;
- q_v denotes the cardinality of the residue field and Frob_v is the geometric Frobenius element at v if v is a finite place of F;
- S_∞ is the set of all infinite places of F;
- $\mathbb{A}_F$ is the ring of adèles over F; $\mathbb{A}_F^S$ is the restricted product of F_v for all $v \notin S$; $\mathbb{A}_F^\infty := \mathbb{A}_F^{S_\infty}$;
- G is a connected reductive group over F;
- $\widehat{G}$ is the dual group, ${}^LG$ is the L-group;
- $G(F_v)^\wedge$ is the unitary dual of $G(F_v)$;
- $\mathrm{Irr}(G(F_v))$ denotes the set of isomorphism classes of irreducible smooth representations of $G(F_v)$; write $\mathrm{Irr}^{\mathrm{temp}}(G(F_v))$ (respectively $\mathrm{Irr}^{\mathrm{ur}}(G(F_v))$) for the subset consisting of tempered (respectively unramified, for a choice[2] of a hyperspecial subgroup of $G(F_v)$ if it exists) representations;
- $\rho \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the half sum of all positive roots when a choice is made of a maximal torus T and a Borel subgroup B such that $T \subset B$ (ρ is also viewed as the half sum of all positive coroots on the dual side, cf. § 2.1 below);
- $\mathcal{H}(H, k)$ denotes the k-algebra of locally constant compactly supported functions on H where H is a locally compact totally disconnected group and k is a field, and $\mathcal{H}_U(H, k)$ the sub-k-algebra of bi-U-invariant functions where U is an open compact subgroup of H (for instance $H = G(\mathbb{A}_F^\infty)$ or $H = G(F_v)$ in the notation above);
- given G as above, hyperspecial subgroups U_v^{hs} are fixed at finite places v outside the set S_{ram} of finitely many v such that G is ramified over F_v; we identify $\mathcal{H}(G(\mathbb{A}_F^\infty), k)$ with the restricted tensor product $\otimes'_{v\nmid\infty}\mathcal{H}(G(F_v), k)$ with respect to $\mathcal{H}_{U_v^{\mathrm{hs}}}(G(F_v), k)$ and decompose an irreducible admissible representation π of $G(\mathbb{A}_F^\infty)$ as $\pi = \otimes'_{v\nmid\infty}\pi_v$; we speak of unramified representations at finite places $v \notin S_{\mathrm{ram}}$ with respect to U^{hs};
- $\varphi_v : W_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ (v finite) and $\varphi_v : W_{F_v} \to {}^LG$ (v infinite) are notation for local L-parameters; the associated local L-packets are denoted $LP(\varphi_v)$ (in the cases where the local Langlands correspondence is established);
- fix a field embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}}_l \to \mathbb{C}$ for each prime l once and for all;
- all twisted characters (and intertwining operators for θ defining them) are normalized as in Arthur's book.

2. Field of rationality

The reader may want to compare the contents of our §§ 2.1 and 2.2 with §§ 3.1 and 7 of [BG11].

[2] Such a choice will always be implicit whenever we mention unramified representations in this article.

2.1 C-algebraicity and coefficient fields

Let $\pi = \otimes'_v \pi_v$ be an automorphic representation of $G(\mathbb{A}_F)$. Let S be a finite set of places of F containing S_∞. We recall the definition of C-algebraicity from [BG11, Definition 3.1.2] (generalizing the notion of algebraicity in [Clo90]). For each infinite place v of F, denote by $\varphi_{\pi_v} : W_{F_v} \to {}^L G$ the associated parameter via the local Langlands correspondence [Lan88].

Definition 2.1. For $v|\infty$, π_v is *C-algebraic* if there exists a maximal torus $\widehat{T}$ of $\widehat{G}$ satisfying $\varphi_{\pi_v}(W_{\mathbb{C}}) \subset \widehat{T} \times W_{\mathbb{C}}$ with the property that $\varphi_{\pi_v}|_{W_{\mathbb{C}}} : W_{\mathbb{C}} \to \widehat{T}$ (via any $\mathbb{R}$-embedding $\sigma : F_v \hookrightarrow \mathbb{C}$, after projecting down to $\widehat{T}$) belongs to $\rho + X_*(\widehat{T})$, where ρ is the half sum of all positive coroots in $\widehat{T}$ with respect to a Borel subgroup $\widehat{B}$ containing $\widehat{T}$. (The latter property is independent of the choice of σ, $\widehat{T}$, and $\widehat{B}$. See [BG11, 2.3].) We say that π_v is *regular* if $\varphi_{\pi_v}|_{W_{\mathbb{C}}}$ is not invariant under any nontrivial element of the Weyl group for $\widehat{T}$ in $\widehat{G}$. If π_v is C-algebraic (respectively regular) for every infinite place v then π is said to be *C-algebraic* (respectively *regular*).

We remark that when $G = \mathrm{GL}_n$, our notion of π being algebraic (respectively regular) coincides with that in [Clo90]. For the next definition we introduce a twist of a complex representation. For $\tau \in \mathrm{Aut}(\mathbb{C})$ and a complex representation (Π, V) of a group Γ, denote by Π^τ the representation of Γ on $V \otimes_{\mathbb{C},\tau^{-1}} \mathbb{C}$ via $\Pi \otimes 1$.

Definition 2.2. The *field of rationality* $\mathbb{Q}(\pi^S)$ is the fixed field of $\mathbb{C}$ under the group $\{\tau \in \mathrm{Aut}(\mathbb{C}) : (\pi^S)^\tau \simeq \pi^S\}$. If $S = S_\infty$, simply write $\mathbb{Q}(\pi)$ for $\mathbb{Q}(\pi^{S_\infty})$. For a finite place v of F, $\mathbb{Q}(\pi_v)$ is defined to be the fixed field under the group $\{\tau \in \mathrm{Aut}(\mathbb{C}) : \pi_v^\tau \simeq \pi_v\}$.

An easy observation is that $\mathbb{Q}(\pi)$ is the composite field of $\mathbb{Q}(\pi_v)$ for all finite v (as a subfield of $\mathbb{C}$).

Remark 2.3. Here is another possible notion of rationality, which will not be used in this paper. We say that π is *defined over* a subfield E of $\mathbb{C}$ if there exists a smooth $E[G(\mathbb{A}_F^\infty)]$-module π_E^∞ such that $\pi_E^\infty \otimes_E \mathbb{C} \simeq \pi^\infty$. Similarly π_v is said to be defined over E for a finite place v if there exists a smooth $E[G(F_v)]$-module $\pi_{v,E}$ such that $\pi_{v,E} \otimes_E \mathbb{C} \simeq \pi_v$. If π (respectively π_v) is defined over E, then clearly $\mathbb{Q}(\pi)$ (respectively $\mathbb{Q}(\pi_v)$) contains E. A natural question is whether π (respectively π_v) can be defined over $\mathbb{Q}(\pi)$ (respectively $\mathbb{Q}(\pi_v)$) itself. If π_v is unramified, it is not hard to see that π_v is defined over E (independently of the choice of a hyperspecial subgroup of $G(F_v)$) if and only if $E \supset \mathbb{Q}(\pi_v)$, cf. [BG11, Lemma 2.2.3, Corollary 2.2.4]. The authors do not know whether the analogue holds for general generic π_v or π^∞. In the case of $G = \mathrm{GL}_n$, this has been shown in [Clo90] using the theory of new vectors.

Remark 2.4. Let v be a finite place of F where π_v is unramified. It is in general false that the Satake parameters of π_v are defined over $\mathbb{Q}(\pi_v)$ (let alone $\mathbb{Q}(\pi)$) in the sense of [BG11, Definition 2.2.2] due to an issue with the square root of q_v.

Definition 2.5. For a finite v, we say π_v is C-arithmetic if $\mathbb{Q}(\pi_v)$ is finite over $\mathbb{Q}$. An automorphic representation π is *C-arithmetic* if $\mathbb{Q}(\pi^S)$ is finite over $\mathbb{Q}$ for some finite set S containing S_∞. It is *strongly C-arithmetic* if $\mathbb{Q}(\pi)$ is finite over $\mathbb{Q}$.

Remark 2.6. Our C-arithmeticity is equivalent to that of [BG11]. It is reasonable to believe that π is C-arithmetic if and only if it is strongly C-arithmetic, but the only if part does not seem easy to prove directly. At least when G is a torus it can be verified that C-arithmeticity is equivalent to strong C-arithmeticity. Indeed the only if part is true if G is a split torus by strong approximation. If G is a general torus the proof is reduced to the split case via a finite extension F'/F splitting G by employing the fact (see the proof of [BG11, Theorem 4.1.9]) that $G(F)$ and

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

the image of $G(\mathbb{A}_{F'}^\infty)$ under the norm map together generate an open and closed subgroup of $G(\mathbb{A}_F^\infty)$ of finite index.

Remark 2.7. Even if π_v is C-arithmetic at every finite v, it may happen that π is not C-arithmetic. For instance, when $G = \mathrm{GL}_1$ over $\mathbb{Q}$ and $\pi = |\cdot|^{1/2}$, we have $\mathbb{Q}(\pi_p) = \mathbb{Q}(p^{1/2})$ for each prime p so π_p is C-arithmetic. However, $\mathbb{Q}(\pi^S)$ is an infinite algebraic extension of $\mathbb{Q}$. Note that $|\cdot|^{1/2}$ is not C-algebraic.

In general, there is no reason to expect that $\mathbb{Q}(\pi)$ is finite or algebraic over $\mathbb{Q}$. In this optic the significance of C-algebraicity stems from Conjecture 2.8 below. An expectedly equivalent conjecture was formulated in [BG11, Conjecture 3.1.6], where they put C-arithmetic in place of strongly C-arithmetic. When G is a torus, the two versions of the conjecture are indeed equivalent (Remark 2.6), so our conjecture is known to be true by [BG11, Theorem 4.1.9] based on work of Weil and Waldschmidt.

Conjecture 2.8. The automorphic representation π is C-algebraic if and only if it is strongly C-arithmetic.

We remark that there are other reasons why C-algebraic automorphic representations stand out. One reason is that C-algebraicity is a natural necessary condition in the cuspidal case (and not too far from being a sufficient condition) to contribute to cohomology, cf. Lemma 2.14 below. Another reason is that l-adic Galois representations are expected to be associated with C-algebraic representations, cf. [BG11, Conjecture 5.3.4]. (In a simpler way Galois representations should also be attached to L-algebraic representations, which differ from C-algebraic ones by 'twisting'. See [BG11, Conjectures 3.2.1 and 3.2.2].)

C-algebraicity and C-arithmeticity are preserved under *unnormalized* parabolic induction. (Compare with [BG11, Lemma 7.1.1] and the paragraph above it.)

Lemma 2.9. *Let M be a Levi subgroup of an F-rational parabolic subgroup P of G. Let Π_M be an automorphic representation of $M(\mathbb{A}_F)$. Suppose that Π is an irreducible subquotient of the unnormalized induction $\mathrm{Ind}_P^G(\Pi_M)$. Then Π_M is C-algebraic if and only if Π is C-algebraic. If Π_M is C-arithmetic (respectively strongly C-arithmetic), then so is Π.*

Remark 2.10. The lemma is in fact purely local and the same argument proves the analogue for $M(F \otimes_{\mathbb{Q}} \mathbb{R})$-representations. For normalized induction, one can prove similar statements with L- in place of C-.

Proof. We may assume $F = \mathbb{Q}$ by reducing the general case via restriction of scalars. Let T be a maximal torus of M over $\mathbb{C}$ and B a Borel subgroup of G over $\mathbb{C}$ containing T. Put $B_M := M \cap B$. Then $\rho, \rho_M \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ are defined. Let $\chi_{\Pi_{M,\infty}}$ (respectively χ_{Π_∞}) denote the character of $X_*(\widehat{T}) = X^*(T)$ associated to $\varphi_{\Pi_{M,\infty}}$ (respectively φ_{Π_∞}) as in Definition 2.1 well-defined up to $W(M,T)$-conjugacy (respectively $W(G,T)$-conjugacy). Let $\lambda_{\Pi_{M,\infty}}$ (respectively λ_{Π_∞}) denote the infinitesimal character of $\Pi_{M,\infty}$ (respectively Π_∞).

The condition of the lemma tells us that λ_{Π_∞} and $\lambda_{\Pi_{M,\infty}} + (\rho - \rho_M)$ are in the same $W(G, T)$-orbit in $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$. On the other hand, $\lambda_{\Pi_{M,\infty}}$ and $\chi_{\Pi_{M,\infty}}$ are in the same $W(M,T)$-orbit and similarly λ_{Π_∞} and χ_{Π_∞} are in the same $W(G,T)$-orbit [Vog93, Proposition 7.4]. Therefore, if Π_M is C-algebraic then so is Π.

We check that Π is strongly C-arithmetic if Π_M is strongly C-arithmetic. Let S be the finite set of places (including S_∞) outside which Π_M is unramified. The assumption tells us that Π_v is a subquotient of $\mathrm{Ind}_P^G(\Pi_{M,v})$ at every finite place v. Hence, Π_v^σ is a subquotient of $\mathrm{Ind}_P^G(\Pi_{M,v}^\sigma)$ at

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

every v for every $\sigma \in \mathrm{Aut}(\mathbb{C})$. (The latter implication fails if normalized induction was used and if σ does not fix $q_v^{1/2}$.) For $v \notin S$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi_M))$ we see that Π_v and Π_v^σ are isomorphic as both of them are the unique unramified subquotient of $\mathrm{Ind}_P^G(\Pi_{M,v})$. For finite $v \in S$, Π_v is C-arithmetic since $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi_M))$ permutes the finitely many irreducible subquotients of $\mathrm{Ind}_P^G(\Pi_{M,v}^\sigma)$. Therefore, $\mathbb{Q}(\Pi)$ is contained in the finite field extension of $\mathbb{Q}(\Pi_M)$ generated by $\mathbb{Q}(\Pi_v)$ for $v \in S$, hence Π is strongly C-arithmetic.

The above proof also shows that if Π_M is C-arithmetic, then Π is C-arithmetic. □

2.2 Rationality for cohomological representations

Temporarily let G be a connected reductive group over $\mathbb{Q}$. Let π be an automorphic representation of $G(\mathbb{A})$. Let K_∞ be a subgroup of $G(\mathbb{R})$ whose image in $G^{\mathrm{ad}}(\mathbb{R})$ is a maximal compact subgroup. Let K_∞^0 be the neutral component of K_∞ with respect to the real topology. Let Q be a parabolic subgroup of $G(\mathbb{C})$ with Levi component $K_{\infty,\mathbb{C}}$. Put $\mathfrak{g} := \mathrm{Lie}\, G(\mathbb{C})$ and $\mathfrak{q} := \mathrm{Lie}\, Q(\mathbb{C})$.

Definition 2.11. We say that π is *cohomological* (respectively *$\overline{\partial}$-cohomological*) if $H^i(\mathfrak{g}, K_\infty^0, \pi_\infty \otimes \xi) \neq 0$ (respectively $H^i(\mathfrak{q}, K_\infty^0, \pi_\infty \otimes \xi) \neq 0$) for some $i \geqslant 0$ and some irreducible algebraic representation ξ of $G(\mathbb{C})$ (respectively $K_{\infty,\mathbb{C}}$). In this case π is said to be ξ-cohomological (respectively ξ-$\overline{\partial}$-cohomological).

Lemma 2.12. *If $G = \mathrm{GL}_n$, then every cuspidal regular C-algebraic automorphic representation π of $G(\mathbb{A}_F)$ is cohomological.*

Proof. Follows from [Clo90, Lemma 3.14]. □

Remark 2.13. If π_∞ is an *arbitrary* regular C-algebraic representation of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, or a product thereof, then there is no reason for π_∞ to have non-vanishing cohomology as in Definition 2.11. What makes the above lemma work is the condition that π_∞ is (essentially) tempered, which is implied by the cuspidality of π, cf. [Clo90, Lemma 4.9].

From now on, let F be a number field and G a connected reductive group over F. By applying the above definition to $\mathrm{Res}_{F/\mathbb{Q}} G$ we define K_∞, Q, $\mathfrak{g}$, $\mathfrak{q}$, and make sense of ($\overline{\partial}$-)cohomological representations. In light of the above remark, a sensible generalization of Lemma 2.12 would be the following assertion: for any connected reductive group G over F, every cuspidal regular C-algebraic automorphic representation of $G(\mathbb{A}_F)$ is cohomological if its infinite component is tempered. We believe that the assertion is true but were not able to verify it. In the converse direction we have the following result.

Lemma 2.14. *Any cohomological automorphic representation π of $G(\mathbb{A}_F)$ is C-algebraic.*

Proof. We may assume $F = \mathbb{Q}$. Let T be a maximal torus over $\mathbb{C}$ and B a Borel subgroup of G over $\mathbb{C}$ containing T. Let $\lambda_{\xi^\vee} \in X^*(T)$ be the highest weight vector for $\xi^\vee$ with respect to (B, T) where ξ is as above. Let $\chi_{\pi_\infty} \in X_*(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{C} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ be the character determined by $\varphi_{\pi_\infty}|_{W_\mathbb{C}}$ as in Definition 2.1. Then χ_{π_∞} is well-defined up to $W(G,T)$-conjugacy. If π is ξ-cohomological, then the infinitesimal character of π_∞ is the same as that of $\xi^\vee$, namely $\lambda_{\xi^\vee} + \rho$. Hence, χ_{π_∞} and $\lambda_{\xi^\vee} + \rho$ are in the same $W(G,T)$-orbit [Vog93, Proposition 7.4]. We conclude that $\chi_{\pi_\infty} - \rho \in X^*(T)$ independently of the choices so far and that π_∞ is C-algebraic. □

Roughly speaking, cohomological (cuspidal) automorphic representations are important in that they are realized in the Betti cohomology (or étale cohomology via comparison theorem) of locally symmetric quotients associated with G. This plays a fundamental role in Clozel's

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

work for $G = \mathrm{GL}_n$, cf. Remark 2.16 below. In work of Blasius, Harris and Ramakrishnan (cf. Proposition 2.19 below) they prove C-arithmeticity by realizing cuspidal automorphic representations in the coherent cohomology of Shimura varieties, which is possible for $\overline{\partial}$-cohomological representations.

We would like to show C-arithmeticity for a large class of cohomological representations by realizing them in the Betti cohomology of locally symmetric quotients with coefficient sheaves defined over number fields. This must be well-known to experts, the idea being similar to [Wal85] and [Clo90], but we provide some details as there does not seem to be a handy reference for the general case.

For any sufficiently small open compact subgroup $U \subset G(\mathbb{A}_F^\infty)$, consider the manifold

$$S_U(G) := G(F)\backslash G(\mathbb{A}_F)/UK_\infty^0$$

with finitely many connected components. Let ξ be an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}G$ over $\mathbb{C}$ and denote by $\mathcal{L}_\xi$ the associated local system of $\mathbb{C}$-vector spaces on $S_U(G)$. (By abuse of notation we omit the reference to U in $\mathcal{L}_\xi$.) Such a ξ admits a model ξ_E over a number field E (so that $\xi_E \otimes_E \mathbb{C} \simeq \xi$) and one can use the highest weight theory to show that $\mathcal{L}_\xi$ also admits a model $\mathcal{L}_{\xi,E}$, a local system of E-vector spaces. For $i \geqslant 0$ define

$$H^i(S(G), \mathcal{L}_\xi) := \varinjlim_{U} H^i(S_U(G), \mathcal{L}_\xi) \tag{2.1}$$

and similarly $H^i(S(G), \mathcal{L}_{\xi,E})$. The usual Hecke action equips $H^i(S(G), \mathcal{L}_\xi)$ (respectively $H^i(S(G), \mathcal{L}_{\xi,E})$) with the structure of admissible $\mathbb{C}[G(\mathbb{A}^\infty)]$-module (respectively $E[G(\mathbb{A}^\infty)]$-module), where admissibility corresponds to the fact that $H^i(S_U(G), \mathcal{L}_\xi) = H^i(S(G), \mathcal{L}_\xi)^U$ is finite dimensional.

Much work has been done to decompose $H^i(S(G), \mathcal{L}_\xi)$ by means of automorphic representations. When $S_U(G)$ are compact, Matsushima's formula does the job. Results in the general case are due to Franke, Harder, Li, Schwermer, and others. This enables us to show C-arithmeticity for cuspidal representations.

Proposition 2.15. *Let π be a cuspidal ξ-cohomological automorphic representation of $G(\mathbb{A}_F)$. Then:*

(i) *π^∞ is a $G(\mathbb{A}_F^\infty)$-module direct summand of $H^i(S(G), \mathcal{L}_\xi)$ for some $i \geqslant 0$;*
(ii) *π is strongly C-arithmetic.*

Remark 2.16. Clozel has shown this for general linear groups [Clo90, Theorem 3.13 and Lemmas 3.14 and 3.15]. We are adapting his ideas to the case of arbitrary reductive groups. (See also the last paragraph of [BG11, § 7] for the case of trivial coefficients.)

Remark 2.17. When $G = \mathrm{GL}_n$ we know moreover that $\mathbb{Q}(\pi)$ is a totally real or CM field, cf. [Pat12, Corollary 6.2.3]. The argument requires to know the subtle point that twists of π^∞ by $\mathrm{Aut}(\mathbb{C})$ are finite parts of automorphic representations of $G(\mathbb{A}_F)$. As this is not known in general, it seems difficult to check whether $\mathbb{Q}(\pi)$ is a totally real or CM field for an arbitrary reductive group. However, see Proposition 2.19(ii) below.

Proof. Part (i) follows from the description of the cuspidal part of $H^i(S(G), \mathcal{L}_\xi)$ via Lie algebra cohomology (see [Sch10, (13.6)] and [FS98]). Note that the cuspidal part is a direct summand, cf. [Sch10, p. 242]. Part (ii) can be shown by arguing as in the proof of [Clo90, Proposition 3.16]. The argument is sketched here for the convenience of the reader.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

Let $U=\prod_{v\nmid\infty} U_v \subset G(\mathbb{A}_F^\infty)$ be a sufficiently small open compact subgroup such that $(\pi^\infty)^U \neq 0$. Then $(\pi^\infty)^U$ is a direct summand of $H^i(S_U(G),\mathcal{L}_\xi)$ and moreover irreducible as a $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\mathbb{C})$-module. (This follows from the irreducibility criterion of [Fla79, p. 179].) Since the field of definition is the same for π^∞ as a $G(\mathbb{A}_F^\infty)$-module and for $(\pi^\infty)^U$ as a $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\mathbb{C})$-module, it is enough to show that the isomorphism class of $(\pi^\infty)^U$ is fixed under a finite index subgroup of $\mathrm{Aut}(\mathbb{C})$.

We start by finding a model of $(\pi^\infty)^U$ on a $\overline{\mathbb{Q}}$-vector space. Burnside's theorem implies that irreducible $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\mathbb{C})$-module subquotients of $H^i(S_U(G),\mathcal{L}_\xi)$ and those of the $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\overline{\mathbb{Q}})$-module $H^i(S_U(G),\mathcal{L}_{\xi,E}\otimes_E\overline{\mathbb{Q}})$ correspond bijectively.[3] In particular, there is an irreducible $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\overline{\mathbb{Q}})$-module subquotient W of $H^i(S_U(G),\mathcal{L}_{\xi,E}\otimes_E\overline{\mathbb{Q}})$ such that $W\otimes_{\overline{\mathbb{Q}}}\mathbb{C}\simeq(\pi^\infty)^U$. Since $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ induces a σ-linear self-automorphism of $H^i(S_U(G),\mathcal{L}_{\xi,E}\otimes_E\overline{\mathbb{Q}})$ as a $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\overline{\mathbb{Q}})$-module, the induced action permutes the irreducible subquotients of $H^i(S_U(G),\mathcal{L}_{\xi,E}\otimes_E\overline{\mathbb{Q}})$ (the point being that $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\overline{\mathbb{Q}})$ has a natural $\mathbb{Q}$-structure). We see from the finite-dimensionality of the latter space that the isomorphism class of W is fixed by a finite index subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ as desired. □

Corollary 2.18. *Let M be a Levi subgroup of an F-rational parabolic subgroup of G. Any automorphic representation of $G(\mathbb{A}_F)$ appearing as a subquotient of an unnormalized parabolic induction of a cuspidal cohomological automorphic representation of $M(\mathbb{A}_F)$ is C-algebraic and strongly C-arithmetic.*

Proof. Immediate from Lemmas 2.9 and 2.14 and Proposition 2.15. □

In the rest of this subsection we briefly recall some results of Blasius, Harris, and Ramakrishnan for the sake of completeness, even though their results will not be used in this paper. Under a restrictive hypothesis (cf. [BHR94, § 0.1]), namely that $\mathrm{Res}_{F/\mathbb{Q}}G$ is of hermitian symmetric type so that $G(F\otimes_{\mathbb{Q}}\mathbb{R})/K_\infty$ admits a $G(F\otimes_{\mathbb{Q}}\mathbb{R})$-invariant complex structure, the three authors have shown the following result.

Proposition 2.19. *Keep the hypothesis in the above paragraph. Let π be any automorphic representation of $G(\mathbb{A}_F)$ such that π_∞ is a nondegenerate limit of discrete series or a discrete series representation of $G(F\otimes_{\mathbb{Q}}\mathbb{R})$ whose restriction to the maximal $\mathbb{R}$-split torus of $(\mathrm{Res}_{F/\mathbb{Q}}G)(\mathbb{R})$ is algebraic. Then:*

(i) *any such π is $\overline{\partial}$-cohomological, C-algebraic; and*
(ii) *if π is moreover cuspidal then $\mathbb{Q}(\pi)$ is either a totally real or a CM field (in particular π is strongly C-arithmetic).*

Remark 2.20. One can extend part (ii) beyond the cuspidal case by applying Lemma 2.9 as it was done in Corollary 2.18.

Proof. This is Theorems 3.2.1 and 4.4.1 of [BHR94] except for the C-algebraicity of π, which is easy to deduce from the description of the infinitesimal character of π_∞ in [BHR94, Theorem 3.2.1] by an argument as in the proof of Lemma 2.14. Note that a subfield of a CM field is either totally real or CM. □

[3] Consider the Jordan–Hölder quotients $M_1,\dots,M_k$ of $H^i(S_U(G),\mathcal{L}_{\xi,E}\otimes_E\overline{\mathbb{Q}})$. By Burnside's theorem, the $\overline{\mathbb{Q}}$-algebra morphism from $\mathcal{H}_U(G(\mathbb{A}_F^\infty),\overline{\mathbb{Q}})$ to $\mathrm{End}_{\overline{\mathbb{Q}}}(M_j)$ is onto. So the Jordan–Hölder quotients remain irreducible after $\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

2.3 Satake parameters under functoriality

Let H and G be connected reductive groups over a number field F. We form their L-groups using the full Galois group over F rather than a finite Galois group or the Weil group. (Later we use the Weil group in the case of even orthogonal groups. In that case the material of this subsection can still be adapted. See § 4.2.) Let $\eta : {}^L H \to {}^L G$ be an L-morphism. Let $(\widehat{B}_H, \widehat{T}_H)$ (respectively $(\widehat{B}, \widehat{T})$) be a pair of a Borel subgroup of $\widehat{H}$ (respectively $\widehat{G}$) and a maximal torus contained in it. We may choose $(\widehat{B}, \widehat{T})$ such that $\eta(\widehat{T}_H) \subset \widehat{T}$ (and $\eta(\widehat{B}_H) \subset \widehat{B}$ but the latter is unnecessary for us). These data determine $\rho_H \in \frac{1}{2}X_*(\widehat{T}_H)$ and $\rho \in \frac{1}{2}X_*(\widehat{T})$ as the half sums of all positive coroots in $\widehat{T}_H$ and $\widehat{T}$, respectively. Moreover, η induces $\eta_* : X_*(\widehat{T}_H) \to X_*(\widehat{T})$.

Definition 2.21. An L-morphism $\eta : {}^L H \to {}^L G$ is said to be *C-preserving* if $\rho - \eta_*(\rho_H)$ at each $v|\infty$ belongs to $X_*(\widehat{T})$ (rather than just $\frac{1}{2}X_*(\widehat{T})$).

In view of Definition 2.1, a C-preserving L-morphism carries L-packets of C-algebraic representations to L-packets of C-algebraic representations at infinite places. The C-preserving property does not depend on the choice of maximal tori and Borel subgroups. Indeed one can go between different maximal tori in $\widehat{H}$ (respectively $\widehat{G}$) by conjugation. Moreover, if $\widehat{T}_H$ is fixed, another choice of $\widehat{B}_H$ changes ρ_H by a Weyl group element w_H for $\widehat{H}$, but clearly $w_H\rho_H - \rho_H \in X_*(\widehat{T}_H)$ so $\rho - \eta_*(\rho_H)$ is shifted by an element of $X_*(\widehat{T})$ (rather than just $\frac{1}{2}X_*(\widehat{T})$). A similar argument shows the independence of the choice of $\widehat{B}$ as well.

The aim of this subsection is to show that for a C-preserving L-morphism, the transfer of unramified representations is compatible with twisting by field automorphisms of $\mathbb{C}$. We begin with some preparation. Let S be a finite set of places of F containing S_∞ such that H, G and η are unramified whenever $v \notin S$. From now on assume $v \notin S$. Let A_v be a maximal F_v-split torus of G, and T_v be the centralizer of A_v in G over F_v. Let B_v be a Borel subgroup of G containing T_v. Define $\rho_v \in \frac{1}{2}X_*(A_v)$ to be the half sum of all F_v-rational B_v-positive roots relative to A_v. Write $q_v^{1/2}$ for the positive square root of q_v. Denote by $\mathrm{sgn}_{\sigma,\rho_v} : T_v(F_v) \to \{\pm 1\}$ a character defined via the following composite map

$$T_v(F_v) \to T_v(F_v)/T_v(\mathcal{O}_v) \simeq X_*(A_v) \to \{\pm 1\}$$

where $\lambda \in X_*(A_v)$ is sent to $\lambda(\varpi_v) \in T_v(F_v)/T_v(\mathcal{O}_v)$ under the isomorphism in the middle and to $(\sigma(q_v^{1/2})/q_v^{1/2})^{\langle \lambda, 2\rho_v\rangle} \in \{\pm 1\}$ under the last map. (In particular, $\mathrm{sgn}_{\sigma,\rho_v}(\lambda) = 1$ if either $q_v^{1/2} \in \mathbb{Q}$ or $\langle \lambda, \rho_v\rangle \in \mathbb{Z}$.) Likewise $A_{H,v}$, $T_{H,v}$, $B_{H,v}$, $\rho_{H,v}$ and $\mathrm{sgn}_{\sigma,\rho_{H,v}}$ are defined for H. Write $\delta_{B_v}^{1/2} : T_v(F_v) \to \mathbb{R}_{>0}^\times$ for the modulus character, which factors through the character $\lambda \mapsto (q_v^{1/2})^{\langle \lambda, 2\rho_v\rangle}$ from $X_*(A_v)$ to $\mathbb{R}_{>0}^\times$.

Lemma 2.22. *Suppose $v \notin S$ and let $\chi_v : T_v(F_v) \to \mathbb{C}^\times$ be a continuous character. If $\pi_v \in \mathrm{Irr}^{\mathrm{ur}}(G(F_v))$ is a subquotient of $\text{n-ind}_{B_v(F_v)}^{G(F_v)}(\chi_v)$, then for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, π_v^σ is a subquotient of $\text{n-ind}_{B_v(F_v)}^{G(F_v)}(\chi_v^\sigma \otimes \mathrm{sgn}_{\sigma,\rho_v})$. The exact analogue holds true for H.*

Proof. Recall that the unnormalized parabolic induction commutes with σ-twisting, cf. Lemma 2.9. So π_v^σ is an unramified subquotient of the following representation (all inductions below are from $B_v(F_v)$ to $G(F_v)$):

$$\text{n-ind}(\chi_v)^\sigma = \mathrm{Ind}(\chi_v \otimes \delta_{B_v}^{1/2})^\sigma = \mathrm{Ind}(\chi_v^\sigma \otimes (\delta_{B_v}^{1/2})^\sigma) = \text{n-ind}(\chi_v^\sigma \otimes (\delta_{B_v}^{1/2})^\sigma/\delta_{B_v}^{1/2})).$$

By definition $(\delta_{B_v}^{1/2})^\sigma/\delta_{B_v}^{1/2} = \mathrm{sgn}_{\sigma,\rho_v}$. Since a principal series representation has a unique unramified subquotient, the first part of the lemma follows. The argument for H is the same. □

We have that η is unramified at $v \notin S$, so it comes from a map on Fr_v-cosets $\widehat{H} \rtimes \mathrm{Fr}_v \to \widehat{G} \rtimes \mathrm{Fr}_v$, again denoted by η. The Satake isomorphism provides a canonical bijection between the set of $\widehat{G}$-conjugacy classes in $\widehat{G} \rtimes \mathrm{Fr}_v$ (respectively ($\widehat{H}$-conjugacy classes in $\widehat{H} \rtimes \mathrm{Fr}_v$) with $\mathrm{Irr}^{\mathrm{ur}}(G(F_v))$ (respectively $\mathrm{Irr}^{\mathrm{ur}}(H(F_v))$). Write

$$\eta_* : \mathrm{Irr}^{\mathrm{ur}}(H(F_v)) \to \mathrm{Irr}^{\mathrm{ur}}(G(F_v))$$

for the map induced by η.

Lemma 2.23. *Let $v \notin S$ and suppose that $\eta : {}^L H \to {}^L G$ is an L-morphism with finite kernel. (So η is unramified.) Then there exists $N \in \mathbb{Z}_{>0}$ such that every fiber of η_* has cardinality at most N.*

Remark 2.24. The N in the lemma can be chosen independently of v. For this observe that the order of the Weyl group in G is clearly bounded independently of v and that the size of the kernel of $\eta_{T,*}$ is also uniformly bounded since there are only finitely many Fr_v-actions on $\widehat{T}_{H,v}$ and $\widehat{T}_v$ as v varies (up to Weyl group actions).

Proof. Obviously the proof is reduced to the case where η is injective, which will be assumed throughout. Let ${}^L B_{H,v}$ be a Borel subgroup of ${}^L H$ relative to the base field F_v (see [Bor79, § 3] for this and other related notions in the proof). Then $\widehat{B}_{H,v} := {}^L B_{H,v} \cap \widehat{H}$ is a Borel subgroup of $\widehat{H}$. Since $\eta(\widehat{B}_{H,v})$ is a closed solvable subgroup of $\widehat{G}$, it is contained in some Borel subgroup $\widehat{B}_v$ of $\widehat{G}$. Then the normalizer ${}^L B_v$ of $\widehat{B}_v$ in ${}^L G$ is a Borel subgroup of ${}^L G$. Let $i_H : {}^L B_{H,v} \hookrightarrow {}^L H$ and $i : {}^L B_v \hookrightarrow {}^L G$ denote the inclusions. Write $\widehat{T}_{H,v}$ and $\widehat{T}_v$ for the maximal tori in $\widehat{B}_{H,v}$ and $\widehat{B}_v$. The normalizer ${}^L T_{H,v}$ of $\widehat{T}_{H,v}$ in ${}^L B_{H,v}$ is a Levi subgroup of ${}^L B_{H,v}$, and similarly we have a Levi subgroup ${}^L T_v$ of ${}^L B_v$. We can identify ${}^L T_{H,v}$ and ${}^L T_v$ with the L-groups for minimal Levi subgroups $T_{H,v}$ and T_v of H and G over F_v, respectively. Clearly we have $\eta({}^L B_{H,v}) \subset {}^L B_v$ and so $\eta({}^L T_{H,v}) \subset {}^L T_v$. Denote the induced map ${}^L T_{H,v} \to {}^L T$ by η_T. Note that i, i_H and η_T are unramified. We have a commutative diagram as below on the left, which induces the following commutative diagram on the unramified spectra.

$$\begin{array}{ccc} {}^L T_{H,v} & \xrightarrow{\eta_T} & {}^L T_v \\ \downarrow{\scriptstyle i_H} & & \downarrow{\scriptstyle i} \\ {}^L H & \xrightarrow{\eta} & {}^L G \end{array} \qquad \begin{array}{ccc} \mathrm{Irr}^{\mathrm{ur}}(T_{H,v}(F_v)) & \xrightarrow{\eta_{T,*}} & \mathrm{Irr}^{\mathrm{ur}}(T_v(F_v)) \\ \downarrow{\scriptstyle i_{H,*}} & & \downarrow{\scriptstyle i_*} \\ \mathrm{Irr}^{\mathrm{ur}}(H(F_v)) & \xrightarrow{\eta_*} & \mathrm{Irr}^{\mathrm{ur}}(G(F_v)) \end{array} \tag{2.2}$$

We know [Bor79, § 10.4] how to describe i_* and $i_{H,*}$ using parabolic induction: $i_*(\chi_v)$ is the unique unramified subquotient of n-ind(χ_v) and the analogue is true for $i_{H,*}$. According to the well-known classification of unramified representations, we know first that i_* and $i_{H,*}$ are surjective and second that the fiber of i_* (respectively $i_{H,*}$) has cardinality at most the order of the Weyl group for T_v in G (respectively for $T_{H,v}$ in H). This order can be bounded uniformly in v. On the other hand $\eta_{T,*}$ has finite fibers. Indeed [Bor79, § 9.4] identifies $\eta_{T,*}$ with a group homomorphism

$$\widehat{T}_{H,v}/(\mathrm{Fr}_v - 1)\widehat{T}_{H,v} \to \widehat{T}_v/(\mathrm{Fr}_v - 1)\widehat{T}_v$$

(Fr_v denoting the geometric Frobenius action), and the above map has finite kernel [Bor79, § 6.3, (2)]. All in all, the fibers of η_* are finite. □

Lemma 2.25. *Suppose that $\eta : {}^L H \to {}^L G$ is C-preserving. Let $v \notin S$. For each $\pi_{H,v} \in \mathrm{Irr}^{\mathrm{ur}}(H(F_v))$:*

(i) $(\eta_* \pi_{H,v})^\sigma = \eta_*(\pi^\sigma_{H,v})$;

(ii) $\mathbb{Q}(\eta_*\pi_{H,v}) \subset \mathbb{Q}(\pi_{H,v})$;

(iii) *if η has finite kernel and N is as in Lemma 2.23, then* $[\mathbb{Q}(\pi_{H,v}) : \mathbb{Q}(\eta_*\pi_{H,v})] \leqslant N!$.

Proof. Let us prove part (i). Adopt the setting in the proof of the last lemma. The first observation is that when $H = T_H$ and $G = T$ are tori, part (i) follows from the fact that η_* is naturally defined over $\mathbb{Q}$ since an algebraic map ${}^LT_H \to {}^LT$ corresponds to a Fr_v-equivariant map $X_*(\widehat{T}_H) = X^*(T_H) \to X_*(\widehat{T}) = X^*(T)$. Now consider the general case. For simplicity of notation only in this proof, we use n-ind to mean the unique unramified subquotient of the normalized induction. Now by surjectivity of $i_{H,*}$ write $\pi_{H,v} = i_{H,*}(\chi_{H,v}) = \text{n-ind}^H_{B_{H,v}}(\chi_{H,v})$ for a smooth character $\chi_{H,v} : T_{H,v}(F_v) \to \mathbb{C}^\times$. Put $\chi_v := \eta_{T,*}(\chi_{H,v})$. From the case of tori we know that

$$\eta_{T,*}(\chi^\sigma_{H,v}) = \chi^\sigma_v.$$

Using Lemma 2.22 and the commutativity of (2.2) we compute

$$(\eta_*\pi_{H,v})^\sigma = (\eta_* i_{H,*}\chi_{H,v})^\sigma = (i_*\chi_v)^\sigma = \text{n-ind}(\chi_v)^\sigma = \text{n-ind}(\chi^\sigma_v \otimes \mathrm{sgn}_{\sigma,\rho_v}). \tag{2.3}$$

Similarly, noting in addition that $\eta_{T,*}$ is a homomorphism,

$$\begin{aligned}\eta_*(\pi^\sigma_{H,v}) &= \eta_*(i_{H,*}(\chi_{H,v})^\sigma) = \eta_*(i_{H,*}(\chi^\sigma_{H,v} \otimes \mathrm{sgn}_{\sigma,\rho_{H,v}})\\ &= \text{n-ind}(\eta_{T,*}(\chi^\sigma_{H,v} \otimes \mathrm{sgn}_{\sigma,\rho_{H,v}})) = \text{n-ind}(\eta_{T,*}(\chi^\sigma_{H,v}) \otimes \eta_{T,*}(\mathrm{sgn}_{\sigma,\rho_{H,v}}))\\ &= \text{n-ind}(\chi^\sigma_v \otimes \mathrm{sgn}_{\sigma,\eta_*(\rho_{H,v})}).\end{aligned} \tag{2.4}$$

Since η is C-preserving, $\mathrm{sgn}_{\sigma,\rho_v} = \mathrm{sgn}_{\sigma,\eta_*(\rho_{H,v})}$ and thus the proof of part (i) is complete.

Part (ii) is clear from part (i). To verify part (iii), put $\pi_v := \eta_*\pi_{H,v}$. By part (i), $\pi^\sigma_v \in \eta^{-1}_*(\pi_v)$ for every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$. This yields a homomorphism from $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$ to the permutation group on $\eta^{-1}_*(\pi_v)$. Since $|\eta^{-1}_*(\pi_v)| \leqslant N$, the kernel has finite index at most $N!$. This proves part (iii). □

Corollary 2.26. *Keep the assumptions of Lemma 2.25.*

(i) *Let $v \notin S$. If $\pi_{H,v} \in \mathrm{Irr}^{\mathrm{ur}}(H(F_v))$ is C-arithmetic, then $\eta_*\pi_{H,v}$ is C-arithmetic. The converse is true if there is a constant κ such that every fiber of $\eta_* : \mathrm{Irr}^{\mathrm{ur}}(H(F_v)) \to \mathrm{Irr}^{\mathrm{ur}}(G(F_v))$ has cardinality at most κ.*

(ii) *Let π_H and π be automorphic representations of $H(\mathbb{A}_F)$ and $G(\mathbb{A}_F)$ such that $\pi_v = \eta_*(\pi_{H,v})$ for all $v \notin S$. If π_H is C-arithmetic, then so is π.*

Proof. This is immediate from parts (ii) and (iii) of Lemma 2.25. □

Remark 2.27. Compare our results with [BG11, Lemmas 6.2 and 6.3], where it is shown that any L-morphism $\eta : {}^LH \to {}^LG$ carries L-algebraic (respectively L-arithmetic) representations to L-algebraic (respectively L-arithmetic) representations. (It is worth noting that they use Galois groups to form the L-groups; it can fail to be true if Weil groups are used.) One could try to derive our results in § 4.2 directly from their results by twisting but this is not automatic for two reasons: some groups lack twisting elements (in the sense of [BG11, § 5.2]) and some others admit no L-algebraic representations at all.

3. Purity and rationality of local components

The contents of this section are purely local and the following notation will be used:

- K is a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, $\mathcal{O}_K$ is its integer ring, and Frob_K is the geometric Frobenius element in $\mathrm{Gal}(K^{\mathrm{ur}}/K)$;

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

- W_K and I_K are the Weil and inertia groups of K;
- Ω is an algebraically closed field of characteristic zero with the same cardinality as $\mathbb{C}$ (usually Ω is taken to be $\mathbb{C}$ or $\overline{\mathbb{Q}}_l$ for a prime l);
- $\mathrm{Frob}_K \in W_K/I_K$ is the geometric Frobenius element;
- $v : W_K \to \mathbb{Z}$ is defined as $W_K \twoheadrightarrow W_K/I_K \simeq \mathbb{Z}$ where the last isomorphism carries Frob_K to 1;
- $|\cdot|_{W_K} : W_K \to \mathbb{Q}^\times$ is a character given by $\tau \mapsto q^{-v(\tau)}$;
- $\mathrm{sc}(\pi)$ denotes the supercuspidal support of $\pi \in \mathrm{Irr}(G(K))$.

3.1 Pure Weil–Deligne representations

Our basic definitions are based on those of [TY07, p. 471]. Their definition is slightly more general in that the weight is allowed to be a real number. For our purpose it suffices to consider only integral weights.

A Weil–Deligne representation (or *WD representation* for simplicity) of W_K (over Ω) is a triple (V, ρ, N) where V is a finite-dimensional Ω-vector space, $\rho : W_K \to GL(V)$ is a group homomorphism such that $\rho(I_K)$ is finite, and $N \in \mathrm{End}_\Omega(V)$ is a nilpotent operator such that $\rho(\tau)N\rho(\tau)^{-1} = |\tau|_{W_K} N$. It is said to be unramified if $\rho(I_K)$ is the identity and $N = 0$, *Frobenius semisimple* (or 'F-ss' for short) if ρ is semisimple, and irreducible if ρ is irreducible and $N = 0$. Let $(V, \rho, N)^{\text{F-ss}} := (V, \rho^{\mathrm{ss}}, N)$ denote the Frobenius semisimplification of (V, ρ, N), where ρ^{ss} is defined as follows: fix a lift $\phi \in W_K$ of Frob_K and let $\rho(\phi) = su$ be the Jordan decomposition with semisimple part s. Then $\rho^{\mathrm{ss}}(\phi^n\tau) := s^n\rho(\tau)$ for all $n \in \mathbb{Z}$ and for all $\tau \in I_K$, which defines ρ^{ss} independently of the choice.

Let $n \in \mathbb{Z}_{\geqslant 1}$. For a continuous l-adic representation $r : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, there is a standard way (depending on whether $l \neq p$ or $l = p$) to associate a Weil Deligne representation $WD(r)$ of W_K as explained on [TY07, pp. 467–470]. (One can view WD as a functor on appropriate categories.)

We recall the key definitions about purity. Let $w \in \mathbb{Z}$. A *q-Weil number* (respectively *integer*) of weight w is an algebraic number α (respectively an algebraic integer α) such that $|\iota(\alpha)| = q^{w/2}$ for any field embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. A WD representation (V, ρ, N) of W_K is *strictly pure* of weight w if every eigenvalue of the image under ρ of some (hence every) lift of Frob_K is a q-Weil number of weight w. We say that (V, ρ, N) is *mixed* if there exists an increasing filtration of sub-WD representations $\{\mathrm{Fil}_i V\}_{i \in \mathbb{Z}}$ on V such that $\mathrm{Fil}_i V = 0$ if $i \ll 0$, $\mathrm{Fil}_i V = V$ if $i \gg 0$ and $\mathrm{gr}_i V := \mathrm{Fil}_i V/\mathrm{Fil}_{i+1} V$ is strictly pure of weight i for every $i \in \mathbb{Z}$. A mixed (V, ρ, N) admits a unique filtration such that $N(\mathrm{Fil}_i V) \subset \mathrm{Fil}_{i-2} V$. Let us say (V, ρ, N) is *pure* of weight w if it is mixed and if $N^i : \mathrm{gr}_{w+i} V \to \mathrm{gr}_{w-i} V$ is an isomorphism for every i with respect to the unique filtration just mentioned. More generally let $\underline{w}$ be a finite multi-set such that the elements of $\underline{w}$ are distinct integers $w_1, \ldots, w_r$ with multiplicities $m_1, \ldots, m_r$. Then (V, ρ, N) is said to be *pure of weight* $\underline{w}$ if $V = \bigoplus_{i=1}^r (V_i, \rho_i, N_i)$ with each (V_i, ρ_i, N_i) being pure of weight w_i and of dimension m_i. (If so, we have in particular $|\underline{w}| = m_1 + \cdots + m_r = \dim V$.) Finally a mixed (V, ρ, N) is *integral* if for some (hence, every) lift $\phi \in W_K$ of Frob_K, every eigenvalue of ϕ on V is an algebraic *integer* (so that every eigenvalue of ϕ on $\mathrm{gr}_i V$ is a q-Weil *integer* of weight i).

The above definitions are motivated by Deligne's weight-monodromy conjecture in its integral form (cf. [Del71] and [Sai03, Conjectures 0.3 and 0.5]). The conjecture is equivalent to the one without Frobenius semisimple/semisimplification in the statement.

Conjecture 3.1 (cf. [Sai03, Conjecture 0.3, 0.5]). Let l be any prime (which could be equal to p). Let (V, ρ, N) be an F-ss WD representation on a $\overline{\mathbb{Q}}_l$-vector space. If (V, ρ, N) is a

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

subquotient of $WD(H^i_{\text{ét}}(X \times_K \overline{K}, \overline{\mathbb{Q}}_l))^{\text{F-ss}}$ for some proper smooth scheme X over K, then it is pure of weight i and integral.

It is worth noting that when X has a proper smooth integral model over $\mathcal{O}_K$ and $l \neq p$, the conjecture is known by Deligne's work on the Weil conjectures. In that case the WD representation below is unramified and strictly pure of weight i. In the non-smooth (bad reduction) case the conjecture is known when $\dim X \leqslant 2$ by Rapoport and Zink and in some special cases, for instance for certain Shimura varieties. A recent breakthrough by Scholze [Sch12] provides a proof for any complete intersection in a projective smooth toric variety. The converse of Conjecture 3.1, which is not as deep as the original conjecture, also seems true. (A proof was announced by Teruyoshi Yoshida but has not appeared in print at the time of writing.)

Motivated by Conjecture 3.1 (as well as its converse) and the Fontaine–Mazur conjecture ([FM95, Conjecture 1]; also see [Tay04, Conjecture 1.3]), we speculate on the following *global* conjecture, which in particular slightly refines the conjecture by Fontaine and Mazur in a sign aspect. More precisely, their conjecture says that parts (i) and (iii) below are equivalent if nonnegativity is dropped in part (i) and a Tate twist of cohomology is allowed in part (iii). The conjecture can be stated for all primes l simultaneously in the language of compatible systems, cf. [Tay04].

Conjecture 3.2. Let F be a number field and $\rho : \text{Gal}(\overline{F}/F) \to \text{GL}_n(\overline{\mathbb{Q}}_l)$ a continuous semisimple representation unramified outside finitely many places. The following are equivalent:

(i) ρ is de Rham at every place $v|l$ with nonnegative Hodge–Tate weights;[4]

(ii) the WD representation associated with ρ at every finite place v is integral and pure of weight $\underline{w}$, which is independent of v, has entries in $\mathbb{Z}$ and satisfies $|\underline{w}| = n$;

(iii) ρ appears as a subquotient of $\bigoplus_{i \geqslant 0} H^i_{\text{ét}}(X \times_F \overline{F}, \overline{\mathbb{Q}}_l)$ for some proper smooth scheme X over F.

When ρ is furthermore irreducible, we may replace the condition in part (ii) with 'integral and pure of weight w for some $w \in \mathbb{Z}$', and the condition in part(iii) with '... a subquotient of $H^i_{\text{ét}}(X \times_F \overline{F}, \overline{\mathbb{Q}}_l)$ for some $i \geqslant 0$ ...'. (For a given ρ the corresponding w and i are expected to be equal, cf. Conjecture 3.1.)

In Proposition 4.1 below we derive a partial result toward Conjecture 3.2 from the well-known results concerning the construction of Galois representations from automorphic representations. That result will be a key to the finiteness result of § 5.3, where the role of integrality will become clear. This was our original motivation. However, Conjecture 3.2 is interesting in its own right and we plan to discuss it in more detail on some other occasion.

Remark 3.3. Part (ii) may be equivalent to part (ii)$'$ below, allowing us to exclude finitely many v:

(ii)$'$ the WD representation associated with ρ at *almost* every finite v is pure and integral.

Conjecture 2.8 suggests that it would also be equivalent to:

(ii)$''$ the WD representation associated with ρ at (almost) every finite v is integral and has its field of rationality contained in some number field E independent of v.

Remark 3.4. We are reduced to a more standard conjecture if we get rid of 'with nonnegative Hodge–Tate weights' in part (i), 'integral' in part (ii), and allow a Tate twist in part (iii).

[4] In our convention the cyclotomic character has Hodge–Tate weight -1 (rather than 1).

In this form we already mentioned that the equivalence of parts (i) and (iii) is exactly the Fontaine–Mazur conjecture.

Remark 3.5. The implication (iii)⇒(i) is known by results of p-adic Hodge theory (the solution of the C_{pst} conjecture and comparison of filtrations in complex and p-adic Hodge theories) and the fact that the Hodge filtration on $H^i_{\text{ét}}(X\times_F \mathbb{C},\overline{\mathbb{Q}}_l)$ has jumps only in nonnegative indices with respect to any field embedding $F\hookrightarrow\mathbb{C}$. According to Conjecture 3.1, part (iii) should imply part (ii). Finally we remark that (i)⇔(ii) may be viewed as the arithmetic analogue of Conjecture 2.8.

Remark 3.6. When ρ is associated with a (classical) cuspidal holomorphic eigenform $f=\sum_{n\geqslant 1} a_n q^n$ of weight $k\in\mathbb{Z}_{\geqslant 1}$ with $a_1=1$ so that a_n are algebraic integers for all $n\geqslant 1$, then (under a suitable normalization) ρ satisfies parts (i), (ii), and (iii) with Hodge–Tate weights 0 and $k-1$. Now assume that $a_n\in\mathbb{Z}$ for all n. The equivalence (i)⇔(ii)′, applied to the twist of ρ by the cyclotomic character, amounts to the assertion that f is ordinary, i.e. a_p is a p-unit for infinitely many primes p.

It is useful to know a preservation property under base field extensions.

Lemma 3.7. *Let $\widetilde{\rho}=(V,\rho,N)$ be a WD representation of W_K, and L/K be a finite extension. Then $\widetilde{\rho}|_{W_L}$ is pure (respectively integral) if and only if $\widetilde{\rho}$ is pure (respectively integral).*

Proof. Straightforward. (The preservation of purity is [TY07, Lemma 1.4.2].) □

Given (V,ρ,N) as above and $s\in\mathbb{Z}_{\geqslant 1}$, one constructs a new Weil–Deligne representation

$$\mathrm{Sp}_s(V):=(V^s,\rho|\cdot|^{s-1}_{W_K}\oplus\cdots\oplus\rho|\cdot|_{W_K}\oplus\rho,N)$$

such that $N:\rho|\cdot|^i_{W_K}\xrightarrow{\sim}\rho|\cdot|^{i-1}_{W_K}$ for $i=1,\ldots,s-1$ and $N=0$ on ρ. Note that $\mathrm{Sp}_s(V)$ is uniquely determined up to isomorphism. If (V,ρ,N) is pure of weight w, then $\mathrm{Sp}_s(V)$ is pure of weight $w+s-1$.

Lemma 3.8. *Let $n\geqslant 1$ and (V,ρ,N) be an n-dimensional F-ss WD representation of W_K. Then there exist:*

- *$m\in\mathbb{Z}_{\geqslant 1}$, $s_1,\ldots,s_m\in\mathbb{Z}_{\geqslant 1}$; and*
- *a collection of irreducible n_i-dimensional F-ss WD representations $(V_i,\rho_i,0)$, $i=1,\ldots,m$;*

such that $V=\bigoplus_{i=1}^m \mathrm{Sp}_{s_i}(V_i)$. Moreover, if (V,ρ,N) is pure of weight $w\in\mathbb{Z}$ then each V_i is strictly pure of weight $w-s_i+1$.

Proof. The first assertion follows from the standard fact that any indecomposable F-ss WD representation is of the form $\mathrm{Sp}_s(V)$ for an irreducible F-ss WD representation V. If (V,ρ,N) is pure of weight w, then so is each $\mathrm{Sp}_{s_i}(V_i)$. From this and the definition of $\mathrm{Sp}_{s_i}(V_i)$ it is elementary to verify that V_i is strictly pure of weight $w-s_i+1$. □

Pure WD representations enjoy a remarkable rationality property of importance to us.

Lemma 3.9. *A pure F-ss WD representation (V,ρ,N) of W_K (of some weight $w\in\mathbb{Z}$) has a number field as a field of rationality.*

Proof. By Lemma 3.8 it suffices to treat $\mathrm{Sp}_s(V)$ when $(V,\rho,0)$ is an irreducible F-ss WD representation which is strictly pure of weight $w\in\mathbb{Z}$. Clearly $\mathrm{Sp}_s(V)$ can be defined over the same number field over which (V_i,ρ_i,N_i) is defined. Hence, we are further reduced to showing that $(V,\rho,0)$ has a number field as a field of rationality when it is irreducible and strictly pure of some weight $w\in\mathbb{Z}$.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

It is enough to verify that the trace function $T := \mathrm{tr}\,\rho : W_K \to \Omega$ has image contained in a finite extension of $\mathbb{Q}$ in Ω. Fix a lift $\phi \in W_K$ of Frob_K. The eigenvalues of $\rho(\phi)$, say $\lambda_1, \ldots, \lambda_n$, are contained in a finite Galois extension E of $\mathbb{Q}$ as they are Weil numbers. We will show that there exists $d \geqslant 1$ such that for every $m \geqslant 0$,

$$\rho(\phi^m \tau)^d = \rho(\phi^m)^d.$$

Then for every $\tau \in W_K$, the eigenvalues of $\rho(\tau)$ are contained in the set of $\alpha \in \Omega$ such that $\alpha^d \in \{\lambda_1^m, \ldots, \lambda_n^m\}$ for some $m \geqslant 1$. The set of such α clearly generates a finite extension of E, in which $T(W_K)$ must be contained.

Let us show the existence of d as above. For $A, B \in \mathrm{GL}_\Omega(V)$ we write A^B for BAB^{-1}. The homomorphism $\tau \mapsto \phi\tau\phi^{-1}$ induces a homomorphism $\theta : \phi^{\mathbb{Z}} \to \mathrm{Aut}(I_K/I_K \cap \ker(\rho))$. Put $i := |I_K/I_K \cap \ker(\rho)| < \infty$ and $j := |\mathrm{Aut}(I_K/I_K \cap \ker(\rho))|$. Then $\rho(\tau)^i = 1$ and $\rho(\tau)^{\rho(\phi^j)} = \rho(\tau)$ for all $\tau \in I_K$. Then (using $\rho(\tau)^{\rho(\phi^j)} = \rho(\tau)$ and $\rho(\tau)^i = 1$ in the second and third equalities, respectively)

$$\begin{aligned} \rho(\phi^m\tau)^{ij}\rho(\phi^m)^{-ij} &= \rho(\tau)^{\rho(\phi^m)}\rho(\tau)^{\rho(\phi^{2m})}\cdots\rho(\tau)^{\rho(\phi^{ijm})} \\ &= (\rho(\tau)^{\rho(\phi^m)}\rho(\tau)^{\rho(\phi^{2m})}\cdots\rho(\tau)^{\rho(\phi^{jm})})^i = 1. \end{aligned} \tag{3.1}$$

Hence, we get the desired d by putting $d := ij$. □

Later we would like to utilize some results of Arthur, in which local L-parameters are used in place of Weil–Deligne representations. We recall the standard way to go between the two. Recall that a local L-parameter for $\mathrm{GL}_n(K)$ is a continuous homomorphism

$$\varphi : W_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$$

for an n-dimensional $\mathbb{C}$-vector space V such that $\varphi|_{W_K}$ is semisimple and $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is an algebraic representation. For such a φ one associates a WD representation $WD(\varphi) := (V, \rho, N)$ such that

$$\rho(\tau) = \varphi\left(\tau, \begin{pmatrix} |\tau|_{W_K}^{1/2} & 0 \\ 0 & |\tau|_{W_K}^{-1/2} \end{pmatrix}\right), \quad N = \varphi\left(1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

The association $\varphi \mapsto WD(\varphi)$ defines a bijection between the set of equivalence classes of L-parameters for $\mathrm{GL}_n(K)$ and the set of isomorphism classes of n-dimensional Frobenius semisimple WD representations. (In fact, it is a categorical equivalence.) In fact, the L-parameter φ can be defined over any Ω in place of $\mathbb{C}$ and the various definitions for WD representations at the beginning of § 3.1 carry over to φ. For instance, φ gives rise to a pure WD representation if and only if $\varphi|_{W_F}$ is strictly pure of integral weight in the sense defined earlier.

3.2 Twists of the local Langlands correspondence

Let rec_K denote the local Langlands bijection for $\mathrm{GL}_n(K)$ as in [HT01] (cf. [Hen00]) so that for each irreducible smooth representation π of $\mathrm{GL}_n(K)$, $\mathrm{rec}_K(\pi)$ denotes the associated n-dimensional Frobenius semisimple Weil–Deligne representation of W_K. Here both representations are considered on $\mathbb{C}$-vector spaces. We introduce a different normalization

$$\mathscr{L}_K(\pi) := \mathrm{rec}_K(\pi) \otimes |\cdot|_{W_K}^{-(n-1)/2}.$$

It was shown in [HT01, Lemma VII.1.6.2] that

$$\mathscr{L}_K(\pi^\sigma) = \mathscr{L}_K(\pi)^\sigma \quad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}). \tag{3.2}$$

(To be precise [HT01] shows (3.2) up to semisimplification, i.e. disregarding N, but this easily implies (3.2) without semisimplification.) Hence, $\mathrm{rec}_K(\pi^\sigma) = \mathrm{rec}_K(\pi)^\sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(q^{1/2}))$.

Lemma 3.10. *Let π be an irreducible smooth representation of $\mathrm{GL}_n(K)$ (on a $\mathbb{C}$-vector space). If $\mathscr{L}_K(\pi)$ is pure of weight $w \in \mathbb{Z}$, then $\mathbb{Q}(\pi)$ is finite over $\mathbb{Q}$.*

Proof. Immediate from Lemma 3.9 and (3.2). □

3.3 Bound on field of rationality implies bound on ramification

For $j \in \mathbb{R}_{\geqslant 0}$ let I_K^j denote the jth ramification subgroup of I_K with respect to the upper numbering. Similarly for any Galois extension M of L (which are extensions of K), we write $\mathrm{Gal}(M/L)^j$ and $\mathrm{Gal}(M/L)_j$ for the upper and lower numbering ramification subgroups of $\mathrm{Gal}(M/L)$. Denote by dep and cond the depth and conductor, which are defined for WD representations of W_K as well as irreducible smooth representations of $\mathrm{GL}_n(K)$. The depth of a WD representation (V, ρ, N) may not be as standard as the others so we recall it here: $\mathrm{dep}(V, \rho, N)$ is defined to be the infimum among the elements $j \in \mathbb{R}_{\geqslant 0}$ such that $\rho(I_K^j)$ is trivial. The infimum is actually attained and the depth is a rational number.

The following lemma will play a key role in the proof of finiteness results of § 5.5. In the proof all extensions of E (which is a subfield of $\mathbb{C}$) are considered in $\mathbb{C}$.

Lemma 3.11.[5] *Fix $n \in \mathbb{Z}_{\geqslant 1}$ and $A \in \mathbb{Z}_{\geqslant 1}$. There exists $d = d_{n,K,A} \in \mathbb{R}_{\geqslant 0}$ (depending on n, A and K) such that for every n-dimensional F-ss WD representation (V, ρ, N) whose field of rationality is an extension of $\mathbb{Q}$ of degree at most A,*

$$\mathrm{dep}(V, \rho, N) \leqslant d.$$

Proof. Consider the representation $\rho|_{I_K} : I_K \to \mathrm{GL}(V)$ with finite image. Let E be the field of rationality of (V, ρ, N). By (3.2), $\wedge^i \rho^\sigma \simeq \wedge^i \rho$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$. We take the trace and see that the degree n characteristic polynomial of $\rho(\tau)$ has coefficients in E. Hence, each eigenvalue λ of $\rho(\tau)$ for $\tau \in I_K$, which are roots of unity, must be contained in a finite extension of E of degree at most n, so in a finite extension of $\mathbb{Q}$ of degree at most nA. Let f be the least common multiple of the order of λ as λ runs over all eigenvalues of $\rho(\tau)$ for $\tau \in I_K$ (i.e. the least power $f \geqslant 1$ such that $\lambda^f = 1$). In particular $\rho(\tau^f)$ is a semisimple element with all eigenvalues equal to 1, i.e. the identity element.

Put $\mu_{\leqslant nA} := \bigcup_{[E'':\mathbb{Q}] \leqslant nA} \mu_\infty(E'')$ where $\mu_\infty(E'')$ denotes the set of all roots of unity in E''. One sees from an elementary theory of cyclotomic fields that $\mu_{\leqslant nA}$ is a finite set (its cardinality is the least common multiple of $m \in \mathbb{Z}_{\geqslant 1}$ such that $\varphi(m) \leqslant nA$, where φ is the Euler totient function). We have $f \leqslant |\mu_{\leqslant nA}|$, an upper-bound which by construction depends only on n and A.

The finite group $H := I_K / \ker \rho$ is equipped with an embedding $\overline{\rho} : H \hookrightarrow GL(V)$ induced by ρ. Let E' be the finite extension of $\mathbb{Q}$ obtained by adjoining all fth roots of unity. As H has exponent dividing f, Brauer's theorem [Ser77, § 12.3, Theorem 24] implies that $\overline{\rho}$ is defined over E', i.e. there exists a representation $\overline{\rho}' : H \hookrightarrow GL(V_{E'})$ on an E'-vector space $V_{E'}$ such that $\overline{\rho}' \otimes_{E'} \mathbb{C} \simeq \overline{\rho}$ as H-representations.

Now choose any prime l relatively prime to f, and a place w of E' above l. Denote by k_w the residue field of E' at w. The l-adic representation $\overline{\rho}' : H \hookrightarrow \mathrm{GL}_n(E'_w)$ has a model over

[5] After furnishing the proof of the lemma, we found that a proof had been given to essentially the same problem in [FM95, § 4.(a)]. We note two differences. First we work with the field of rationality rather than the field of definition. Second we obtain an explicit bound on $d_{n,A,K}$ which is not immediately available from [FM95]. We also mention an analogous result for crystalline representations, cf. [CE04, § 4].

$\mathrm{GL}_n(\mathcal{O}_{E'_w})$ in the sense that the model becomes isomorphic to $\overline{\rho}'$ after extending scalars to E'_w.[6] We denote the model by the same symbol $\overline{\rho}'$. The kernel of the map $\flat : \mathrm{GL}_n(\mathcal{O}_{E'_w}) \to \mathrm{GL}_n(k_w)$ taking matrix entries modulo the maximal ideal of $\mathcal{O}_{E'_w}$ is a pro-l group, which must have trivial intersection with H. Hence, $\flat \circ \overline{\rho}'$ is an injection $H \hookrightarrow \mathrm{GL}_n(k_w)$. The upshot is that

$$|I_K/\ker\ \rho| = |H| \quad \text{divides} \quad |\mathrm{GL}_n(k_w)|. \tag{3.3}$$

The cardinality $|\mathrm{GL}_n(k_w)|$ can be made to depend only on nA. Indeed f and E' depend only on nA by construction. By choosing the minimal prime l coprime to f, and w above l minimalizing $|k_w|$, we arrange that the cardinality $|\mathrm{GL}_n(k_w)|$ depends only on nA, cf. (3.4) below.

Now it is enough to verify the existence of $d \in \mathbb{Z}_{\geqslant 0}$ with the following property:

$$\mathrm{Gal}(L/\widehat{K}^{\mathrm{ur}})^d = \{1\}$$

for all finite Galois extensions L of $\widehat{K}^{\mathrm{ur}}$ such that $[L : \widehat{K}^{\mathrm{ur}}]$ divides $|\mathrm{GL}_n(k_w)|$, where $\widehat{K}^{\mathrm{ur}}$ denotes the completion of the maximal unramified extension of K. This is a standard exercise. Indeed, writing e_L (respectively e_K) for the absolute ramification index of L (respectively K) so that elements of $L^\times$ take valuations exactly on $(1/e_L)\mathbb{Z}$ (if p is normalized to have valuation 1), we know that $\mathrm{Gal}(L/\widehat{K}^{\mathrm{ur}})_{d'} = \{1\}$ for all $d' > e_L/(p-1)$ by [Ser79, IV.2, Exercise 3.c]. The same is certainly true for the d'th upper numbering group since the latter is identified with the d''th lower numbering group for some $d'' \geqslant d'$. Since $e_L \leqslant |\mathrm{GL}_n(k_w)|e_K$, we conclude that the choice of $d = |\mathrm{GL}_n(k_w)|e_K/(p-1)$ satisfies the desired property. □

Corollary 3.12. *Fix $n, A \in \mathbb{Z}_{\geqslant 1}$. There exists $d \in \mathbb{Z}_{\geqslant 0}$ (depending on n, A and K) such that for every C-algebraic $\pi \in \mathrm{Irr}(\mathrm{GL}_n(K))$ satisfying $[\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A$, we have that*

$$\mathrm{dep}(\pi) \leqslant d, \quad \mathrm{cond}(\pi) \leqslant dn.$$

Proof. Keeping (3.2) in mind, we apply Lemma 3.11 to find $d \in \mathbb{Z}_{\geqslant 0}$ such that for every π as above, the WD representation $\mathscr{L}_K(\pi) := (V, \rho, N)$ has depth at most d. Since $\mathrm{dep}(\pi) = \mathrm{dep}(V, \rho, N)$ by [Yu09, Theorem 2.3.6.4], we see that $\mathrm{dep}(\pi) \leqslant d$. The assertion on conductor holds true thanks to Lemma 3.13 below. □

The following lemma may be well-known but we present a proof here.

Lemma 3.13. *For every $\pi \in \mathrm{Irr}(\mathrm{GL}_n(K))$, $\mathrm{cond}(\pi) \leqslant n \cdot (\mathrm{dep}(\pi) + 1)$.*

Proof. Let $(V, \rho, N) := \mathscr{L}_K(\pi)$. Since rec_K and thus $\mathscr{L}_K$ preserve conductor, we obtain from the formula for the Artin conductor of (V, ρ, N) that

$$\mathrm{cond}(\pi) = \mathrm{codim}(V^{I_K})^{N=0} + \int_0^\infty \mathrm{codim} V^{I_K^j}\, dj.$$

Each codimension is certainly less than n and by definition the integral is supported on the interval $0 \leqslant j \leqslant \mathrm{dep}(V, \rho, N)$. The inequality again follows. □

It is worth emphasizing that the constructive nature of the proof of Lemma 3.11 makes it possible to find an explicit bound in that lemma (and also in Corollary 3.12). The first step of the proof was to adjoin all roots of unity of degree at most $n[E : \mathbb{Q}]$. This yields the explicit bound $f \leqslant |\mu_{n[E:\mathbb{Q}]}| \leqslant (2n[E : \mathbb{Q}])!$. To obtain an effective bound of better quality we establish the following result.

[6] This is true even for continuous l-adic representations of any profinite group. The main point is that the $\mathcal{O}_{E'_w}$-module generated by finitely many translations of an $\mathcal{O}_{E'_w}$-lattice is still an $\mathcal{O}_{E'_w}$-lattice.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

Lemma 3.14. *For each $n \geqslant 1$ there is a constant $c_n > 0$ such that the following holds. For any number field E let F be the finite extension of E generated by all of the roots of unity that are of degree at most n over E. Then*

$$[F : E] \leqslant c_n[E : \mathbb{Q}].$$

Proof. We have $F = E(\zeta_N)$ for some $N \geqslant 1$. Write $N = \prod_{p^r||N} p^r$. It is not difficult to show that $\phi(p^r)$ divides $n![E : \mathbb{Q}]$ but we shall derive a more precise estimate below.

Since the extensions $E(\zeta_{p^r})$ are linearly disjoint over E,

$$[F : E] = \prod_{p^r||N} [E(\zeta_{p^r}) : E].$$

Let $E(p) := E \cap \mathbb{Q}(\zeta_{p^r})$. Then similarly $[E : \mathbb{Q}] \geqslant \prod_{p|N}[E(p) : \mathbb{Q}]$.

Since $\mathbb{Q}(\zeta_{p^r})$ is Galois, it is linearly disjoint from E over $E(p)$. Thus, we have $[\mathbb{Q}(\zeta_{p^r}) : E(p)] = [E(\zeta_{p^r}) : E]$ and therefore

$$\phi(p^r) = [E(\zeta_{p^r}) : E][E(p) : \mathbb{Q}].$$

Since $[E(\zeta_{p^r}) : E] \leqslant n$ we can deduce the inequality

$$\frac{[E(\zeta_{p^r}) : E]}{[E(p) : \mathbb{Q}]} \leqslant \min\left(\phi(p^r), n, \frac{n^2}{\phi(p^r)}\right).$$

Taking a product we deduce the estimate

$$\frac{[F : E]}{[E : \mathbb{Q}]} \leqslant \prod_{p^r||N} \min\left(\phi(p^r), n, \frac{n^2}{\phi(p^r)}\right) \leqslant c_n.$$

In the last inequality we could extend the product to all prime numbers and the choice $c_n = n^n$ is admissible. This concludes the proof. □

Using the Lemma 3.14 we have the bound $f \leqslant c_n A$ in the proof of Lemma 3.11. Since there is a prime between f and $2f$ (if $f \geqslant 2$) by Chebyshev's theorem, we can choose the prime $l < 2f$. On the other hand $[k_w : \mathbb{F}_l] \leqslant [\mathbb{Q}(\mu_f) : \mathbb{Q}] \leqslant f$, hence $|\mathrm{GL}_n(k_w)| \leqslant n^2 l^f \leqslant n^2(2f)^f$. So a value

$$d = \frac{e_K}{p-1} A^{O_n(A)}$$

is admissible for the conclusion of Lemma 3.11 to hold. Actually we shall establish an improved bound using a more efficient argument.

Lemma 3.15. (i) *Let H be a finite subgroup of GL_n whose traces generate a field E with $[E : \mathbb{Q}] \leqslant A$. The order of a p-Sylow of H is at most $\leqslant c_{n,p}A^n$ for some constant $c_{n,p}$ depending only on n and p.*

(ii) *In the statement of Lemma 3.11 the constant*

$$d_{n,K,A} \leqslant c_{n,K} A^n \tag{3.4}$$

is admissible, where $c_{n,K}$ depends only on n and K.

Proof. (i) Let H_p be a p-Sylow subgroup of H. By a result of Roquette, the Schur indices of H_p are 1 if $p \neq 2$ and 1 or 2 if $p = 2$ (see [Yam79] for a direct proof using the Hasse invariant). Since the traces of elements of $H_p \subset H$ are in E, this shows that the representation $H_p \subset GL(V)$ can be realized over E if $p \neq 2$ and over $E(\sqrt{-1})$ if $p = 2$.

There remains the problem of estimating the order of a finite p-group H_p inside $\mathrm{GL}_n(E)$. Minkowski obtained optimal bounds when $E = \mathbb{Q}$. The method can be extended to a general number field E and Schur gave a different proof using character theory. Serre [Ser07] treated the case of an arbitrary E. If $p \neq 2$, then

$$\frac{\log |H_p|}{\log p} \leqslant m \left\lfloor \frac{n}{t} \right\rfloor + \left\lfloor \frac{n}{pt} \right\rfloor + \left\lfloor \frac{n}{p^2 t} \right\rfloor + \cdots$$

where $t = [E(\zeta_p) : E]$ and $m \in \mathbb{Z}_{\geqslant 1}$ is the largest integer such that $\zeta_{p^m} \in E(\zeta_p)$. There are some subtle modifications if $p = 2$ to get a sharp bound, but this is not relevant for us since we are interested in the behavior for A large.

Certainly $\phi(p^m) \leqslant [E(\zeta_p) : \mathbb{Q}] \leqslant tA$. Therefore, there is a constant c_n depending on n such that $|H_p| \leqslant c_n (tA)^{n/t}$. In particular $|H_p| \leqslant c_{n,p} A^n$ for some constant $c_{n,p}$ depending only on n and p.

(ii) This relies on the basic structure of the inertia group I_K. For any Galois extension M of L and any integer $j \geqslant 1$, the quotients $\mathrm{Gal}(M/L)_j / \mathrm{Gal}(M/L)_{j+1}$ can be identified with an additive subgroup of the residue field of M. Since these are abelian group of p-power order we have that $\mathrm{Gal}(M/L)_1$ is a p-group [Ser79, IV.2, Corollary 3]. We are in position to apply the assertion (i). Then similarly to Lemma 3.11 we can conclude the proof of the estimate (3.4). □

4. Automorphic representations of classical groups

In this section we recall the endoscopy and the associated Galois representations for automorphic representations of symplectic, orthogonal and unitary groups. A key input is the integrality proposition in § 4.1 coming from an arithmetic geometry study of Shimura varieties.

4.1 Galois representations associated to automorphic representations

Let $n \in \mathbb{Z}_{\geqslant 1}$. Let F^+ be a totally real field and consider the following two cases:

(CM) F is a CM quadratic extension of F^+ with complex conjugation c (so that $F^+ = F^{c=1}$); or

(TR) $F = F^+$.

Let Π be a regular C-algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that Π_∞ has the same infinitesimal character as an irreducible algebraic representation Ξ of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$. The highest weight for Ξ may be written as $a(\Xi) = (a_{\sigma,i})_{\sigma \in \mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{C}), 1 \leqslant i \leqslant n}$ with $a_{\sigma,i} \in \mathbb{Z}$, viewed as a character of the standard diagonal torus of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$. We further assume in each of the above cases that:

(CM) $\Pi^\vee \simeq \Pi \circ c$;

(TR) $\Pi^\vee \simeq \Pi \otimes (\det \circ \chi)$ for $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that $\chi_v(-1)$ is the same for every $v \in S_\infty$.

In case (CM), fix a subset $\Phi^+ \subset \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$ such that $\Phi^+ \coprod \Phi^+ \circ c = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$ (called a CM-type). Recall that various notation and notion about WD representations were introduced in § 3.1 and the twist of the local Langlands correspondence $\mathscr{L}_{F_v}(\Pi_v)$ in § 3.2.

Proposition 4.1. *There exists a family of l-adic representations (for varying l)*

$$\{R_{l,\iota_l}(\Pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)\}_{l,\iota_l}, \quad l \text{ is a prime and } \iota_l : \overline{\mathbb{Q}}_l \simeq \mathbb{C}$$

such that for every finite place v of F,

$$\mathscr{L}_{F_v}(\Pi_v) = \iota_l WD(R_{l,\iota_l}(\Pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{\text{F-ss}},$$

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

$\mathscr{L}_{F_v}(\Pi_v)$ is a pure WD representation of weight $n-1$, and Π_v are essentially tempered. Moreover, there exists $s(\Pi_\infty) \in \mathbb{Z}$ (depending only on Π_∞ and F) such that for every finite v, $\mathscr{L}_{F_v}(\Pi_v)|\cdot|^{-s(\Pi_\infty)/2}$ is integral.

Remark 4.2. When $v \notin S$ and $v \nmid l$, the proposition tells us that $\mathscr{L}_{F_v}(\Pi_v)$ is unramified, *strictly* pure of weight $n-1$ and the eigenvalues of Frob_v on $\mathscr{L}_{F_v}(\Pi_v)$ are q_v-Weil numbers of weight $n-1$ which become algebraic integers after multiplying $q_v^{s(\Pi_\infty)}$.

Remark 4.3. A more elementary method seems available in some cases by exploiting the integral structure on the space of algebraic modular forms (cf. [Gro99]) on classical groups which are compact at infinity, without the need of arithmetic geometry and Galois representations. We have not adopted it here as we do not know how to cover all cases with that approach.

Proof. Except for the last assertion the proposition is a result of combined effort: see [BGGT14, Theorem A] as well as [Shi11, Theorem 1.2], [CH13, Theorem 1.4], [Car12a, Theorem 1.1] and [Car12b, Theorem 1.1].

The last assertion on integrality remains to be justified. The main idea is that the Galois representation $R_l(\Pi) = R_{l,\iota_l}(\Pi)$ of the proposition is essentially realized in the cohomology of a certain $(n-1)$-dimensional compact Shimura variety Sh to which some general results in arithmetic geometry (such as [Sai03, Corollary 0.6.(1)]) apply. A precise argument requires us to import lots of notation and various pieces of results. For any difficulty caused by this when reading our proof we apologize. We will recall at least some of the important notation and facts as we go along.

In case (TR) it is possible to choose a CM quadratic extension L of F and an algebraic Hecke character $\phi : L^\times\backslash\mathbb{A}_L^\times \to \mathbb{C}^\times$ such that $\phi\phi^c = \chi\chi^c$ and $BC_{L/F}(\Pi)$ is cuspidal. (One can make a choice to ensure cuspidality by arguing as in [Clo13, § 1].) Then $\Pi' := BC_{L/F}(\Pi)\otimes\phi$ is conjugate self-dual, regular, and C-algebraic. Thus, the proof of the integrality assertion is reduced to case (CM) for L and Π' via Lemma 3.7.

From now on we put ourselves in case (CM). Starting with the case where n is odd, we will derive the integrality result as a consequence of [Shi11] and [TY07]. It is desirable to reconcile notation with [Shi11] at the outset to avoid confusion. Only in this proof G denotes the unitary similitude group as in [Shi11]. Our Π corresponds to Π^1 in that paper. The notation Π there, designating a representation of $G(\mathbb{A}_E)$, will be written as Π' here. (So the finite part of the representation Π' of $G(\mathbb{A}_E) \simeq \mathrm{GL}_1(\mathbb{A}_E)\times\mathrm{GL}_n(\mathbb{A}_F)$ descends to the finite part of an automorphic representation of $G(\mathbb{A})$.)

Since integrality may be checked after a series of finite cyclic base changes (Lemma 3.7) we may and will assume that conditions (i)–(v) of § 6.1 and the five assumptions at the start of [Shi11, § 7.1] are satisfied. In particular, F contains an imaginary quadratic field E. Fix an embedding $\tau_E : E \hookrightarrow \mathbb{C}$ and choose Φ^+ to be the set of $F \hookrightarrow \mathbb{C}$ extending τ_E. Recall that $(a_{\sigma,i})_{\sigma\in\mathrm{Hom}(F,\mathbb{C}),1\leqslant i\leqslant n}$ is associated with Ξ. Choose a Hecke character $\psi : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ as in [Shi11, Lemma 7.2], cf. [HT01, Lemma VI.2.10]. The proof in the latter reference shows that $\psi_\infty(z) = \tau_E(z)^{a_0}$, where $a_0 := -\sum_{\sigma\in\Phi^+,1\leqslant i\leqslant n} a_{\sigma,i}$. Let ξ be the irreducible algebraic representation of $G\times_{\mathbb{Q}}\mathbb{C} \simeq \mathrm{GL}_1(\mathbb{C})\times\prod_{\sigma\in\Phi^+}\mathrm{GL}_n(F_\sigma)$ of highest weight $(a_0,(a_{\sigma,i})_{\sigma\in\Phi^+,1\leqslant i\leqslant n})$. Put

$$d_\sigma := \max_{1\leqslant i\leqslant k}(\max(0,a_{\sigma,i})),\quad s_\sigma := \sum_{1\leqslant i\leqslant k}|a_{\sigma,i}-d_\sigma|,\quad t_\xi := a_0 + \sum_{\sigma\in\Phi^+} kd_\sigma,\quad m_\xi = \sum_{\sigma\in\Phi^+}(s_\sigma + kd_\sigma),$$

and $s(\Pi_\infty) := 2t_\xi - \min(0,a_0)$. Note that $s(\Pi_\infty)$ depends only on the data defining Π_∞. One can check that $2t_\xi - m_\xi = 2a_0 + \sum_{\sigma,i} a_{\sigma,i}$, cf. [HT01, p. 98] and [TY07, p. 476].

Consider the étale cohomology

$$H^{n-1}_{\text{ét}}(\mathrm{Sh}, \mathcal{L}_\xi) := \varinjlim_{U\subset G(\mathbb{A}^\infty)} H^{n-1}_{\text{ét}}(\mathrm{Sh}_U \times_F \overline{F}, \mathcal{L}_\xi)$$

as U runs over sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$. The limit is a $\overline{\mathbb{Q}}_l[G(\mathbb{A}^\infty)\times \mathrm{Gal}(\overline{F}/F)]$-module, which is admissible (respectively continuous) with respect to the $G(\mathbb{A}^\infty)$-(respectively $\mathrm{Gal}(\overline{F}/F)$-)action. For small enough U, $H^{n-1}_{\text{ét}}(\mathrm{Sh}_U \times_F \overline{F}, \mathcal{L}_\xi)$ is finite dimensional and

$$H^{n-1}_{\text{ét}}(\mathrm{Sh}_U \times_F \overline{F}, \mathcal{L}_\xi) = a_\xi H^{n-1+m_\xi}_{\text{ét}}(\mathcal{A}^{m_\xi} \times_F \overline{F}, \overline{\mathbb{Q}}_l(t_\xi))$$

where a_ξ is an idempotent of [TY07, pp. 476–477], which gives rise to an element of the Chow group $\mathrm{CH}^{n-1+m_\xi}(\mathrm{Sh}_U \times_F \mathrm{Sh}_U)_{\mathbb{Q}}$. (The subscript $\mathbb{Q}$ indicates that the coefficient ring is taken to be $\mathbb{Q}$.) On the other hand, $H^{n-1}_{\text{ét}}(\mathrm{Sh}_U \times_F \overline{F}, \mathcal{L}_\xi)$ is the direct sum

$$\Bigg(\bigoplus_{BC(\pi^{S,\infty})\simeq(\Pi')^{S,\infty}} R^{n-1}_l(\pi^\infty)\otimes(\pi^\infty)^U\Bigg) \oplus \Bigg(\bigoplus_{BC(\pi^{S,\infty})\not\simeq(\Pi')^{S,\infty}} R^{n-1}_l(\pi^\infty)\otimes(\pi^\infty)^U\Bigg) \tag{4.1}$$

where the first (respectively second) sum runs over π^∞, the finite part of discrete automorphic representations of $G(\mathbb{A})$, such that $BC(\pi^{S,\infty}) \simeq (\Pi')^{S,\infty}$ holds (respectively does not hold). According to [Shi11, Corollary 6.8], there is a positive integer C_G and a $\mathrm{Gal}(\overline{F}/F)$-representation $\widetilde{R}'_l(\Pi')$ such that $C_G\widetilde{R}'_l(\Pi') = \bigoplus_{\pi^\infty} R^{n-1}_l(\pi^\infty)$ where the sum is taken over the same set as in the first sum of (4.1). Corollary 6.10 of [Shi11] tells us that $R_l(\Pi)$ in the proposition is given by

$$R_l(\Pi) := \widetilde{R}'_l(\Pi') \otimes \mathrm{rec}_l(\psi). \tag{4.2}$$

The decomposition (4.1) allows us to find an idempotent b_ξ in the Chow group $\mathrm{CH}^{n-1}(\mathrm{Sh}_U\times_F \mathrm{Sh}_U)_{\mathbb{Q}}$ such that $C_G\cdot\widetilde{R}'_l(\Pi') \simeq b_\xi H^{n-1}_{\text{ét}}(\mathrm{Sh}_U\times_F\overline{F}, \mathcal{L}_\xi)$, equivariant for the $\mathrm{Gal}(\overline{F}/F)$-action. (First, find an idempotent separating each π^∞-part as in the [TY07, proof of Lemma 2.3]. Then one uses a Hecke algebra element which acts with trace 1 on each π^∞ such that $BC(\pi^{S,\infty}) \simeq (\Pi')^{S,\infty}$.) Write c_ξ for the pullback of b_ξ along the projection from $\mathcal{A}^{m_\xi}_U \times \mathcal{A}^{m_\xi}_U$ to $\mathrm{Sh}_U \times \mathrm{Sh}_U$. Since $H^j_{\text{ét}}(\mathrm{Sh}_U \times_F \overline{F}, \mathcal{L}_\xi)$ for $j \neq n-1$ is linearly independent from $(\pi^\infty)^K$ for any π^∞ as in (4.1) by [Shi11, Corollary 6.5.(i)], we may construct a correspondence Γ coming from $\mathrm{CH}(\mathcal{A}^{m_\xi}\times\mathcal{A}^{m_\xi})_{\mathbb{Q}}$ such that Γ acts on $H^j_{\text{ét}}(\mathcal{A}^{m_\xi}\times_F\overline{F}, \overline{\mathbb{Q}}_l(t_\xi))$ as $c_\xi \circ a_\xi$ if $j = n-1+m_\xi$ and zero if $j \neq n-1+m_\xi$. By construction we have an isomorphism of $\mathrm{Gal}(\overline{F}/F)$-representations

$$C_G \cdot \widetilde{R}'_l(\Pi')(-t_\xi) \simeq \Gamma \cdot H^{n-1+m_\xi}_{\text{ét}}(\mathcal{A}^{m_\xi} \times_F \overline{F}, \overline{\mathbb{Q}}_l).$$

Finally we apply the argument of [Sai03, Proposition 3.5],[7] noting that his condition (3) is satisfied for Γ as above. The conclusion is that any lift ϕ_v of Frob_v on $H^{n-1+m_\xi}_{\text{ét}}(\mathcal{A}^{m_\xi}_U\times_F\overline{F}, \overline{\mathbb{Q}}_l)$ has algebraic *integers* as eigenvalues. Hence, the WD representations associated with $H^{n-1}_{\text{ét}}(\mathrm{Sh}_U\times_{F_v}\overline{F}_v, \mathcal{L}_\xi)(-t_\xi)$ as well as $\widetilde{R}'_l(\Pi')(-t_\xi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ are integral. By (4.2), $WD(R_l(\Pi)(-t_\xi)\otimes \mathrm{rec}(\psi)^{-1}|_{\mathrm{Gal}(\overline{F}_v/F_v)})$ is integral. We have

$$\begin{aligned}\mathscr{L}_{F_v}(\Pi_v)|\cdot|_v^{-s(\Pi_\infty)/2} &= \iota_l WD(R_{l,\iota_l}(\Pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{\text{F-ss}}|\cdot|_v^{-s(\Pi_\infty)/2}\\ &= \iota_l WD(R_l(\Pi)(-t_\xi)\otimes\mathrm{rec}(\psi)^{-1}|_{\mathrm{Gal}(\overline{F}_v/F_v)})\otimes\iota_l WD(\mathrm{rec}(\psi_v)^{-1}|\cdot|_v^{\min(0,a_0)}).\end{aligned}$$

[7] The difference is that Saito considers the whole $H^{n-1+m_\xi}_{\text{ét}}$ of $\mathcal{A}^{m_\xi}_U \times_F \overline{F}$ whereas we argue only on its subrepresentation. Still it is easy to adapt his argument to our situation.

Since $WD(\mathrm{rec}(\psi_v)^{-1}|\cdot|_v^{-\max(0,a)})$ is integral by Lemma 4.4 below we are done with verifying the integrality of $\mathscr{L}_{F_v}(\Pi_v)|\cdot|_v^{-s(\Pi_\infty)/2}$ when n is odd.

It remains to justify integrality when n is even. The argument is essentially the same as above, so it would suffice to point out what modifications are needed. In this case we put ourselves in the setting of [Car12b], which shows that $R_l(\Pi)^{\otimes 2}$ is realized up to an explicit twist in $H^{2n-2}_{\text{ét}}(X,\mathcal{L}_\xi)$ for the $(2n-2)$-dimensional Shimura varieties therein. By arguing as above, we obtain $s(\Pi_\infty)\in\mathbb{Z}$ such that $WD(R_l(\Pi)^{\otimes 2}|_{\mathrm{Gal}(\overline{F}_v/F_v)})|\cdot|_v^{-s(\Pi_\infty)}$ is integral at every finite place v. But the latter implies that $WD(R_l(\Pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})|\cdot|_v^{-s(\Pi_\infty)/2}$ is integral as well. □

LEMMA 4.4. *Let F be any number field and $\psi: F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$ a Hecke character. At each infinite place v, suppose that there are some $m_v\in\mathbb{Z}_{\leqslant 0}$ and some continuous character $F_v^\times\to\mathbb{C}^\times$ such that $\psi_v(z)=\tau_v(z)^{m_v}$ for all $z\in(F_v^\times)^0$. Then for every finite place v and every uniformizer ϖ_v of F_v, $\psi_v(\varpi_v)$ is an algebraic integer.*

Proof. There exists an open compact subgroup U of $\widehat{\mathcal{O}}_F=\prod_{w\nmid\infty}\mathcal{O}_w^\times$ such that $\psi|_U\equiv 1$. Fix a finite place v and a uniformizer ϖ_v. By strong approximation there exists $a\in F^\times$ such that $a\in\varpi_v U$ in $(\mathbb{A}_F^\infty)^\times$. Since $a\in\widehat{\mathcal{O}}_F$, a is an algebraic integer. Now

$$\psi_v(\varpi_v)=\psi^\infty(a)=\psi_\infty(a)^{-1}=\pm\prod_{w|\infty}\tau_w(a)^{-m_v}$$

where the sign comes from the character $F_\infty^\times/(F_\infty^\times)^0$ with values in $\{\pm 1\}$. The lemma follows. □

4.2 Quasi-split classical groups

Later on several results will be established concerning automorphic representations of quasi-split[8] classical groups. To this end we would like to introduce basic data for symplectic, orthogonal, and unitary groups. Let F^+ be a totally real field. We take F to be F^+ in the symplectic and orthogonal cases and a CM quadratic extension of F^+ in the unitary case. Both G and $\mathbf{G}$ below will be connected reductive *quasi-split* groups over F^+. Let us suppress the choice of the symplectic, symmetric, or hermitian pairings.

Define $c\in\mathrm{Gal}(F/F^+)$ to be the identity if $F=F^+$ and the nontrivial element if $F\neq F^+$. For $n\geqslant 1$ let J_n denote the matrix with $(-1)^i$ in $(i,n+1-i)$th entry for $1\leqslant i\leqslant n$ and zeros off the anti-diagonal. Write θ_n (respectively $\widehat{\theta}_n$) for the automorphism $g\mapsto J_n{}^tg^{-c}J_n^{-1}$ of $\mathrm{Res}_{F/F^+}\mathrm{GL}_n$ over F^+ (respectively $g\mapsto J_n{}^tg^{-1}J_n^{-1}$ of $\mathrm{GL}_n(\mathbb{C})$). The standard embedding of a symplectic, special orthogonal, or general linear group will be denoted std.

In all cases below $n\in\mathbb{Z}_{\geqslant 1}$, $\mathbf{G}=\mathrm{Res}_{F/F^+}\mathrm{GL}_n$ (which is just GL_n except the unitary case), $\theta=\theta_n\in\mathrm{Aut}_{F^+}(\mathbf{G})$, $\widehat{\theta}=\widehat{\theta}_n\in\mathrm{Aut}_{F^+}(\widehat{\mathbf{G}})$, $s\in\widehat{G}$, and $\eta:{}^LG\hookrightarrow{}^L\mathbf{G}$ is an L-morphism. We will describe G, s, and η case-by-case. Only for even orthogonal groups, we use the Weil group form of an L-group in order to accommodate a half-integral twist, which is needed for η to be C-preserving.

(i) Symplectic groups: n is odd, $G=\mathrm{Sp}_{n-1}$, $s=1$,

$$\eta=(\mathrm{std},\mathrm{id}):\mathrm{SO}_n(\mathbb{C})\times\Gamma_F\hookrightarrow\mathrm{GL}_n(\mathbb{C})\times\Gamma_F.$$

(ii) Orthogonal groups: n is even, $s=1$,

[8] The analogous results for non quasi-split groups are sketched in the last chapter of [Art13] but might require a few more years for a complete proof.

(a) type B: $G = \mathrm{SO}_{n+1}$, $s = 1$,

$$\eta = (\mathrm{std}, \mathrm{id}) : \mathrm{Sp}_n(\mathbb{C}) \times \Gamma_F \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times \Gamma_F;$$

(b) type D: let $\delta \in F^\times/(F^\times)^2$ be the discriminant of the underlying quadratic form and $F_\delta := F(\delta^{1/2})$.

- We have $G = \mathrm{SO}_n$, $\delta = 1$ so that G is a split group, $s = 1$, $\eta_0 = (\mathrm{std}, \mathrm{id}) : \mathrm{Sp}_n(\mathbb{C}) \times \Gamma_F \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times \Gamma_F$ and define

$$\eta : \mathrm{SO}_n(\mathbb{C}) \times W_F \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times W_F, \quad \eta := \eta_0 |\cdot|^{1/2}$$

  where $|\cdot|$ is the modulus character on W_F.
- We have $G = \mathrm{SO}_n$, $\delta \neq 1$ so that G is a non-split group, ${}^L G = \mathrm{SO}_n(\mathbb{C}) \rtimes \Gamma_F$ (with Γ_F acting through $\mathrm{Gal}(F_\delta/F)$ on $\mathrm{SO}_n(\mathbb{C})$ via order 2 outer automorphism); $s = \mathrm{diag}(-I_n, I_n)$, $\eta_0 : \mathrm{SO}_n(\mathbb{C}) \rtimes \Gamma_F \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times \Gamma_F$ is an extension of the standard embedding $\mathrm{SO}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ defined on [Wal10, p. 51] (the map ${}^L\xi$ in case $d^- = n$, $d^+ = 1$, and $\delta^- = \delta \neq 1$). Define

$$\eta : \mathrm{SO}_n(\mathbb{C}) \rtimes W_F \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times W_F, \quad \eta := \eta_0 |\cdot|^{1/2}.$$

(ii)$'$ This is a subcase of case (ii); in case (ii)(a) it is the same as above; in case (ii)(b) further assume that $\delta = 1$ if $n/2$ is even and $\delta \neq 1$ if $n/2$ is odd.

(iii) Unitary groups: $G = U_n$, $s = 1$, ${}^L G = \mathrm{GL}_n(\mathbb{C}) \rtimes \Gamma_{F^+}$ (with Γ_{F^+} acting through $\mathrm{Gal}(F/F^+) = \{1, c\}$, the c-action being $\widehat{\theta}_n$),$\theta = \theta_n$,

$$\eta : \mathrm{GL}_n(\mathbb{C}) \times \Gamma_F \hookrightarrow (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \rtimes \Gamma_F, \quad g \times \gamma \mapsto (g, J_n {}^t g^{-1} J_n^{-1}) \rtimes \gamma.$$

Set $\epsilon := 0$ in case (i), (ii)(a), and (iii) and $\epsilon := 1$ in case (ii)(b). This auxiliary constant accounts for the modulus character in the definition of η.

The reason for introducing case (ii)$'$ is the following: a classical group G over F^+ admits discrete series at real places (equivalently admits compact maximal tori) exactly when G belongs to case (i), (ii)$'$, or (iii). In each of cases (i), (ii), and (iii), (G, s, η) is a twisted endoscopic datum for $\mathbf{G} \rtimes \langle\theta\rangle$ in the sense of [KS99], cf. [Art13, § 1.2] for symplectic and orthogonal groups. Observe that in all cases

$$\mathbf{G}(\mathbb{A}_{F^+}) = \mathrm{GL}_n(\mathbb{A}_F).$$

Lemma 4.5. *Put ourselves in case (i), (ii), or (iii) as above. Let v be an infinite place of F^+ as above:*

- *if $\varphi_v : W_{F_v^+} \to {}^L G$ is regular (i.e. the restriction $\varphi_v|_{W_{\overline{F}_v^+}}$ is not invariant under any nontrivial Weyl element, cf. Definition 2.1), then $\eta\varphi_v$ is also regular;*
- *η is C-preserving (Definition 2.21).*

Proof. Both assertions are checked by explicit computations with root data. We will verify the first assertion in case (i) and leave it to the reader in the other cases. We may choose maximal tori $\widehat{T}$ and $\widehat{\mathbf{T}}$ of G and GL_n and $\mathbb{Z}$-bases $\{e_i\}$ and $\{f_j\}$ for the cocharacter groups $(1 \leqslant i \leqslant (n-1)/2$, $1 \leqslant j \leqslant n)$ and such that η restricts to $\widehat{T} \hookrightarrow \widehat{\mathbf{T}}$ inducing $e_i \to f_i - f_{n+1-i}$ on the cocharacter

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

groups. It suffices to show that every regular element of $X_*(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ maps to a regular element of $X_*(\widehat{\mathbf{T}}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Let $\sum_i a_i e_i$ with $a_i \in \mathbb{C}^\times$ be regular. Since the Weyl group is generated by permutation of the indices i and sign changes $e_i \to -e_i$, the regularity means that a_i's are distinct and that $a_i \neq 0$. Then the image $\sum_i a_i(f_i - f_{n+1-i})$ has the property that the coefficients of f_j are all distinct, i.e. has trivial stabilizer under the Weyl group action in GL_n. This shows that regularity is preserved under η in case (i).

It is easy to compute the half sum of positive roots to verify C-preservation in each case. We only deal with case (ii)(b) to explain the role of the extra half-power twist there. Choose $\widehat{T}$, $\widehat{\mathbf{T}}$, $\{e_i\}$, and $\{f_j\}$ ($1 \leqslant i \leqslant n/2$, $1 \leqslant j \leqslant n$) similarly such that η_0 induces $e_i \to f_i - f_{n+1-i}$ on $X_*(\widehat{T}) \to X_*(\widehat{\mathbf{T}})$. The Borel subgroups can be chosen such that the half sum of positive roots is $\rho_G = (n/2-1)e_1 + (n/2-2)e_2 + \cdots + e_{n/2-1}$ for G (respectively $\rho_{\mathrm{GL}_n} = ((n-1)/2)f_1 + ((n-3)/2)f_2 + \cdots + ((1-n)/2)f_n$). So $\eta_0(\rho_G) = (n/2-1)(f_1 - f_n) + \cdots + (f_{n/2-1} - f_{n/2+1})$, and $\eta(\rho_G) = \eta_0(\rho_G) + \frac{1}{2}(f_1 + f_2 + \cdots + f_n)$. Hence, $\rho_{\mathrm{GL}_n} - \eta(\rho_G)$ has integral coefficients in f_j, showing that η is C-preserving (but note that η_0 is not). □

Lemma 4.6. *Assertions (i), (ii), and (iii) of Lemma 2.25 hold true in the even orthogonal case (ii)(b) (even though η does not satisfy the hypothesis in that lemma).*

Proof. This is proved by the same argument in the proof of assertion (i) in Lemma 2.25 for $\eta_0 : {}^L G \to {}^L \mathbf{G}$ □

4.3 Twisted endoscopic transfer for classical groups

We would like to recall elements of local twisted endoscopy at a *non-archimedean* place v of F^+ as these will be important to us. (The corresponding theory at archimedean places is well known.) Kottwitz, Langlands, and Shelstad [LS87, KS99] defined transfer factors $\Delta_v(\gamma_G, \gamma_{\mathbf{G}})$ for all strongly regular semisimple elements $\gamma_G \in G(F_v^+)$ and $\gamma_{\mathbf{G}} \in \mathbf{G}(F_v^+)$ at every place v of F^+. In fact, we will use the Whittaker normalization of transfer factors, to be denoted Δ_v^{Wh}, which were defined in [KS99, § 5.3] in the quasi-split case.[9] We say that $\phi_v \in C_c^\infty(G(F_v^+))$ is a Δ_v^{Wh} transfer of $f_v \in C_c^\infty(\mathbf{G}(F_v^+))$ if

$$STO_\delta^{\mathbf{G}(F_v^+)}(f_v) = \sum_{\gamma \sim_{\mathrm{st}} \delta} \Delta_v^{\mathrm{Wh}}(\gamma, \delta) O_\gamma^{G(F_v^+)}(\phi_v) \tag{4.3}$$

for every pair $(\gamma_G, \gamma_{\mathbf{G}})$ of strongly regular semisimple elements. The proof of the fundamental lemma by Ngô, Waldspurger, and others (see [Ngo10, Wal97, Wal06, Wal08]) ensures that a Δ_v^{Wh}-transfer of f_v exists for every f_v as above.

Proposition 4.7. *In cases (i), (ii), and (iii) of the previous subsection,*

$$\Delta_v^{\mathrm{Wh}}(\gamma_G, \gamma_{\mathbf{G}}) \in \mathbb{Q} \tag{4.4}$$

for all strongly regular semisimple elements $\gamma_G \in G(F_v^+)$ and $\gamma_{\mathbf{G}} \in \mathbf{G}(F_v^+)$. (In fact, $\Delta_v^{\mathrm{Wh}}(\gamma_G, \gamma_{\mathbf{G}}) \in \{0, \pm 1\}$ with the exception of case (ii)(b).)

Proof. We will only sketch the argument. Since Δ_v^{Wh} differs from Δ_0 of [KS99, § 5.3] by ± 1 [KS99, p. 65] it suffices to prove the claim for Δ_0. The transfer factors for classical groups were

[9] Here Δ_v^{Wh} depends on the extra choice of Whittaker data of [KS99, § 5.3], which will be chosen globally. The reference to this choice will be suppressed as the transfer factors are only affected by sign, cf. [KS99, p. 65], and do not affect the asserted rationality of transfer factors.

computed in [Wal10]. In the cases of interest, it is shown that the transfer factor $\Delta_I\Delta_{II}\Delta_{III}$ belongs to $\{0,\pm1\}$ for (G,s,η) as in cases (i), (ii)(a), (iii) and for (G,s,η_0) as in case (ii)(b). (See the cases of twisted linear groups in [Wal10, § 1.10], noting that χ is a character of order dividing two in the odd twisted linear case and that μ^- and μ^+ may be chosen to be trivial in the case of base change for unitary groups.) Note that Waldspurger suppressed Δ_{IV} in his formulas but the transfer factor Δ_0 is $\Delta_I\Delta_{II}\Delta_{III}\Delta_{IV}$. In cases (i), (ii)(a), and (iii) we see $\Delta_{IV}=1$ following the definition of [KS99, § 4.5], so the values of Δ_0 range in $\{0,\pm1\}$. In case (ii)(b) Δ_{IV} is a nontrivial function involving a half-power of the modulus character (cf. [KS99, (4.5.1)]) so Δ_0 for (G,s,η_0) takes values in $\mathbb{Q}(q_v^{1/2})$ but replacing η_0 with η twists the transfer factor by an extra half-power of the modulus character. As a result Δ_0 with respect to (G,s,η) has values in $\mathbb{Q}$. □

From here until § 6.5 we will restrict our attention to cases (i) and (ii) above. Case (iii) is excluded until there only because our understanding of representations of unitary groups is still limited. Nevertheless we will treat all three kinds of classical groups on an equal footing at the expense of burdening notation (e.g. we distinguish between F and F^+, which is unnecessary in cases (i) and (ii)) so that the results in this article apply to unitary groups as soon as the analogue of [Art13] for unitary groups is worked out. In fact, our results already produce some partial results in the case of unitary groups by appealing to the progress on twisted endoscopy (base change) for unitary groups in [KK05], [Moe07], and [Lab11] among others.

To use results for automorphic representations on quasi-split classical groups as in [Art13] (symplectic and orthogonal) and [Mok12] (unitary), we assume that the following hypothesis holds.[10]

Hypothesis 4.8. Suppose that the twisted trace formula for GL_n and twisted even orthogonal groups can be stabilized in the sense of [Art13, Hypothesis 3.2.1] and [Mok12, Hypo 4.2.1].

Even though we do not to strive to extract an optimal partial result from the current knowledge, see § 6.5 for some unconditional results not replying on the above hypothesis in case G is unitary. Now recall from § 4.2 that $\epsilon=0$ except for case (ii)(b) where $\epsilon=1$.

Proposition 4.9. *For every (finite and infinite) place v of F^+, every $f_v\in C_c^\infty(\mathbf{G}(F_v^+))$, every Δ_v^{Wh}-transfer $\phi_v\in C_c^\infty(G(F_v^+))$ of f_v, and every tempered L-parameter $\varphi_v: W_{F_v^+}\times\mathrm{SL}_2(\mathbb{C})\to {}^LG$, we have an identity*

$$\sum_{\pi_v\in\widetilde{LP}(\varphi_v)}\Theta_{\pi_v}(\phi_v)=\Theta_{\Pi_v,\theta}(f_v)\quad\text{where }\Pi_v=\mathrm{rec}^{-1}(\eta\varphi_v)|\cdot|^{\epsilon/2}\tag{4.5}$$

[10] One can be optimistic that the hypothesis will become unnecessary before long. At the time of revision, Waldspurger has released a series of five preprints (more to come) on the stabilization of the general twisted trace formula. For an extra careful reader, we remark that both [Art13] and [Mok12] depend on the papers [A25] and [A26] of [Art13], which have not appeared up to now, and that the proof of the weighted fundamental lemma has not been completely written up, cf. the footnote in [BMM11, Appendix A]. [Mok12, Proposition 8.2.5] asserts that Ban's result, cited as [Ban] there and proved only for split groups, extends to quasi-split unitary groups but this appears to be a nontrivial point to be justified. Arthur, as well as Mok, refers to work in progress by Mezo and Shelstad on twisted endoscopy for real groups and by Waldspurger on the local twisted trace formula. This seems fine: The former is basically addressed in the preprints cited as [Me] and [S8] in [Art13]; they have been updated or expanded since Arthur's book was published. The latter appeared in the preprint 'La formule des traces locales tordue'.

for a unique finite subset $\widetilde{LP}(\varphi_v)$ *of* $\mathrm{Irr}^{\mathrm{temp}}(G(F_v^+))$ *(independent of* f_v *and* ϕ_v*). The subsets* $\widetilde{LP}(\varphi_v)$ *give a partition of* $\mathrm{Irr}^{\mathrm{temp}}(G(F_v^+))$ *where* $\widetilde{LP}(\varphi_v)$ *and* $\widetilde{LP}(\varphi'_v)$ *coincide exactly when* $\eta\varphi_v$ *is equivalent to* $\eta\varphi'_v$ *as L-parameters for* $\mathbf{G}(F_v^+)$ *(and are disjoint otherwise).*

Remark 4.10. Even though this would be clear to the reader, let us clarify the meaning of Π_v in the proposition when v splits as ww^c in F, which can only happen in the unitary case (then $G(F_v^+)$ is a general linear group). Then $F_v = F \otimes_{F^+} F_v^+ \simeq F_w \times F_{w^c}$, thereby one may write $\Pi_v = \Pi_w \otimes \Pi_{w^c}$. On the other hand, $\eta\varphi_v : W_{F_v^+} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_{F^+}(F,\mathbb{C})}$ determines an L-parameter Φ_w for $\mathrm{GL}_n(F_w)$ and an L-parameter Φ_{w^c} for $\mathrm{GL}_n(F_{w^c})$. Then $\Pi_v = \mathrm{rec}^{-1}(\eta\varphi_v)$ is defined by $\Pi_w = \mathrm{rec}^{-1}(\Phi_w)$ and $\Pi_{w^c} = \mathrm{rec}^{-1}(\Phi_w)$ in the usual sense. Actually in this case, $\Pi_v = \pi_v \otimes \pi_v$ where π_v is the unique member of $\widetilde{LP}(\varphi_v)$. Similarly if $v = ww^c$ in F in the setting of Corollary 4.16, we interpret $\mathscr{L}_{F_v}(\Pi_v)$ and $|\cdot|_v$ as $\mathscr{L}_{F_w}(\Pi_w) \otimes \mathscr{L}_{F_{w^c}}(\Pi_{w^c})$ and $|\cdot|_w| \cdot |_{w^c}$, respectively.

Remark 4.11. The set $\widetilde{LP}(\varphi_v)$ is the local L-packet for φ_v except when G is an even orthogonal group, in which case it is a union of one or two L-packets. See the discussion above and below the [Art13, Theorem 1.5.1]. Our notation $\widetilde{LP}(\varphi_v)$ corresponds to his $\widetilde{\Pi}_\phi$.

Remark 4.12. Arthur also proved that when φ_v is a non-tempered A-parameter, the analogue of (4.5) holds true if $\Theta_{\pi_v}(\phi_v)$ are summed with suitable signs. We will not need this for our theorems.

Proof. This is part of the main local theorem by Arthur [Art13, Theorems 1.5.1 and 2.2.1] when G is symplectic or orthogonal and by Mok [Mok12, Theorems 2.5.1 and 3.2.1] when G is unitary. □

The above proposition tells us that for each $\pi_v \in \mathrm{Irr}^{\mathrm{temp}}(G(F_v^+))$ there is a unique (up to equivalence) tempered L-parameter φ_v such that $\pi_v \in \widetilde{LP}(\varphi_v)$. In this case we will write

$$\eta_*(\pi_v) := \mathrm{rec}^{-1}(\eta\varphi_v)|\cdot|^{\epsilon/2}, \tag{4.6}$$

cf. (4.5). Namely $\eta_* : \mathrm{Irr}^{\mathrm{temp}}(G(F_v^+)) \to \mathrm{Irr}^{\mathrm{temp}}(\mathbf{G}(F_v^+))$ denotes the local functorial lifting given by η.

Proposition 4.13. *There exists* $m_G \in \mathbb{Z}_{\geqslant 1}$ *such that for every finite place* v *and every tempered L-parameter* φ_v*,* $|\widetilde{LP}(\varphi_v)| \leqslant m_G$*. To be explicit, one can choose* $m_G = 2^n$*.*

Proof. When G is symplectic or orthogonal, the [Art13, Theorem 2.2.1.(b)] says that there is a bijection between $\widetilde{LP}(\varphi_v)$ and the set of characters on the group $\mathcal{S}_{\varphi_v}$ (denoted $\mathcal{S}_\psi$ in therein). According to [Art13, (1.4.9)] $\mathcal{S}_{\varphi_v}$ is an abelian group whose order divides 2^n so the proposition follows. In the case of unitary groups one argues similarly using [Mok12, (2.4.14)]. □

Now we summarize some results on the global functoriality for classical groups that we will need. (A good number of cases also follow from the method of converse theorem and integral representations but we do not discuss them here.) We will cite only [Art13] (which treats symplectic/orthogonal groups) in the remainder of this subsection without further comments on the unitary group case, believing that the reader understands by now that the completely analogous results in the latter case can be found easily in [Mok12].

Let us introduce some new notation, which is mostly consistent with that of [Art13] but not always. Write $\widetilde{\Psi}_{\mathrm{ell}}(\mathrm{GL}_n)$ for the set of quadruples

$$\psi = (r, \{(n_i, \Pi_i, \nu_i)\}_{i=1}^r)$$

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

(in which (n_i, Π_i, ν_i) are unordered relative to the index i) where:

- $r \in \mathbb{Z}_{\geqslant 1}$, $n_i \in \mathbb{Z}_{\geqslant 1}$, $\nu_i \in \mathbb{Z}_{\geqslant 1}$, $\sum_{i=1}^r n_i\nu_i = n$;
- Π_i are cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, and such that $\Pi_i^\vee \simeq \Pi_i^c$ for every i and $\Pi_i \not\simeq \Pi_j$ for ever pair $i \neq j$.

Let $\widetilde{\mathcal{E}}^{\mathrm{ell}}(\mathrm{GL}_n)$ denote the set of isomorphism classes of (twisted) endoscopic data for $\mathbf{G} \rtimes \langle\theta\rangle$ as defined in [Art13, § 1.2]. Whether we are in case (i), (ii), or (iii), (G, s, η) belongs to $\widetilde{\mathcal{E}}^{\mathrm{ell}}(\mathrm{GL}_n)$. (In case (iii) our η corresponds to the L-morphism ξ_{χ_+} of [Mok12]. His ξ_{χ_-} is not used in our paper.) According to a classification of self-dual parameters as in [Art13, § 1.2], there is a natural decomposition

$$\widetilde{\Psi}_{\mathrm{ell}}(\mathrm{GL}_n) = \coprod_{H \in \widetilde{\mathcal{E}}^{\mathrm{ell}}(\mathrm{GL}_n)} \widetilde{\Psi}_2(H)$$

so that ψ belongs to $\widetilde{\Psi}_2(H)$ if, loosely speaking, it satisfies the characterizing properties of the parameters coming from H. See the paragraph preceding [Art13, (1.4.7)].

Let us explain the construction of local parameters from $\psi \in \widetilde{\Psi}_{\mathrm{ell}}(\mathrm{GL}_n)$. Put $\mathcal{L}_{F_v^+} := W_{F_v^+} \times \mathrm{SL}_2(\mathbb{C})$ if $v \nmid \infty$ and $\mathcal{L}_{F_v^+} := W_{F_v^+}$ if $v|\infty$. Define

$$\psi_v : \mathcal{L}_{F_v^+} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$$

to be the L-parameter for $\mathbf{G}(F_v^+)$ given by $\bigoplus_{i=1}^r \mathrm{rec}(\Pi_i) \otimes \mathrm{Sym}^{\nu_i - 1}(\mathbb{C}^2)$, where each direct summand is the exterior tensor product of $\mathrm{rec}(\Pi_i)$ on $\mathcal{L}_{F_v^+}$ and $\mathrm{Sym}^{\nu_i-1}(\mathbb{C}^2)$ on $\mathrm{SL}_2(\mathbb{C})$. If $\psi \in \widetilde{\Psi}_2(G)$, then it is a nontrivial theorem that ψ_v (or an isomorphic parameter thereof) factors through only $\eta : {}^LG \hookrightarrow {}^L\mathbf{G}$ and no embedding of other elliptic endoscopic groups. (See the Theorem 1.4.2 and the discussion above [Art13, (1.5.3)].) This determines $\psi_v^\flat : \mathcal{L}_{F_v^+} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ such that $\eta\psi_v^\flat \simeq \psi_v$ canonically up to $\mathrm{Out}(G)$-action. (The outer automorphism group has order 1 or 2. See [Art13, § 1.2] for details.) It turns out that ψ_v always lands in $\widetilde{\Psi}^+_{\mathrm{unit}}(G(F_v^+))$ in the notation of Arthur, which is designed to accommodate local components of discrete automorphic representations of G. The precise definition of $\widetilde{\Psi}^+_{\mathrm{unit}}(G(F_v^+))$ is not needed for our purpose so not recalled here.

Now we turn to the purely local setting and explain some local inputs beyond the tempered objects to be used in this paper. Arthur associates to each $\psi_v \in \widetilde{\Psi}^+_{\mathrm{unit}}(G(F_v^+))$ (which may not come from a global parameter ψ) a finite set $\widetilde{\Pi}_{\psi_v}$ consisting of finite length $G(F_v^+)$-representations by extending the definition of tempered L-packets, i.e. $\widetilde{\Pi}_{\psi_v}$ is the tempered L-packet (cf. the paragraph above Proposition 4.13) if ψ_v is a tempered L-parameter. Although $\widetilde{\Pi}_{\psi_v}$ is designed to play the role of local A-packets, it should be noted that members of $\widetilde{\Pi}_{\psi_v}$ may be reducible or non-unitary. Let us define $\widetilde{AP}(\psi_v)$ to be the set consisting of irreducible subquotients of the members of $\widetilde{\Pi}_{\psi_v}$.

PROPOSITION 4.14. *Consider cases (i), (ii), or (iii) of § 4.2. Suppose that π is a discrete automorphic representation of $G(\mathbb{A}_{F^+})$ unramified outside a finite set S. Then there exists a unique $\psi = (r, \{(n_i, \Pi_i, \nu_i)\}_{i=1}^r) \in \widetilde{\Psi}_2(G)$ such that:*

(i) *$\pi_v \in \widetilde{AP}(\psi_v)$ at every place v of F^+;*

(ii) *if π_v is tempered and all ν_i are trivial, then $\eta_*(\pi_v) = \boxplus_{i=1}^r \Pi_{i,v}|\cdot|^{\epsilon/2}$ at each place v of F^+;*

(iii) *at every finite place $v \notin S$, π_v is isomorphic to the unramified member of $\widetilde{AP}(\psi_v)$, which is unique (relative to the fixed hyperspecial subgroup U_v^{hs}).*

Remark 4.15. In case some ν_i is nontrivial so that we are in the nontempered case, one knows from [Art13] only an equality of infinitesimal characters (i.e. supercuspidal support when v is a

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

finite place) in case (ii). If we knew the Ramanujan conjecture for general linear groups, it would be enough assume in case (ii) only that ν_i are trivial.

Proof. The first assertion is implied by [Art13, Theorem 1.5.2]. In case (ii) $\eta_*(\pi_v)$ is characterized by Proposition 4.9, so the assertion follows from [Art13, (2.2.3)].

The last assertion is deduced from the [Art13, Theorem 1.5.1], which implies that $\widetilde{AP}(\psi_v)$ possesses at most one unramified representation. (One can identify π_v a little more explicitly. When π_v is unramified, ψ_v is also unramified (i.e. all Π_i are unramified at v). Then $\widetilde{AP}(\psi_v)$ contains a local L-packet for the unramified L-parameter given by ψ, cf. [Art13, Proposition 7.4.1], so π_v is that corresponding to the latter L-parameter via the unramified Langlands correspondence.) $\square$

Corollary 4.16. *In the setting of Proposition 4.14, let $\psi = (r, \{(n_i, \Pi_i, \nu_i)\}_{i=1}^r)$ be the associated data to π and suppose that π is ξ-cohomological. Then there exists $s(\xi) \in \mathbb{Z}_{\geqslant 0}$ depending only on G and ξ such that:*

- *for every finite place v, $\mathscr{L}_{F_v}(\Pi_{i,v}|\cdot|^{(\epsilon+n_i-n/2)})|\cdot|_v^{-s(\xi)/2}$ is pure of weight $s(\xi)+n-1-\epsilon$ and integral.*

If moreover the highest weight of ξ is regular, then:

- *π_v are tempered at all places v;*
- *$\eta\varphi_{\pi_v}$ are pure WD representations of weight $-\epsilon$ for all finite places v;*
- *$\eta\varphi_{\pi_v}$ are unramified and strictly pure of weight $-\epsilon$ if $v \notin S$.*

Proof. Let us begin by proving the first assertion. Proposition 4.14(i) at infinite places implies, by the comparison of infinitesimal characters, that $\eta\psi_w|_{W_{\overline{F_w}}}$ is isomorphic to the direct sum over all infinite places w of F of the L-parameter for $\Pi_{i,w}$ restricted to $W_{\overline{F_w}}$. (Of course $\overline{F_w} \simeq \mathbb{C}$ for $w|\infty$.) Since π is ξ-cohomological thus regular and C-algebraic, Lemma 4.5 implies that $\eta\psi_w$ is a regular C-algebraic parameter. From this it follows that $\Pi_{i,w}|\cdot|^{(\epsilon+n_i-n)/2}$ at $w|\infty$ are regular and C-algebraic. One deduces from Proposition 4.1 that there exists $s(\Pi_{i,\infty}) \in \mathbb{Z}_{\geqslant 0}$ depending only on the infinite component $\Pi_{i,\infty}$ of Π_i such that $\mathscr{L}_{F_y}(\Pi_{i,y}|\cdot|^{(\epsilon+n_i-n)/2)})|\cdot|_y^{-s(\Pi_{i,\infty})/2}$ is pure of weight $s(\Pi_{i,\infty})+n-1-\epsilon$ and integral for each $1 \leqslant i \leqslant r$ for every finite place y of F. Clearly there are only finitely many $W_{\overline{F_w}}$-subrepresentation of $\eta\psi_w|_{W_{\overline{F_w}}}$, so the number of all possible infinitesimal characters for $\Pi_{1,w}, \ldots, \Pi_{r,w}$ is finite at each $w|\infty$. Since there are only finitely many irreducible representations of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ with fixed $m \in \mathbb{Z}_{\geqslant 1}$ and fixed infinitesimal character, there are only finitely many possibilities for $\Pi_{i,\infty}$. The proof of the first assertion is complete as soon as $s(\xi)$ is taken to be the maximum of $s(\Pi_{i,\infty})$ over all possible $\{\Pi_{i,\infty}\}_{1\leqslant i\leqslant r}$.

Now suppose that the highest weight of ξ is regular. According to a standard result on Lie algebra cohomology, π_v at $v|\infty$ must be discrete series to be ξ-cohomological. Considering infinitesimal characters for ψ_v at $v|\infty$, we see that $\nu_i = 1$ for all $1 \leqslant i \leqslant r$. Since $\Pi_i|\cdot|^{(\epsilon+n_i-n)/2}$ is of type (TR) or (CM) for each i, Proposition 4.1 tells us that $\Pi_{i,v}$ are essentially tempered at all finite places v. Since $\Pi_{i,v}$ is already known to be unitary, $\Pi_{i,v}$ is tempered. Hence, ψ_v is tempered and $\widetilde{AP}(\psi_v)$ is nothing but the tempered L-packet $\widetilde{LP}(\psi_v|_{\mathcal{L}_{F_v^+}})$ at each $v \nmid \infty$, cf. Proposition 4.9. In particular, $\pi_v \in \widetilde{AP}(\psi_v)$ is tempered. Since

$$\eta\varphi_{\pi_v} = \bigoplus_{i=1}^{r} \mathrm{rec}(\Pi_{i,v})|\cdot|^{\epsilon/2} = \bigoplus_{i=1}^{r} \mathscr{L}_{F_v^+}(\Pi_{i,v}|\cdot|^{(\epsilon+n_i-n)/2)})|\cdot|^{n-1/2}, \quad v \nmid \infty,$$

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

Proposition 4.1 and Remark 4.2 allow us to verify the properties of $\eta\varphi_{\pi_v}$ in the corollary. □

5. Finiteness results

The first two subsections prove local finiteness results for unramified and arbitrary representations. After stating a global finiteness conjecture (Conjecture 5.10 below) for C-algebraic representations with bounded coefficient fields in a fairly general setting, we establish the conjecture for general linear groups and quasi-split classical groups.

5.1 Finiteness for unramified representations

Put ourselves in the setting of § 4.2.

Lemma 5.1. *Fix $s \in \mathbb{Z}_{\geqslant 0}$, $A \in \mathbb{Z}_{\geqslant 1}$, and a finite place v of F^+. There are only finitely many $\pi_v \in \mathrm{Irr}^{\mathrm{ur}}(G(F_v^+))$ such that:*

- $\mathscr{L}_{F_v^+}(\eta_*\pi_v)|\cdot|_v^{-s/2}$ *is strictly pure of weight $n-1+s-\epsilon$ and integral; and*
- $[\mathbb{Q}(\pi_v):\mathbb{Q}] \leqslant A$.

Proof. Since the map $\eta_* : \mathrm{Irr}^{\mathrm{ur}}(G(F_v^+)) \to \mathrm{Irr}^{\mathrm{ur}}(\mathbf{G}(F_v^+))$ has finite fibers (Lemma 2.23) it suffices to prove the finiteness of the set of $\Pi_v \in \mathrm{Irr}^{\mathrm{ur}}(\mathrm{GL}_n(F_v))$ such that $\mathscr{L}_{F_v^+}(\Pi_v)|\cdot|_v^{-s/2}$ is strictly pure of weight $n-1+s$ and integral with $[\mathbb{Q}(\Pi_v):\mathbb{Q}] \leqslant A$. A first observation is that any $\Pi_v = \eta_*\pi_v$ for π_v as in the lemma lands in the set just defined, where the inequality follows from Lemma 2.25(ii). Next consider the bijection $\mathscr{S} : \mathrm{Irr}^{\mathrm{ur}}(\mathrm{GL}_n(F_v)) \to (\mathbb{C}^\times)^n/S_n$ coming from the Satake isomorphism for GL_n. Then each complex number appearing in $\mathscr{S}(\Pi_v|\cdot|_v^{-(n-1+s-\epsilon)/2})$ must be a root of an (irreducible) monic polynomial $x^m + a_{m-1}x^{m-1} + \cdots + a_0$ with

$$1 \leqslant m \leqslant A, \quad a_{m-1},\ldots,a_0 \in \mathbb{Z} \tag{5.1}$$

by integrality and the bound on $[\mathbb{Q}(\Pi_v):\mathbb{Q}]$. The condition on purity and weight ('Weil bounds') implies that $|\lambda|_v \leqslant q_v^{-(n-1+s-\epsilon)/2}$ for all roots $\lambda \in \mathbb{C}$ of the above polynomial, imposing a constraint

$$|a_i|_v \leqslant \binom{m}{i} q_v^{-(n-1+s-\epsilon)/2} \quad \text{for all } 0 \leqslant i \leqslant m-1. \tag{5.2}$$

As there are only finitely many polynomials satisfying (5.1) and (5.2), we are done. □

5.2 Rationality of endoscopic transfer

Keep the notation of the previous subsection. We start by studying the behavior of the functorial lifting η_* relative to automorphisms of $\mathbb{C}$.

Proposition 5.2. *Let $\pi_v \in \mathrm{Irr}^{\mathrm{temp}}(G(F_v^+))$. Then*

$$\eta_*(\pi_v^\sigma) = (\eta_*\pi_v)^\sigma \quad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}). \tag{5.3}$$

If moreover $\mathbb{Q}(\pi_v)$ is finite over $\mathbb{Q}$ then:

(i) *$\mathbb{Q}(\eta_*\pi_v)$ is also finite over $\mathbb{Q}$;*

(ii) *$\mathbb{Q}(\pi_v)$ contains $\mathbb{Q}(\eta_*\pi_v)$ and is contained in a finite extension of $\mathbb{Q}(\eta_*\pi_v)$ of degree at most $m_G!$; in particular, $[\mathbb{Q}(\eta_*\pi_v):\mathbb{Q}] \leqslant [\mathbb{Q}(\pi_v):\mathbb{Q}] \leqslant m_G!\,[\mathbb{Q}(\eta_*\pi_v):\mathbb{Q}]$.*

Remark 5.3. Only the left inequality in part (ii) will be needed in our main results. The proposition extends Lemma 2.25 from unramified (possibly non-tempered) representations to tempered representations in the case of classical groups.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

Proof. Put $\Pi_v := \eta_*\pi_v$. Since part (i) is an immediate consequence of part (ii), it suffices to verify part (ii).

When π_v is tempered, we would like to verify (5.3). For any $f_v \in C_c^\infty(\mathbf{G}(F_v^+))$ let $\phi_v \in C_c^\infty(G(F_v^+))$ be its Δ_v^{Wh}-transfer. For every $\sigma \in \mathrm{Aut}(\mathbb{C})$ we obtain from Proposition 4.7 and (4.3) that

$$STO_\delta^{\mathbf{G}(F_v^+)}(f_v^\sigma) = \sum_{\gamma\sim_{\mathrm{st}}\delta} \Delta_v^{\mathrm{Wh}}(\gamma,\delta) O_\gamma^{G(F_v^+)}(\phi_v^\sigma).$$

Hence, ϕ_v^σ is a KLS transfer of f_v^σ. On the other hand, twisting (4.5) by σ leads to an identity

$$\Theta_{\Pi_v^\sigma,\theta}(f_v^\sigma) = \sum_{\eta_*(\rho_v)=\Pi_v} \Theta_{\rho_v^\sigma}(\phi_v^\sigma). \tag{5.4}$$

Plugging in $f_v^{\sigma^{-1}}$ and $\phi_v^{\sigma^{-1}}$ in place of f_v and ϕ_v (noting that $f_v^{\sigma^{-1}}$ is a Δ_v^{Wh}-transfer of $\phi_v^{\sigma^{-1}}$) we derive

$$\Theta_{\Pi_v^\sigma,\theta}(f_v) = \sum_{\eta_*(\rho_v)=\Pi_v} \Theta_{\rho_v^\sigma}(\phi_v). \tag{5.5}$$

Comparing with $\Theta_{\Pi_v^\sigma,\theta}(f_v) = \sum_{\eta_*(\rho_v)=\Pi_v^\sigma} \Theta_{\rho_v}(\phi_v)$, cf. (4.5), we obtain an equality of stable characters (evaluated on elements of $C_c^\infty(G(F_v^+))$)

$$\sum_{\eta_*(\rho_v)=\Pi_v} \Theta_{\rho_v^\sigma} = \sum_{\eta_*(\rho_v)=\Pi_v^\sigma} \Theta_{\rho_v} \tag{5.6}$$

since Δ_v^{Wh}-transfers of $C_c^\infty(\mathbf{G}(F_v^+))$ generate the space of stable distributions on $G(F_v^+)$. (In the language of Remark 1 below [Art13, Theorem 2.2.1], the map $\widetilde{f} \mapsto \widetilde{f}^G$ is *onto*.) Then (5.6) holds true also as the equality of finite character sums. Since $\Theta_{\pi_v^\sigma}$ appears as a summand on the left-hand side, it should also on the other side by linear independence of characters. We have established (5.3).

Formula (5.3) readily implies that if $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$ then $\Pi_v^\sigma = \eta_*(\pi_v^\sigma) = \eta_*\pi_v = \Pi_v$. Therefore, $\mathbb{Q}(\Pi_v) \subset \mathbb{Q}(\pi_v)$ and, in particular, $\mathbb{Q}(\Pi_v)$ is finite over $\mathbb{Q}$.

Now if $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi_v))$, then $\eta_*(\pi_v^\sigma) = \Pi_v^\sigma = \Pi_v$. One deduces from (5.6) that $\pi_v, \pi_v^\sigma \in \eta_*^{-1}(\Pi_v)$. Thereby one obtains a group homomorphism

$$\Upsilon : \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi_v)) \to \mathrm{Perm}(\eta_*^{-1}(\Pi_v)), \quad \Upsilon(\sigma) : \pi_v \mapsto \pi_v^\sigma$$

where $\mathrm{Perm}(\cdot)$ denotes the permutation group. Since $|\eta_*^{-1}(\Pi_v)| \leqslant m_G$ by Proposition 4.13, the kernel of Υ has index $\leqslant m_G!$ in $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi_v))$. Then the fixed field of $\ker \Upsilon$ is a finite extension of $\mathbb{Q}(\Pi_v)$ of degree $\leqslant m_G!$ and contains $\mathbb{Q}(\pi_v)$. The proof of part (ii) is finished. □

Remark 5.4. Alternatively (5.3) may be proved using a global argument (explained to us by Wee Teck Gan): reduce to the case where π_v is a discrete series. When π_v is discrete, globalize π_v to a π. Consider $\eta_*(\pi)$ and $\eta_*(\pi^\sigma)$ (assumed isobaric). By comparing $\eta_*(\pi)^\sigma$ and $\eta_*(\pi^\sigma)$ at almost all unramified places, one deduces from strong multiplicity one that $\eta_*(\pi_v)^\sigma = \eta_*(\pi_v^\sigma)$ at the place v of interest. (Use Clozel's result that the Langlands quotient is compatible with σ.)

5.3 Sparsity of arithmetic points in the unitary dual

PROPOSITION 5.5. *Fix $A \geqslant 1$, a finite place v of F^+, an open compact subgroup $K_v \subset \mathbf{G}(F_v^+)$ and an irreducible algebraic $\mathbf{G}_\infty$-representation Ξ. The set of $\Pi_v \in \mathrm{Irr}(\mathbf{G}(F_v^+))$ satisfying conditions*

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

(i) and (ii) below is finite:

(i) Π_v *appears as the* v*-component of some* Ξ*-cohomological isobaric representation* $\Pi = \boxplus_{i=1}^{s}\Pi_i$ *of* $\mathbf{G}(\mathbb{A}_{F^+})$ *such that* Π_i *are cuspidal and* $\Pi_i^\vee \simeq \Pi_i^c \otimes (\det \circ \chi_i)$ *for* $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ *such that* $\chi_v(-1)$ *is the same for every* $v \in S_\infty$*;*

(ii) $[\mathbb{Q}(\Pi_v) : \mathbb{Q}] \leqslant A$.

Remark 5.6. Since C-algebraicity is incompatible with $\boxplus$ (which is locally the Langlands quotient for the normalized induction), Π being C-algebraic implies not Π_i but $\Pi_i|\cdot|^{(n_i-n)/2}$ is C-algebraic in condition (i).

Proof. By condition (ii) and Corollary 3.13 the depth (or conductor) of Π_v is bounded, so the set of Π_v is contained in finitely many Bernstein components. We may show that the set of Π_v satisfying conditions (i) and (ii) is finite in each Bernstein component $\mathcal{B}$. Suppose that $\Pi_v^0 \in \mathcal{B}$ satisfies conditions (i) and (ii). Write $\mathscr{L}_{F_v^+}(\mathrm{sc}(\Pi_v^0)) = \bigoplus_{i=1}^k V_i$ where V_i are irreducible WD representations. For any other $\Pi_v \in \mathcal{B}$,

$$\mathscr{L}_{F_v^+}(\mathrm{sc}(\Pi_v)) = \bigoplus_{i=1}^{k} V_i \otimes \mathrm{unr}(\lambda_i)$$

where $\mathrm{unr}(\lambda_i) : \mathrm{GL}_1(F_v^+) \to \mathbb{C}^\times$ is the unramified character mapping every uniformizer of F_v^+ to $\lambda_i \in \mathbb{C}^\times$. Since $\mathbb{Q}(\bigoplus_{i=1}^k V_i) \subset \mathbb{Q}(\Pi_v^0)$ (cf. (3.2)), we have $[\mathbb{Q}(\bigoplus_{i=1}^k V_i) : \mathbb{Q}] \leqslant A$. Put $E := \mathbb{Q}(V_1)\mathbb{Q}(V_2)\cdots\mathbb{Q}(V_k)$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\bigoplus_{i=1}^k V_i))$ acts on $\{V_1, \ldots, V_k\}$ (a multi-set) as permutations,

$$[E : \mathbb{Q}] \leqslant k!A \leqslant n!A.$$

Consider the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the unordered set $\{V_i \otimes \mathrm{unr}(\lambda_i)\}_{i=1}^k$. Clearly there exists an extension E'/E of degree $\leqslant k!$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/E')$ fixes the isomorphism class of $V_i \otimes \mathrm{unr}(\lambda_i)$ for every i. Observe that every $\lambda \in \mathbb{C}^\times$ such that $V_i \otimes \mathrm{unr}(\lambda) \simeq V_i$ satisfies $\lambda^n = 1$. (For this consider the equality of the determinants.) Setting $E'' := E(\mu_n)$, we conclude that $\mathrm{Gal}(\overline{\mathbb{Q}}/E'')$ fixes λ_i for every i. In particular,

$$[\mathbb{Q}(\lambda_i) : \mathbb{Q}] \leqslant k!n!\varphi(n)A \quad \text{for all } 1 \leqslant i \leqslant k. \tag{5.7}$$

By condition (i) and Proposition 4.1 there exists $s > 0$ (depending on Ξ) such that for every $\Pi_v \in \mathcal{B}$ satisfying conditions (i) and (ii), $\mathscr{L}_{F_v^+}(\mathrm{sc}(\Pi_v))|\cdot|^{-s/2}$ is pure of weight $s + (n-1)$ and integral. (To deduce this, apply Proposition 4.1 to each $\Pi_i|\cdot|^{(n_i-n)/2}$, cf. Remark 5.6.) As Lemma 3.8 applies to the present situation with all s_i in the lemma equal to one, we see that both $V_i|\cdot|^{-s}$ and $V_i|\cdot|^{-s} \otimes \mathrm{unr}(\lambda_i)$ are strictly pure of weight in $s + (n-1)$ and integral. We claim that λ_i is a Weil q_v-number of weight zero such that $q_v^{s+(n-1)}\lambda_i$ is integral. Indeed, for any eigenvalue ω of a lift of geometric Frobenius on V_i, we know from the above that $q_v^{s+n-1/2}/\omega$ and $\omega\lambda_i$ are integral. In view of the integrality of $q_v^{s+(n-1)}\lambda_i$ and (5.7), an argument as in Lemma 5.1 shows that there are only finitely many λ_i with these properties. Therefore, the set of Π_v as above is finite. □

Let G be as in cases (i), (ii), or (iii) of § 4.2. Recall that Arthur and Mok associate to $\pi_v \in \mathrm{Irr}^{\mathrm{temp}}(G(F_v^+))$ a tempered L-parameter $\varphi_{\pi_v} : W_{F_v^+} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$. In a standard manner this extends to the construction of all L-packets via Langlands quotients. Now we are about to state a result providing a crucial input for the main results of § 6.1.

COROLLARY 5.7. *Fix* $A \geqslant 1$ *and an irreducible algebraic* $\mathrm{Res}_{F^+/\mathbb{Q}}G$ *representation* ξ*. Suppose that* ξ *has regular highest weight. Then the set of* $\pi_v \in \mathrm{Irr}^{\mathrm{temp}}(G(F_v^+))$ *satisfying the two*

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

properties below is finite:

(i) π_v *appears as the* v*-component of some* ξ*-cohomological discrete automorphic representation* π *of* $G(\mathbb{A}_{F^+})$*;*

(ii) $[\mathbb{Q}(\pi_v) : \mathbb{Q}] \leqslant A$.

Remark 5.8. In particular, the set of such π_v is measurable with respect to the Plancherel measure on $G(F_v^+)^\wedge$. We caution the reader that its measure may not be zero. Indeed it has positive Plancherel measure precisely when it contains discrete series.

Remark 5.9. The regularity assumption on ξ should be unnecessary for the corollary to be true. We imposed it for simplicity and also for the reason that the same hypothesis will be in place for applications in § 6.

Proof. Let $C(\xi, A)$ be the set of π_v as above. We need to show $|C(\xi, A)| < \infty$. Since η_* is a finite-to-one map, we will be done if η_* is shown to map $C(\xi, A)$ into the union of the sets of Proposition 5.5 for some Ξ, where Ξ depends only on ξ.

Each $\pi_v \in C(\xi, A)$ is the v-component of some π as in the corollary. Let $\psi = (r, \{(n_i, \Pi_i, \nu_i)\}_{i=1}^r$ be the data associated with π and put

$$\Pi := \boxplus_{i=1}^r ((\Pi_i \otimes |\cdot|^{1-\nu_i/2}) \boxplus (\Pi_i \otimes |\cdot|^{3-\nu_i/2}) \boxplus \cdots \boxplus (\Pi_i \otimes |\cdot|^{\nu_i - 1/2})) \otimes |\cdot|^{\epsilon/2}.$$

The infinitesimal character of π_v at each infinite place v of F^+ is the same as that of $\xi_v^\vee$, where $\xi = \bigotimes_{v|\infty} \xi_v$ is a tensor product of irreducible representations with regular highest weights. Remark 4.15 tells us that this infinitesimal character transfers via η to that of Π_v. The regularity on the former implies via the explicit description of η that the infinitesimal character of Π_v is the same as that of (the tensor product of two) irreducible algebraic representations of $\mathbf{G}$ of regular highest weight. In particular, ν_i must be all trivial in ψ.

So $\Pi = \boxplus_{i=1}^r \Pi_i |\cdot|^{\epsilon/2}$ and $\Pi_v = \eta_*(\pi_v)$ by Proposition 4.14(ii), cf. Remark 4.15. Thanks to Proposition 5.2 we know $[\mathbb{Q}(\Pi_v) : \mathbb{Q}] \leqslant A$, which is Proposition 5.5(ii). It remains to verify condition (i) of that proposition. This is clear except possibly the property that Π is cohomological (for some Ξ), which we now explain. Since π_∞ is ξ-cohomological it is regular C-algebraic. This implies that $\Pi_\infty = \eta_*(\pi_\infty)$ is also regular C-algebraic (for GL_n) by Lemma 4.5 (and the sentence right below Definition 2.21). Now Lemma 2.12 tells us that Π_∞ is Ξ-cohomological for some Ξ. Moreover, Ξ is determined by the infinitesimal character of Π_∞, hence also by that of π_∞, or just by ξ. We are done. □

5.4 A finiteness conjecture

We think this is a good place to state an interesting finiteness conjecture on automorphic representations in the spirit of the Shafarevich conjecture (Theorem 1.2). Earlier Fontaine and Mazur proposed the analogue of the Shafarevich conjecture for l-adic Galois representations [FM95, I.§ 3]. While their conjecture is still mostly open in dimension greater than one to the best of the authors' knowledge, we are able to verify *our* conjecture in many cases including $G = \mathrm{GL}_n$. This opens up the possibility for an automorphic proof of Fontaine–Mazur's finiteness conjecture via the Langlands correspondence. At the moment we are unable to get many cases of their conjecture since the correspondence is established in only limited cases. We wish to return to this problem in the future.

In the conjecture G is allowed to be an arbitrary connected reductive group over any number field F. Let S_{ram} be the finite set of finite places v such that $G \times_F F_v$ is ramified (i.e. either non-quasi-split over F_v or non-split over any finite unramified extension of F_v). Recall that hyperspecial subgroups outside S_{ram} are fixed as in § 1.6 once and for all, and unramified representations are considered with respect to this data. Denote by $Z(\mathfrak{g})$ the center of the universal enveloping algebra of $\operatorname{Lie} G(F \otimes_{\mathbb{Q}} \mathbb{C})$.

Conjecture 5.10. Fix $A \in \mathbb{Z}_{\geqslant 1}$, S a finite set of places of F containing S_{ram} and all infinite places, and a $\mathbb{C}$-algebra character $\chi_\infty : Z(\mathfrak{g}) \to \mathbb{C}$. Then the set of discrete automorphic representations π of $G(\mathbb{A}_F)$ with the following properties is finite:

- π^S is unramified;
- π_∞ has infinitesimal character χ_∞; and
- $[\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A$.

Remark 5.11. To state a more modest conjecture, one may replace the condition $[\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A$ by the condition that $\mathbb{Q}(\pi)$ is contained in a fixed finite extension of $\mathbb{Q}$ in $\mathbb{C}$.

Remark 5.12. Since there are up to isomorphism only finitely many π_∞ with a fixed infinitesimal character χ_∞, one may replace the above condition on the infinitesimal character by the condition that π_∞ is isomorphic to a fixed irreducible $G(F \otimes_{\mathbb{R}} \mathbb{C})$-representation π_∞^0.

Remark 5.13. The π as in the conjecture should be C-algebraic according to the 'if' part of Conjecture 2.8, which may well belong to the realm of transcendental number theory and would be difficult to check. Fortunately we can still verify the conjecture in many cases without a priori knowledge that π is C-algebraic; cf. § 5.5 below.

Remark 5.14. It should be stressed that no bound on ramification is imposed at places in S. (Otherwise the conjecture would be uninteresting.) Such a bound is only a consequence of the condition that $\mathbb{Q}(\pi) \subset E$, at least in the setting of § 5.5 below. The conjecture is certainly false if the condition $\mathbb{Q}(\pi) \subset E$ is omitted, as it is often well known that there are infinitely many discrete automorphic representations if arbitrary ramification is allowed at one place, cf. [Shi12].

Remark 5.15. On the Galois side (as opposed to the automorphic side) the analogues of the Fontaine–Mazur finiteness conjecture for complex and mod l Galois representations have been proposed and investigated by [ABCZ94] and [Kha00]. The result of Anderson, Blasius, Coleman and Zettler [ABCZ94] is as follows: given a number field K, there are finitely many complex representations of the Weil group W_K of bounded degree and bounded Artin conductor (their proof uses Jordan theorem that finite subgroups of $\mathrm{GL}(d, \mathbb{C})$ are virtually abelian). Their result confirms some very special cases of the Fontaine–Mazur finiteness conjecture.

5.5 Results on the finiteness conjecture

The aim of this section is to prove Conjecture 5.10 in some important cases. Namely the conjecture will be established first in the case of general linear groups taking Lemma 3.3 and Harish-Chandra's finiteness theorem (Proposition 5.16 below) as crucial inputs, and next in the case of classical groups via functorial transfer to general linear groups.

Proposition 5.16. *For any $\mathbb{C}$-algebra character $\chi_\infty : Z(\mathfrak{g}) \to \mathbb{C}$ and for any open compact subgroup U of $G(\mathbb{A}_F^\infty)$, the set of isomorphism classes of discrete automorphic representations π of $G(\mathbb{A}_F)$ satisfying the following is finite:*

- *π^∞ has a nonzero U-fixed vector; and*
- *π_∞ has infinitesimal character χ_∞.*

Remark 5.17. We are using a weaker version of Harish-Chandra's theorem in that our attention is restricted to the discrete automorphic spectrum.

Proof. The proposition results immediately from Harish-Chandra's theorem 1 in [Har68]. (The proof in [Har68] for semisimple groups is extended to the case of reductive groups as explained in [Bor07, Theorem 7.4].) □

Theorem 5.18. *Conjecture 5.10 is true in the case of $G = \mathrm{GL}_n$ for any $n \in \mathbb{Z}_{\geqslant 1}$ and any ground field F.*

Proof. Suppose that π satisfies the condition of Conjecture 5.10. Corollary 3.12 tells us that π_v has bounded conductor (depending only on A, F_v, and n) at every $v \in S$. Therefore, the cardinality of such π is finite by Proposition 5.16. □

Theorem 5.19. *Conjecture 5.10 is true for quasi-split classical groups as in cases (i), (ii), and (iii) of § 4.2 (if Hypothesis 4.8 is assumed).*

Proof. Write $C(G, S, \chi_\infty, A)$ for the set of Conjecture 5.10 (but we adopt the notation of § 4.2 in this proof, so F^+ plays the role of F in the conjecture). Consider the association

$$C(G, S, \chi_\infty, A) \to \widetilde{\Psi}_2(G)$$
$$\pi \mapsto \psi = (r, \{(n_i, \Pi_i, \nu_i)\}_{i=1}^r$$

as in Proposition 4.14. Since π is unramifiedoutside S, the associated Π_i enjoys the same property for every i. Since η is C-preserving by inspection, Lemma 2.25, Proposition 4.14(iii), and the strong multiplicity one theorem imply that $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))$ permutes the set $\{\Pi_1, \ldots, \Pi_r\}$. Hence, there exists some $E \supset \mathbb{Q}(\pi)$ with $[E : \mathbb{Q}(\pi)] \leqslant r!$ such that $\mathrm{Aut}(\mathbb{C}/E)$ fixes all $\Pi_1, \ldots, \Pi_r$. Note that $[E : \mathbb{Q}] \leqslant r![\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant n!A$.

Let us make some observation about infinitesimal characters. It is standard that χ_∞ corresponds to a collection of complex L-parameters $\varphi_{\chi_\infty, w} : W_{\mathbb{C}} \to \widehat{G}$ where w runs over the infinite places of F. For each infinite place v of F^+, it follows from $\pi_v \in \widetilde{AP}(\psi_v)$ that $\psi_v|_{W_{\overline{F_v^+}}}$ is isomorphic to $\varphi_{\chi_\infty, w}$ up to $\mathrm{Out}(\widehat{G})$-action when $v|w$. Let X_v be the set of all infinitesimal characters of $\mathrm{GL}_m(F \otimes_{F^+} F_v^+)$ with $1 \leqslant m \leqslant n$ corresponding to a $W_{\overline{F_v^+}}$ subrepresentation of $\eta\psi_v|_{W_{\overline{F_v^+}}}$ at each $v|\infty$. Clearly $X := \prod_{v|\infty} X_v$ is a finite set. Proposition 4.14, cf. the proof of Corollary 4.16, tells us that the infinitesimal character of $\Pi_{i,v}$ belongs to X_v.

Let $D(\mathrm{GL}_{\leqslant n}, S, X, n!A)$ be the set of cuspidal automorphic representations Π of $\mathrm{GL}_m(\mathbb{A}_F)$ with $1 \leqslant m \leqslant n$ which are unramified outside S, have the infinitesimal character of Π_∞ in X, and satisfy $[\mathbb{Q}(\Pi) : \mathbb{Q}] \leqslant n!A$. According to Theorem 5.18, $D(\mathrm{GL}_{\leqslant n}, S, X, n!A)$ is a finite set. We have seen that for any $(r, \{(n_i, \Pi_i, \nu_i)\}_{i=1}^r \in \widetilde{\Psi}_2(G)$ coming from $\pi \in C(G, S, \chi_\infty, A)$, every Π_i belongs to $D(\mathrm{GL}_{\leqslant n}, S, X, n!A)$. Hence, the image of $C(G, S, \chi_\infty, A)$ in $\widetilde{\Psi}_2(G)$ is finite. The proof boils down to showing that each fiber of the arrow $C(G, S, \chi_\infty, A) \to \widetilde{\Psi}_2(G)$ is finite. This is a consequence of Proposition 4.14(iii): if π is in the preimage of ψ then π_v at every finite place $v \notin S$ is determined to the unique unramified member of $\widetilde{AP}(\psi_v)$. For each $v \in S$ or $v|\infty$, π_v must lie in the finite set $\widetilde{AP}(\psi_v)$. □

6. Growth of fields of rationality in automorphic families

Let G be a quasi-split classical group as in cases (i), (ii)′, or (iii) of § 4.2 from here up to § 6.4. In particular F^+ denotes the base field of G. Note that this is different from the convention of [ST12], which will be frequently cited in this section, where F is the base field. (We restrict from case (ii) to (ii)′ since we need results from [ST12] established under the assumption that G has discrete series at the real places of F^+.) In § 6.5 we concisely explain how the earlier part of this section can be adapted to obtain an unconditional result for (non-quasi-split) unitary groups. Throughout this section it is assumed that G is nontrivial so that the absolute rank of G is at least one.

6.1 Growth of fields of rationality in level aspect

We start by recalling the level aspect families $\mathcal{F}_\xi(U_x)$ of automorphic representations of $G(\mathbb{A}_{F^+})$ of weight ξ and level subgroups U_x as in [ST12, § 9.3] or [Shi12].

Let U_x be a sequence of level $\mathfrak{n}_x$-subgroups of $G(\mathbb{A}^\infty_{F^+})$. Here $\mathfrak{n}_x$ is a sequence of integral ideals of $\mathcal{O}_{F^+}$ such that $\mathbb{N}(\mathfrak{n}_x) := [\mathcal{O}_{F^+} : \mathfrak{n}_x] \to \infty$. When G is a split group over F, the sequence U_x is defined as

$$U_x = \ker(G(\mathcal{O}_{F^+}) \to G(\mathcal{O}_{F^+}/\mathfrak{n}_x))$$

using the Chevalley group scheme for G over $\mathbb{Z}$. In general, we refer the reader to [ST12, § 8] for the precise definition via Moy-Prasad filtrations. We have a product decomposition $U_x = \prod_{v\nmid\infty} U_{x,v}$ such that each $U_{x,v}$ is a compact open subgroup of $G(F^+_v)$. Set $U^v_x := \prod_{w\nmid\infty, w\neq v} U_{x,w}$.

A variant of the level sequence would be a tower of bounded depth in the sense of [DH99], which corresponds to $U_{x+1} \subset U_x$ or $\mathfrak{n}_x \mid \mathfrak{n}_{x+1}$. But here we prefer to work more generally with the condition that $\mathbb{N}(\mathfrak{n}_x) \to \infty$.

Let ξ be an irreducible algebraic representation of $\mathrm{Res}_{F^+/\mathbb{Q}} G$ over $\mathbb{C}$, which can be viewed as a representation of $\prod_{v|\infty} G \times_{F^+} F^+_v$ where v runs over infinite places of F^+ (see § 2.2). In this section we assume, except for Corollary 6.8, that the highest weight for every representation of $G \times_{F^+} F^+_v$ induced by ξ is *regular*. The regularity assumption is made mainly because the equidistribution theorems as in [Shi12] and [ST12] rely on it. (This is why we also made the assumption earlier for simplicity, cf. Remark 5.9).

Let S_0 be a (possibly empty) finite set of places disjoint from $\mathfrak{n}_x$ for all x. Let $\widehat{f}_{S_0}$ be a well-behaved function on the unitary dual of $G(F^+_{S_0})$ in the sense of [Sau97, § 7] and [ST12, § 9.1]. (Such functions are very useful in prescribing interesting local conditions. Namely we can impose that π_{S_0} belongs to a bounded measurable subset of $G(F^+_{S_0})^\wedge$ whose boundary has zero Plancherel measure and whose image in the Bernstein variety ($\Theta(G)$ in [Sau97, pp. 164–165]) has compact closure. In fact, there is essentially no loss of generality in assuming that $\widehat{f}_{S_0}$ is a characteristic function of such a subset.) Henceforth we will assume that $\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{f}_{S_0}) > 0$ and that $\widehat{f}_{S_0}$ takes nonnegative real values on the unitary dual.

Let $\mathcal{F}_x := \mathcal{F}(U_x, \widehat{f}_{S_0}, \xi)$ be the set (or family) of discrete automorphic representations π of $G(\mathbb{A}_{F^+})$ such that:

- π_{S_0} belongs to the support of $\widehat{f}_{S_0}$, i.e. $\widehat{f}_{S_0}(\pi_{S_0}) \neq 0$;
- π^∞ has a nonzero U_x-fixed vector; and
- π_∞ is ξ-cohomological (Definition 2.11).

To be precise this is a multi-set with a density function $a_{\mathcal{F}_x}(\pi)$ as in [RS87], [Shi12, § 3.3] and [ST12, § 9.2]. In the present case we have for all $\pi \in \mathcal{F}_x$,

$$a_{\mathcal{F}_x}(\pi) := m_{\mathrm{disc}}(\pi)\widehat{f}_{S_0}(\pi_{S_0})\dim((\pi^{S_0,\infty})^{U^{S_0}_x}) \tag{6.1}$$

where $m_{\mathrm{disc}}(\pi)$ is the multiplicity of π in the discrete automorphic spectrum. The exact nature of the formula for $a_{\mathcal{F}_x}(\pi)$ does not play an explicit role in what follows but is needed in [ST12], which we are going to cite from there. Note that it is nonnegative and if $\widehat{f}_{S_0}$ is a characteristic

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

function, it takes integral values. The cardinality of a subset of the multi-set $\mathcal{F}_x$ is considered in the obvious sense. For instance, $|\mathcal{F}_x|$ is defined to be the number of π in $\mathcal{F}_x$ counted with multiplicities $a_{\mathcal{F}_x}(\pi)$.

Theorem 6.1. *Let $v \notin S_0$ be a fixed place of F^+ such that either:*

(i) *$U_{x,v}$ is maximal hyperspecial for all but finitely many x; or*
(ii) *$\mathrm{ord}_v(\mathfrak{n}_x) \to \infty$ as $x \to \infty$,*

Then $|\{\pi \in \mathcal{F}_x : [\mathbb{Q}(\pi_v) : \mathbb{Q}] \leqslant A\}|/|\mathcal{F}_x|$ tends to zero as $x \to \infty$.

Remark 6.2. We recall that the case $G = \mathrm{GL}(2)_{\mathbb{Q}}$ under assumption (i) is due to Serre [Ser97], see Theorem 1.1.

Proof. We have seen in Corollary 4.16 that π_v is tempered for all $\pi \in \mathcal{F}_x$. We are in position to apply Corollary 5.7 which implies that the set $\mathcal{Z}^{\mathrm{ur}}$ (respectively $\mathcal{Z}$) of all $\pi_v \in G(F_v^+)^{\wedge,\mathrm{ur}}$ (respectively $\pi_v \in G(F_v^+)^{\wedge}$) for $\pi \in \mathcal{F}_x$ with $[\mathbb{Q}(\pi_v) : \mathbb{Q}] \leqslant A$ is finite. For part (i), concerning $\mathcal{Z}^{\mathrm{ur}}$, we could use alternatively the easier fact that there are only finitely many associated Weil numbers (Lemma 5.1). Since $\mathcal{Z}^{\mathrm{ur}}$ and $\mathcal{Z}$ are finite, they are certainly a $\widehat{\mu}_v^{\mathrm{pl}}$-regular relatively compact subset of $G(F_v^+)^{\wedge}$.

We will follow the notation of [ST12] and all measures are chosen as in that paper. We have

$$\begin{aligned} |\{\pi \in \mathcal{F}_x : [\mathbb{Q}(\pi_v) : \mathbb{Q}] \leqslant A\}| &= |\{\pi \in \mathcal{F}_x : \pi_v \in \mathcal{Z}^{\mathrm{ur}}\}| \\ &= \frac{\tau'(G)\dim\xi}{\mathrm{vol}(U_x^v)} \widehat{\mu}_x(\mathcal{Z}^{\mathrm{ur}}), \end{aligned} \tag{6.2}$$

where $\tau'(G)$ is the volume of $G(F^+)\backslash G(\mathbb{A}_{F^+})/A_{G,\infty}$, and $\widehat{\mu}_x(\mathcal{Z}^{\mathrm{ur}})$ is the automorphic counting measure for $\mathcal{F}_x$. (See § 6.6 and (9.5) with $S_0 = \{v\}$ and $S_1 = \emptyset$ in [ST12].) The same as (6.2) holds true for $x \gg 1$ with $\mathcal{Z}$ in place of $\mathcal{Z}^{\mathrm{ur}}$. (We want $x \gg 1$ so that every member of $\mathcal{Z}$ has level at most $\mathfrak{n}_x$ at v.) A key ingredient for both (i) and (ii) is the automorphic Plancherel equidistribution theorem [ST12, Corollary 9.22] (see also [Shi12, Theorem 4.4]), stating that $\lim_{x\to\infty} \widehat{\mu}_x(\mathcal{Z}^{\mathrm{ur}}) = \widehat{\mu}_v^{\mathrm{pl}}(\mathcal{Z}^{\mathrm{ur}})$ and the same for $\mathcal{Z}$ in place of $\mathcal{Z}^{\mathrm{ur}}$.

(i) According to [ST12, Corollary 9.25], $\lim_{x\to\infty} (\tau'(G)\dim\xi/\mathrm{vol}(U_x^v)|\mathcal{F}_x|) = 1$. Hence, the limit in the theorem is nothing but $\widehat{\mu}_v^{\mathrm{pl}}(\mathcal{Z}^{\mathrm{ur}})$, which is zero. (Note that $\mathcal{Z}^{\mathrm{ur}}$ is a finite subset of the unramified unitary dual which is a torus of positive dimension and the restriction of the Plancherel measure is absolutely continuous with respect to the Lebesgue measure.)

(ii) By [Shi12, Proposition 5.2] and its extension to the setting of [ST12] by the same argument, we have

$$\lim_{x\to\infty} \frac{\tau'(G)\dim\xi}{\mathrm{vol}(U_x)|\mathcal{F}_x|} = 1. \tag{6.3}$$

([ST12, Corollary 9.25] cannot be applied as it assumes that $\mathfrak{n}_x$ is prime to v. Note that (6.3) is consistent with the formula in the proof of part (i), in which case $\mathrm{vol}(U_{x,v}) = 1$.) Therefore,

$$\lim_{x\to\infty} \frac{\tau'(G)\dim\xi}{\mathrm{vol}(U_x^v)|\mathcal{F}_x|} \widehat{\mu}_x(\mathcal{Z}) = \lim_{x\to\infty} \mathrm{vol}(U_{x,v})\widehat{\mu}_x(\mathcal{Z}) = 0$$

since we have that $\mathrm{vol}(U_{x,v}) \to 0$ from that $\mathrm{ord}_v(\mathfrak{n}_x) \to \infty$. (Note that $\widehat{\mu}_x(\mathcal{Z})$ tends to $\widehat{\mu}_v^{\mathrm{pl}}(\mathcal{Z})$, which may not be zero due to discrete series in $\mathcal{Z}$ but has bounded value.) The proof is concluded. ◻

Remark 6.3. It would be interesting to improve on the condition that $U_{x,v}$ be maximal hyperspecial. This is a question of Serre [Ser97, § 6.1] in the GL(2) case. The main obstruction is the presence of square-integrable representations $\pi_v \in \mathcal{Z}$ with $\pi \in \mathcal{F}_x$. The proof does not extend to these representations because $\widehat{\mu}^{\mathrm{pl}}(\pi_v) > 0$.

For convenience we introduce the multi-set

$$\mathcal{F}_x^{\leqslant A} := \{\pi \in \mathcal{F}_x : [\mathbb{Q}(\pi):\mathbb{Q}] \leqslant A\}.$$

It is to be understood that if $\pi \in \mathcal{F}_x^{\leqslant A}$, then π appears with the same multiplicity $a_{\mathcal{F}_x}(\pi)$ in $\mathcal{F}_x^{\leqslant A}$. This way we make sense of $|\mathcal{F}_x^{\leqslant A}|$.

Corollary 6.4. *Under the same assumptions, $|\mathcal{F}_x^{\leqslant A}|/|\mathcal{F}_x|$ tends to zero as $x \to \infty$.*

Proof. Obviously $[\mathbb{Q}(\pi_v):\mathbb{Q}] \leqslant [\mathbb{Q}(\pi):\mathbb{Q}]$. □

6.2 Quantitative estimates

One may wonder about the precise size of $\mathcal{F}_x^{\leqslant A}$ relative to that of $\mathcal{F}_x$. For instance, the following generalizes another Serre's question for families of modular forms (Remarques 2 below Théorème 6 of [Ser97]).

Question 6.5. Does there exist $\delta < 1$ such that $|\mathcal{F}_x^{\leqslant A}| = O(|\mathcal{F}_x|^{\delta})$?

As a weaker variant (cf. Remark 5.11), for a fixed finite extension E of $\mathbb{Q}$ in $\mathbb{C}$ one may ask whether there exist $\delta < 1$ such that $|\{\pi \in \mathcal{F}_x : \mathbb{Q}(\pi) \subset E\}| = O(|\mathcal{F}_x|^{\delta})$. We establish the following estimate towards a positive answer to Question 6.5. Define S_{unr} to be the set of finite places v of F^+ such that $U_{x,v}$ is hyperspecial at v for all large enough k. Let R_{unr} be the sum of the F_v^+-ranks of $G(F_v^+)$ for all $v \in S_{\mathrm{unr}}$ (it could be infinity).

Theorem 6.6. *Suppose that S_{unr} is not empty (but it could be an infinite set). Then, as $x \to \infty$,*

$$|\mathcal{F}_x^{\leqslant A}| \ll_R \frac{|\mathcal{F}_x|}{(\log|\mathcal{F}_x|)^R}, \tag{6.4}$$

for all $R \leqslant R_{\mathrm{unr}}$.

Example 6.7. In a typical example if U_x is a principal congruence subgroup of prime level $\mathfrak{n}_x$, then the set S_{unr} contains all finite places and R_{unr} is infinite: the statement holds for all $R > 0$ which is an indication for an affirmative answer to Question 6.5 in this case. If $R_{\mathrm{unr}} \geqslant 1$ is finite, then it is best to choose $R = R_{\mathrm{unr}}$. Note that the possibility $R_{\mathrm{unr}} = 0$ is excluded from the proposition because S_{unr} is not empty; this is the case discussed in Remark 6.3.

Proof. We fix a finite set of unramified places $S_1 \subset S_{\mathrm{unr}}$ disjoint from S_0. Let $\mathcal{R}$ be a rectangle in $G(F_{S_1}^+)^{\wedge,\mathrm{unr},\mathrm{temp}}$. Lemma 6.16 yields the existence of $\phi_{S_1} \in \mathcal{H}^{\mathrm{unr}}(G(F_{S_1}^+))^{\leqslant c\kappa}$ which is such that $\widehat{\phi}_\kappa$ approximates the characteristic function of $\mathcal{R}$. (The definition of $\mathcal{H}^{\mathrm{unr}}(G(F_{S_1}^+))^{\leqslant c\kappa}$ is recalled in § 6.4 below. The constant $c > 0$ depends on a choice fixed once and for all for G.)

Applying the automorphic Plancherel theorem with error bound [ST12, Theorem 9.16] to the family $\mathcal{F}_x$, we deduce that for all integer $\kappa \geqslant 1$,

$$\widehat{\mu}_{\mathcal{F}_x,S_1}(\mathcal{R}) = \widehat{\mu}_{S_1}^{\mathrm{pl}}(\mathcal{R}) + O(q_{S_1}^{A_l+B_l\kappa}|\mathcal{F}_x|^{-C_l}) + O(\kappa^{-R})$$

where $A_l, B_l, C_l > 0$ are absolute constants and $R \geqslant 0$ is the sum of the ranks of $G(F_v^+)$ for $v \in S_1$. Note that by choosing $S_1 \subset S_{\mathrm{unr}}$ arbitrary large, the integer R is arbitrary large subject to the condition that $R \leqslant R_{\mathrm{unr}}$.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

The optimal choice is $\kappa = O(\log |\mathcal{F}_x|)$, which yields

$$\widehat{\mu}_{\mathcal{F}_x,S_1}(\mathcal{R}) = \widehat{\mu}^{\mathrm{pl}}_{S_1}(\mathcal{R}) + O((\log |\mathcal{F}_x|)^{-R}). \tag{6.5}$$

Note that the constant in the remainder term does not depend on $\mathcal{R}$. In particular, $\mathcal{R}$ can be chosen to be a single element in which case $\widehat{\mu}^{\mathrm{pl}}_{S_1}(\mathcal{R}) = 0$ since the Plancherel measure is atomless. We deduce that the following estimate holds for any finite set $\mathcal{Z}$ in $G(F^+_{S_1})^{\wedge,\mathrm{unr,temp}}$,

$$\widehat{\mu}_{\mathcal{F}_x,S_1}(\mathcal{Z}) \ll \frac{|\mathcal{Z}|}{(\log |\mathcal{F}_x|)^R}.$$

We apply this to the set

$$\mathcal{Z} := \{\pi_{S_1} :\ \pi \in \mathcal{F}_x, [\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A\},$$

since it follows as before from Corollary 5.7 (or alternatively from Lemma 5.1 and the first assertion of Corollary 4.16) that $\mathcal{Z}$ is a finite set. Thus, we can conclude the proof of the proposition since

$$|\{\pi \in \mathcal{F}_x : [\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A\}| \ll |\mathcal{F}_x| \widehat{\mu}_{\mathcal{F}_x,S_1}(\mathcal{Z}).$$ □

We now consider the case where the automorphic family admits ramification at only finitely many fixed places S. Theorem 6.6 applies for any $R > 0$ since R_{unr} is infinite, but we can prove a stronger bound. Indeed Theorem 5.18 may be rephrased as a strong answer to Question 6.5. For this it is *unnecessary* to assume that the highest weight of ξ is regular (thus, π_∞ may not be a discrete series). In fact, we will prescribe a condition at infinity which is weaker than the ξ-cohomological condition. For a $\mathbb{C}$-algebra morphism $Z(\mathfrak{g}) \to \mathbb{C}$ (cf. § 5.5) and an open compact subgroup $U_x \subset G(\mathbb{A}^\infty_{F^+})$, define $\mathcal{F}(U_x, \chi_\infty)$ to be the set of discrete automorphic representations π of $G(\mathbb{A}_{F^+})$ such that (for the corollary it is unimportant to think of $\mathcal{F}(U_x, \chi_\infty)$ as a multi-set, i.e. the multiplicity of each member may be taken to be one):
- $(\pi^\infty)^{U_x} \neq 0$;
- π_∞ has infinitesimal character χ_∞.

Corollary 6.8. *Fix $A \in \mathbb{Z}_{\geqslant 1}$. Let G be either:*
- *$G = \mathrm{GL}_n$ over an arbitrary number field F; or*
- *G is a quasi-split classical group of § 4.2 over a totally real field F.*[11]

Suppose that there exists a finite set S such that for every x, the level subgroup U_x has the form $U_x = U_{S,x} U^{S,\infty}$, where $U^{S,\infty}$ is a product of hyperspecial subgroups of $G(F_v)$ for all finite $v \notin S$. Then there is a constant $C = C(A, G, \chi_\infty, S)$ such that for all x

$$|\{\pi \in \mathcal{F}(U_x, \chi_\infty) : [\mathbb{Q}(\pi) : \mathbb{Q}] \leqslant A\}| \leqslant C.$$

Proof. Immediate from Theorems 5.18 and 5.19. □

For instance, when $G = \mathrm{GL}_2$ over $\mathbb{Q}$, the theorem applies to C-algebraic automorphic representations arising from Maass forms, namely those with Laplace eigenvalue $\frac{1}{4}$. It is worth comparing our results in this subsection with previous work in the case of elliptic curves.

Remark 6.9. We briefly discuss the most basic case of $G = \mathrm{GL}(2)_{\mathbb{Q}}$, weight two and $\mathbb{Q}(\pi) = \mathbb{Q}$ (that is $A = 1$). See also the remarks following [Ser97, Théorème 7]. Modular forms of weight two with integer coefficients are attached to elliptic curves and thus more precise results than (6.4) are available.

[11] For uniformity of notation we write F rather than F^+ here.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

For an integer $N \in \mathbb{Z}_{\geqslant 1}$, let $Ell(N)$ be the number of isogeny classes of elliptic curves over $\mathbb{Q}$ of conductor N. The following is currently known [DK00, § 3.1]:

$$X^{5/6} \ll \sum_{1 \leqslant N \leqslant X} Ell(N) \ll_\epsilon X^{1+\epsilon}.$$

On the other hand, by counting S-integral points on curves of given genus, it is shown by Helfgott–Venkatesh [HV06, § 4.2] that $Ell(N) = O(N^\delta)$ for some $\delta < \frac{1}{2}$, improving earlier bounds by Evertse, Silverman, and Brumer. The numerical value is improved further in [EV07] into $\delta = 0.169\ldots$.

6.3 Order of growth

It follows from Theorem 6.1 that there are automorphic representations $\pi_x \in \mathcal{F}_x$ such that $[\mathbb{Q}(\pi_x) : \mathbb{Q}] \to \infty$ as $x \to \infty$. It is interesting to study the order of growth of $[\mathbb{Q}(\pi_x) : \mathbb{Q}]$ as $x \to \infty$. We establish the following which generalizes a result of Royer [Roy00, Theorem 1.1] in the case of $G = \mathrm{GL}(2)_{\mathbb{Q}}$. By the degree of a Weil number α (or any algebraic number) we will mean $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Proposition 6.10. *Let assumptions be as in Theorem 6.1(i). Then as $x \to \infty$ there exists an automorphic representation $\pi_x \in \mathcal{F}_x$ such that*

$$[\mathbb{Q}(\pi_x) : \mathbb{Q}] \gg (\log\log \mathbb{N}(\mathfrak{n}_x))^{1/2}. \tag{6.6}$$

Proof. Consider the set of local representations $\pi_v \in G(F_v^+)^{\wedge,unr}$ as π_v ranges over $\mathcal{F}_x$. We see from (6.5) that there are $\gg \log \mathbb{N}(\mathfrak{n}_x)$ distinct such representations π_v. On the other hand, the number of q_v-Weil integers of weight one and degree at most d is at most $q_v^{O(d^2)}$. (The $O(d^2)$-bound is easily seen from the argument of the last paragraph in the proof of Lemma 5.1.) □

In the depth aspect, that is under condition (ii) in Theorem 6.1, we can also give a lower bound for the order of growth. Suppose that $\mathfrak{n}_x$ is supported on a fixed finite set of primes. Then using the estimate in (3.4) we can deduce that there exists $\pi_x \in \mathcal{F}_x$ such that

$$[\mathbb{Q}(\pi_x) : \mathbb{Q}] \gg (\log \mathbb{N}(\mathfrak{n}_x))^{1/n}.$$

We have removed a logarithm compared with the order of growth (6.6) obtained in the level aspect.

The remainder of this subsection is devoted to discuss the case of $G = \mathrm{GL}(2)_{\mathbb{Q}}$ and weight two forms, where interestingly there is another method to establish the bound (6.6). This is based on the following result about curves over finite fields which is of independent interest.

Proposition 6.11 (Serre [Ser97, § 7]). *There are only finitely many curves over $\mathbb{F}_q$ whose Jacobian is isogenous to a product of abelian varieties of dimension at most d.*

The method of Serre is effective, see [Ser97, p. 93] for the example of $q = 2$ and $d = 1$. It does not produce immediately an explicit upper-bound in general but there have been several works in this direction, in particular we quote the following.

Proposition 6.12 (Elkies *et al.* [EHR14]). *Let $S \subset [0, \pi]$ be a finite set. If $C/\mathbb{F}_q$ is a curve of genus g with Frobenius angles in S, then*

$$g \leqslant 23|S|^2 q^{2|S|} \log q.$$

The proof of Proposition 6.11 and of the effective bounds such as in Proposition 6.12 is based on trigonometric inequalities. Precisely one uses the fact that there are $\theta_j \in [0, \pi]$, $1 \leqslant j \leqslant g$, such that $q^{1/2}e^{i\theta_j}$ (and also $q^{1/2}e^{-i\theta_j}$) are q-Weil integers of weight one and

$$2q^{n/2}\sum_{j=1}^{g}\cos(n\theta_j) \leqslant q^n + 1 \quad \text{for any integer } n \geqslant 1.$$

(The θ_j are the Frobenius angles and this holds because the right-hand side of the inequality minus the left-hand side is equal to $\#C(\mathbb{F}_{q^n})$, the number of points of C over $\mathbb{F}_{q^n}$.)

Proposition 6.12 implies the following effective estimate in the case of simple isogeny factors of dimension at most d.

Corollary 6.13. *If the Jacobian of a curve of genus g over $\mathbb{F}_q$ is isogenous to a product of abelian varieties of dimension at most d, then*

$$g \leqslant q^{q^{O(d^2)}}.$$

The underlying constant in $O(d^2)$ is absolute (independent of q and d^2).

Example 6.14. Let $q = p$ be a prime number and $r \in \mathbb{Z}_{\geqslant 1}$. The Fermat curve

$$C_r : X^{p^r+1} + Y^{p^r+1} + Z^{p^r+1} = 0$$

is such that all eigenvalues of Frobenius are $2r$th roots of $-p^r$ (see [GR78]). Thus, $\mathrm{Jac}(C_r)$ is isogenous to a product of abelian varieties over $\mathbb{F}_p$ of dimension at most $4r$. On the other hand, C_r has genus $p^r(p^r-1)/2$. Also it may be verified that the exponent of the class group $\mathrm{Jac}(C)(\mathbb{F}_p)$ is at most p^r+1, which is asymptotically the square-root of the genus and may be compared with (6.7) below. Note that C_r is a *hermitian curve* over $\mathbb{F}_{p^{2r}}$ and it is a maximal curve in the sense that $C_r(\mathbb{F}_{p^{2r}})$ is of cardinality $1 + p^{2r} + 2gp^r$ which achieves equality in the Weil bound.

In fact, the same result as in Corollary 6.13 was established around 2000 by A. de Jong using a different method. We would like to thank de Jong for explaining his result to us which had remained unpublished.

Alternative proof of Corollary 6.13 (de Jong). A theorem of Madan and Madden [MM77] states that the exponent E of the class group of a curve C of genus g over $\mathbb{F}_q$ satisfies

$$E \gg \left(\frac{g}{\log^3 g}\right)^{1/4}. \tag{6.7}$$

(Note that their arguments do apply uniformly in q and thus the above multiplicative constant is absolute, although this is not explicitly stated in their paper. Precisely it can be verified that each estimate in their proof improves when q gets large.)

On the other hand let Fr_q be the qth power Frobenius endomorphism of $\mathrm{Jac}(C)$ and let $P \in \mathbb{Z}[X]$ be its minimal monic polynomial. Note that P has integral coefficients because Fr_q is an element of the endomorphism ring of $\mathrm{Jac}(C)$ which is an order in a semisimple algebra. Since Fr_q is a semisimple endomorphism by Tate's theorem, $P(\mathrm{Fr}_q)$ acts as zero on $\mathrm{Jac}(C)$. Since Fr_q acts as the identity on $\mathrm{Jac}(C)(\mathbb{F}_q)$, we deduce that $P(1) \in \mathbb{Z}$ acts as zero on $\mathrm{Jac}(C)(\mathbb{F}_q)$. Therefore,

$$E \mid P(1).$$

The polynomial P divides the product of the characteristic polynomials of Frobenius on the abelian varieties which are the simple isogeny factors of $\mathrm{Jac}(C)$. By assumption these abelian varieties have dimension at most d and there are $q^{O(d^2)}$ isogeny classes of them by counting the Weil q-integers of weight one given via Honda–Tate theory, cf. the proof of Proposition 6.10. Thus,

$$P(1) \leqslant q^{q^{O(d^2)}}.$$

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

Note that $P(1) \neq 0$ because Fr_q is always a nontrivial endomorphism. Combining the three estimates we conclude the proof of the proposition. □

Alternative proof of Proposition 6.10 for $G = \mathrm{GL}(2)_{\mathbb{Q}}$ in weight two. Consider the modular curve $X_0(N)$ which is a smooth algebraic curve over $\mathbb{Q}$ of genus $g_0(N)$. Let $(A_i)_{i\in I}$ be the simple isogeny factors of its Jacobian $J_0(N)$, counted with multiplicity, so that there exists an injective isogeny $\prod_{i\in I} A_i \hookrightarrow J_0(N)$ over $\mathbb{Q}$ (see [Mil86, Proposition 10.1]). By the theorem of Eichler–Shimura we are reduced to finding a lower bound for the maximal dimension

$$d := \max_{i\in I} \dim A_i.$$

Suppose that the fixed prime p does not divide N. From now on we work over $\mathbb{Q}_p$ and with a small abuse of notation we still write $X_0(N)$, $J_0(N)$, and A_i for their base change $X_0(N) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $J_0(N) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and $A_i \otimes_{\mathbb{Q}} \mathbb{Q}_p$, respectively.

There exists an integral model $\mathcal{X}_0(N)$ over $\mathbb{Z}_p$ and its reduction modulo p is smooth irreducible over $\mathbb{F}_p$. Also there exists a relative Picard scheme $\mathcal{J}_0(N)$ which is a smooth abelian group scheme over $\mathbb{Z}_p$. The generic fiber $\mathcal{J}_0(N) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ can be identified with $J_0(N)$. Since $J_0(N)$ has good reduction at p, the Néron–Ogg–Shafarevich criterion tells us that A_i has good reduction at p for each $i \in I$. Let $\mathcal{A}_i$ denote the integral model of A_i over $\mathbb{Z}_p$ which is an abelian scheme. By the property of a Néron model, the injection $\prod_{i\in I} A_i \hookrightarrow J_0(N)$ extends to an injection $\prod_{i\in I} \mathcal{A}_i \hookrightarrow \mathcal{J}_0(N)$. (The latter is an injection because the kernel is flat over $\mathbb{Z}_p$ with trivial group scheme as the generic fiber.) As an injection between abelian schemes of the same dimension, it is also an isogeny.

Reducing modulo p we find that $\mathcal{J}_0(N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is isogenous to the product $\prod_{i\in I} \mathcal{A}_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Each simple isogeny factor of $\mathcal{J}_0(N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a factor of $\mathcal{A}_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ for some $i \in I$. In particular, all isogeny factors of $\mathcal{J}_0(N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ have dimension at most d.

Since $\mathcal{X}_0(N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is an irreducible smooth curve of genus $g_0(N)$ whose Jacobian can be identified with $\mathcal{J}_0(N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ we are in position to apply Corollary 6.13 which yields

$$g_0(N) \leqslant p^{p^{O(d^2)}}.$$

Since $g_0(N) \asymp N$ as $N \to \infty$, this concludes the proof of Proposition 6.10 for $G = \mathrm{GL}(2)_{\mathbb{Q}}$. □

6.4 Uniform approximation in the unitary dual

In this subsection we record some lemmas on approximation by functions in the local Hecke algebra of bounded degree. Only in this subsection let G be a connected reductive group over a p-adic field K. Write U^{hs} for a fixed hyperspecial subgroup of $G(K)$ and Ω_K for the Weyl group for G relative to K.

We begin with the classical problem of approximating periodic functions by trigonometric polynomials. The following result is a version with sharp constants that comes from the work of Beurling in the 1930s and rediscovered by Selberg in the context of the large sieve inequality. We identify $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the unit circle S^1 inside $\mathbb{C}$. Thus, a trigonometric polynomial is viewed as an element of $\mathbb{C}[z, z^{-1}$.

Lemma 6.15 (Vaaler [Vaa85]). *Let f be a function on $\mathbb{T}$ of bounded variation $V(f) \in \mathbb{R}_{\geqslant 0}$. For every integer $\kappa \in \mathbb{N}$ there are trigonometric polynomials $P_\kappa^{\pm}$ of degree κ such that $P_\kappa^- \leqslant f \leqslant P_\kappa^+$ and*

$$\int_{\mathbb{T}} P_\kappa^+ - P_\kappa^- = \frac{V(f)}{\kappa+1}. \tag{6.8}$$

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

In particular, $\|P_\kappa^\pm\|_1 \leqslant \|f\|_1 + V(f)/\kappa + 1$ *by the triangle inequality. Also the* n*th coefficients of* $P_\kappa^\pm$ *are uniformly bounded by much less than* $V(f)/|n|$ *for all* $n \neq 0$.

Proof. This is [Vaa85, Theorem 19] where it is also shown that the constants are sharp if f is a sign function. We briefly recall the the construction of the polynomials:

$$P_\kappa^\pm(z) = \sum_{|n|\leqslant\kappa} \left[\widehat{J}\left(\frac{n}{\kappa+1}\right)\widehat{f}(n) \pm \frac{n}{(2\kappa+2)}\widehat{K}\left(\frac{n}{\kappa+1}\right)\widehat{g}(n)\right] z^n,$$

for all $z \in \mathbb{T}$. Here $\widehat{f}(n)$ (respectively $\widehat{g}(n)$) are the Fourier coefficients of f (respectively the variation function of f). The Beurling functions J and K are entire of exponential type 2π with Fourier transform:

$$\widehat{J}(t) := \pi t(1-|t|)\cot(\pi t) + |t|, \quad \widehat{K}(t) := 1 - |t|, \quad |t| < 1.$$

The properties of J and K and some arguments in Fourier analysis imply the first two assertions of the lemma. Since $\widehat{f}(n), \widehat{g}(n) \ll V(f)/|n|$ for all $n \neq 0$, we deduce the third and final assertion on the decay of coefficients. □

The Satake isomorphism induces a topological isomorphism $G(K)^{\wedge,\mathrm{unr,temp}} \simeq \widehat{A}_c/\Omega_K$ where $\widehat{A}_c \simeq \mathbb{T}^r$ is a complex torus with r the K-rank of G. For $\phi \in \mathcal{H}^{\mathrm{unr}}(G(K))$ we write $\widehat{\phi}$ for the corresponding function on the real torus $\widehat{A}_c$ or its quotient $\widehat{A}_c/\Omega_K$. The truncated Hecke algebra $\mathcal{H}^{\mathrm{unr}}(G(K))^{\leqslant\kappa}$ is defined in [ST12, § 2] so that the following holds (which is all we need to know here): there exists a constant $c > 0$ (depending on a fixed choice of basis in the character group of a maximal torus in G over $\overline{K}$) such that for every $\kappa \in \mathbb{Z}_{>0}$, the set of $\phi \in \mathcal{H}^{\mathrm{unr}}(G(K))$ such that $\widehat{\phi}$ is a (Ω_K-invariant) polynomial of degree at most κ on $\widehat{A}_c$ contains $\mathcal{H}^{\mathrm{unr}}(G(K))^{\leqslant\kappa/c}$ and is contained in $\mathcal{H}^{\mathrm{unr}}(G(K))^{\leqslant c\kappa}$. (Use [ST12, § 2.4] to see this.)

Lemma 6.16. *Let* $c > 0$ *be as above. For every integer* $\kappa \geqslant 1$, *and every rectangle* $\mathcal{R} \subset G(K)^{\wedge,\mathrm{unr,temp}}$, *there is a Hecke function* $\phi_\kappa \in \mathcal{H}^{\mathrm{unr}}(G(K))^{\leqslant c\kappa}$ *such that* $\widehat{\phi}_\kappa \geqslant 0$ *on* $G(K)^{\wedge,\mathrm{unr,temp}}$, $\widehat{\phi}_\kappa \geqslant 1$ *on* $\mathcal{R}$ *while* $\widehat{\mu}^{\mathrm{pl}}(\widehat{\phi}_\kappa) \ll \widehat{\mu}^{\mathrm{pl}}(\mathcal{R}) + \kappa^{-r}$ *and* $|\phi_\kappa| \ll 1$. *Here* r *is the rank of* $G(K)$.

Proof. We can apply Lemma 6.15 to the characteristic function $\mathbb{1}_I$ of any interval I of $\mathbb{T}$, in which case the total variation is $V(\mathbb{1}_I) = 2$. Then it is not difficult to deduce the following statement in higher dimension. Let $\mathcal{R} = I_1 \times \cdots \times I_r$ be a rectangle in $\mathbb{T}^r$. There are trigonometric polynomials $P_\kappa^\pm$ of degree at most κ in r variables such that $P_\kappa^- \leqslant \mathbb{1}_\mathcal{R} \leqslant P_\kappa^+$ and

$$\int_{\mathbb{T}^d} P_\kappa^+ - P_\kappa^- \ll \kappa^{-r}. \tag{6.9}$$

We choose $\widehat{\phi}_\kappa$ to be the Ω_K-average of P_κ^+. Then $\phi_\kappa \in \mathcal{H}^{\mathrm{unr}}(G(K))$ giving rise to $\widehat{\phi}_\kappa$ via the Satake isomorphism belongs to $\mathcal{H}^{\mathrm{unr}}(G(K))^{\leqslant c\kappa}$. Note that the first two assertions follow from the inequality $\mathbb{1}_\mathcal{R} \leqslant P_\kappa^+$.

The estimate of $\widehat{\mu}^{\mathrm{pl}}(\widehat{\phi}_\kappa)$ follows from (6.9) and the fact that the Plancherel density on $G(K)^{\wedge,\mathrm{unr,temp}}$ given by Macdonald formula is uniformly bounded below (see [ST12, Proposition 3.3]). In other words we used that the Lebesgue measure on $\widehat{A}_c/\Omega_K$ is absolutely continuous with respect to the Plancherel measure.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

The Harish-Chandra Plancherel formula $\phi(1) = \widehat{\mu}^{\mathrm{pl}}(\widehat{\phi})$ holds for all smooth functions ϕ, thus in particular for all $\phi \in \mathcal{H}^{\mathrm{unr}}(G(K))$. In the unramified case (see [Wal03, Theorem VIII.1.1] for the general case) we have more generally the relation

$$\phi(g) = \int_{G(K)^{\wedge,\mathrm{unr},\mathrm{temp}}} \widehat{\phi}(\pi) M_\pi(g)\, d\widehat{\mu}^{\mathrm{pl}}(\pi), \quad g \in G(K), \tag{6.10}$$

where $M_\pi(g) = (v_\circ, g v_\circ)$ is a spherical matrix coefficient of π, that is $v_\circ$ is a unit U^{hs}-fixed vector in the representation space V_π. Let us justify formula (6.10) by computing the trace of $\pi(\phi)\circ\pi(g)^{-1}$ on V_π and the Plancherel formula for $\phi(1)$. Note that $\pi(\phi)$ has image in $\mathbb{C}v_\circ$ because ϕ is left U^{hs}-invariant. Using also the right U^{hs}-invariance, we infer that

$$\pi(\phi) w = \widehat{\phi}(\pi)(w, v_\circ) v_\circ$$

for all vector $w \in V_\pi$. Thus, $\pi(\phi) g^{-1} v_\circ = \widehat{\phi}(\pi) M_\pi(g) v_\circ$. Since $\pi(\phi)\circ\pi(g)^{-1}$ maps V_π into $\mathbb{C}v_\circ$, this implies that its trace is $\widehat{\phi}(\pi) M_\pi(g)$.

From (6.10) we deduce that $|\phi_\kappa(g)| \leqslant \phi_\kappa(1)$ for all $g \in G(K)$. Thus, we deduce from the estimate for $\widehat{\mu}^{\mathrm{pl}}(\widehat{\phi_\kappa})$ that $|\phi_\kappa| \ll 1$. □

6.5 The case of unitary groups

In this subsection let G be a unitary group as in § 4.2 or its inner form and assume that $[F^+ : \mathbb{Q}] \geqslant 2$. We would like to explain unconditional results on the growth of field of rationality which are already available from our current knowledge. Let us be brief: eventually complete unconditional results for non-quasi-split unitary (respectively symplectic/orthogonal) groups will follow from our earlier arguments once the unitary group analogue of [Mok12] (respectively [Art13]) is extended to inner forms and Hypothesis 4.8 is verified.

We assert below that Theorems 6.1(i) and 6.6 hold true for unitary groups without any hypothesis. Let $\mathcal{F}_x = \mathcal{F}(U_x)$ be a level aspect family constructed for G, now a unitary group, as in § 6.1. Let us define S_{unr} and R_{unr} for G and $\mathcal{F}_x$ as in Theorem 6.6.

Theorem 6.17. *Suppose that the highest weight of ξ is regular, that $[F^+ : \mathbb{Q}] \geqslant 2$, and that $S_{\mathrm{unr}} \neq \emptyset$ so that R_{unr} is defined. Then for all $R \leqslant R_{\mathrm{unr}}$,*

$$|\mathcal{F}_x^{\leqslant A}|/|\mathcal{F}_x| \ll_R |\mathcal{F}_x|/(\log|\mathcal{F}_x|)^R \quad \text{as } x \to \infty.$$

The argument is the same as in Theorem 6.6 (also see Theorem 6.1). The theorem relies on some of the earlier results, which we need to justify for unitary groups, but this is not so complicated as we are concerned only with the unramified local components here. The necessary results are provided by [Lab11, Corollary 5.3], especially the weaker analogue of Proposition 4.14 (here 'weaker' means that no information is available at finitely many v where π, η or the extension F/F^+ is ramified at v). In Corollary 4.16, only the first assertion is needed and derived from the latter substitute. Then the methods of proof for Theorems 6.1 and 6.6 justify Theorem 6.17 once it is noted that the final main ingredients, namely Lemma 5.1 and the level aspect Plancherel equidistribution theorem with error terms [ST12], are still valid for unitary groups.

6.6 Concluding remarks

As we have noted earlier, the arguments and main results of this paper should apply to non-quasi-split classical groups as soon as the work [Art13] and [Mok12] are extended to those groups.

There are several directions in which our work may be generalized. An obvious problem is to deal with other reductive groups. As for the growth of field of rationality, we raised the question of removing the hypotheses from Theorem 1.7 and power saving in Question 6.5. Any quantitative refinement such as power saving would be of arithmetic significance, already in the case of weight two modular forms and field of rationality $\mathbb{Q}$, cf. Remark 6.9. Another widely open question is how much of § 6 remains valid for families in the weight aspect (for instance, as defined in [ST12]). In this respect even the case of modular forms is still unsolved (Maeda's conjecture). Note that the finiteness of Weil numbers in the argument for Theorem 1.1 fails if weight grows to infinity. Finally we would like to mention Hida's recent study of field of rationality ('Hecke field' in his terminology) for p-adic families of modular forms and arithmetic applications [Hid11, Hid12], providing a perspective different from ours.

Acknowledgements

We are grateful to Wee Teck Gan, Peter Sarnak, Jean-Pierre Serre, and David Vogan for their helpful comments and the referee for a careful reading. The authors acknowledge support from the National Science Foundation under agreement numbers DMS-1162250 and DMS-1200684.

References

ABCZ94 G. Anderson, D. Blasius, R. Coleman and G. Zettler, *On representations of the Weil group with bounded conductor*, Forum Math. **6** (1994), 537–545.

Art13 J. Arthur, *Orthogonal and symplectic groups*, in *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61 (American Mathematical Society, Providence, RI, 2013).

BGGT14 T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Local–global compatibility for $l = p$. II*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), 165–179.

BMM11 N. Bergeron, J. Millson and C. Moeglin, *Hodge type theorems for arithmetic manifolds associated to orthogonal groups*, Ann. of Math. (2), submitted. Preprint (2011), arXiv:1110.3049.

BHR94 D. Blasius, M. Harris and D. Ramakrishnan, *Coherent cohomology, limits of discrete series, and Galois conjugation*, Duke Math. J. **73** (1994), 647–685.

Bor79 A. Borel, *Automorphic L-functions*, in *Automorphic forms, representations and L-functions (Oregon State University, Corvallis, OR, 1977, Part 2)*, Proceedings of Symposia in Pure Mathematics, vol. XXXIII (American Mathematical Society, Providence, RI, 1979), 27–61.

Bor07 A. Borel, *Automorphic forms on reductive groups*, in *Automorphic forms and applications*, IAS/Park City Mathematics Series, vol. 12 (American Mathematical Society, Providence, RI, 2007), 7–39.

BG11 K. Buzzard and T. Gee, *The conjectural connections between automorphic representations and Galois representations*, in *Proc. LMS Durham Symp.* (2011), to appear. Preprint (2010), arXiv:1009.0785.

CE04 F. Calegari and M. Emerton, *The Hecke algebra T_k has large index*, Math. Res. Lett. **11** (2004), 125–137.

Car12a A. Caraiani, *Monodromy and local–global compatibility for $l = p$*, Preprint (2012), arXiv:1202:4683.

Car12b A. Caraiani, *Local–global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), 2311–2413.

CH13 G. Chenevier and M. Harris, *Construction of automorphic Galois representations, II*, Cambridge Math. J. **1** (2013), 53–73.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

Clo90 L. Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, in *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988*, Perspectives in Mathematics, vol. 10 (Academic Press, Boston, MA, 1990), 77–159.

Clo13 L. Clozel, *Purity reigns supreme*, Int. Math. Res. Not. IMRN (2013), 328–346.

DH99 A. Deitmar and W. Hoffman, *Spectral estimates for towers of noncompact quotients*, Canad. J. Math. **51** (1999), 266–293.

Del71 P. Deligne, *Théorie de Hodge. I*, in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1* (Gauthier-Villars, Paris, 1971), 425–430.

DK00 W. Duke and E. Kowalski, *A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations*, Invent. Math. **139** (2000), 1–39; with an appendix by Dinakar Ramakrishnan.

EHR14 N. D. Elkies, E. W. Howe and C. Ritzenthaler, *Genus bounds for curves with fixed Frobenius eigenvalues*, Proc. Amer. Math. Soc. **142** (2014), 71–84.

EV07 J. S. Ellenberg and A. Venkatesh, *Reflection principles and bounds for class group torsion*, Int. Math. Res. Not. IMRN **2007** (2007), doi:10.1093/imrn/rnm002.

Fal86 G. Faltings, *Finiteness theorems for abelian varieties over number fields*, in *Arithmetic geometry (Storrs, CT, 1984)* (Springer, New York, 1986), 9–27; translated from the German original [Invent. Math. **7**3 (1983, no. 3, 349–366; Invent. Math. **7**5 (1984), no. 2, 381; MR 85g:11026ab] by Edward Shipz.

Fla79 D. Flath, *Decomposition of representations into tensor products*, in *Proceedings of Symposia in Pure Mathematics*, vol. 33(1) (American Mathematical Society, Providence, RI, 1979), 179–183.

FM95 J.-M. Fontaine and B. Mazur, *Geometric Galois representations*, in *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, Series in Number Theory, vol. I (International Press, Cambridge, MA, 1995), 41–78.

FS98 J. Franke and J. Schwermer, *A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups*, Math. Ann. **311** (1998), 765–790.

GJS99 A. Gamburd, D. Jakobson and P. Sarnak, *Spectra of elements in the group ring of* SU(2), J. Eur. Math. Soc. (JEMS) **1** (1999), 51–85.

Gro99 B. H. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93.

GR78 B. H. Gross and D. E. Rohrlich, *Some results on the Mordell–Weil group of the Jacobian of the Fermat curve*, Invent. Math. **44** (1978), 201–224.

Har68 Harish-Chandra, *Automorphic forms on semisimple Lie groups. Notes by J. G. M. Mars*, Lecture Notes in Mathematics, vol. 62 (Springer, Berlin, 1968).

HT01 M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151 (Princeton University Press, Princeton, NJ, 2001); with an appendix by Vladimir G. Berkovich.

HV06 H. A. Helfgott and A. Venkatesh, *Integral points on elliptic curves and 3-torsion in class groups*, J. Amer. Math. Soc. **19** (2006), 527–550; (electronic).

Hen00 G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps* p*-adiques*, Invent. Math. **139** (2000), 439–455.

Hid11 H. Hida, *Hecke fields of analytic families of modular forms*, J. Amer. Math. Soc. **24** (2011), 51–80.

Hid12 H. Hida, *A finiteness property of abelian varieties with potentially ordinary good reduction*, J. Amer. Math. Soc. **25** (2012), 813–826.

Kha00 C. Khare, *Conjectures on finiteness of mod* p *Galois representations*, J. Ramanujan Math. Soc. **15** (2000), 23–42.

KK05 H. H. Kim and M. Krishnamurthy, *Stable base change lift from unitary groups to* GL_n, IMRP Int. Math. Res. Pap. **1** (2005), 1–52.

KS99 R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), vi+190.

Lab11 J.-P. Labesse, *Changement de base CM et séries discrètes*, in *On the stabilization of the trace formula*, The Stable Trace Formula, Shimura Varieties, and Arithmetic Applications, vol. 1 (International Press, Somerville, MA, 2011), 429–470.

Lan88 R. Langlands, *The classification of representations of real reductive groups*, Mathematics Surveys and Monographs, vol. 31 (American Mathematical Society, Providence, RI, 1988).

LS87 R. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219–271.

MM77 M. L. Madan and D. J. Madden, *Note on class groups of algebraic function fields*, J. Reine Angew. Math. **295** (1977), 57–60.

Mil86 J. S. Milne., *Abelian varieties*, in *Arithmetic geometry (Storrs, CT, 1984)* (Springer, New York, 1986), 103–150.

Moe07 C. Moeglin, *Classification et changement de base pour les séries discrètes des groupes unitaires padiques*, Pacific J. Math. **233** (2007), 159–204.

Mok12 C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc., to appear. Preprint (2012), arXiv:1206.0882.

Ngo10 B. C. Ngô., *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), 1–169.

Pat12 S. Patrikis, Variations on a theorem of Tate, Preprint (2012), arXiv:1207.6724v1 [math.NT].

RS87 J. Rohlfs and B. Speh, *On limit multiplicities of representations with cohomology in the cuspidal spectrum*, Duke Math. **55** (1987), 199–211.

Roy00 E. Royer, *Facteurs* $\mathbf{Q}$*-simples de* $J_0(N)$ *de grande dimension et de grand rang*, Bull. Soc. Math. France **128** (2000), 219–248.

Sai03 T. Saito, *Weight spectral sequences and independence of* l, J. Inst. Math. Jussieu **2** (2003), 583–634.

Sar02 P. Sarnak, *Maass cusp forms with integer coefficients*, in *A panorama of number theory or the view from Baker's garden (Zürich, 1999)* (Cambridge University Press, Cambridge, 2002), 121–127.

Sau97 F. Sauvageot, *Principe de densité pour les groupes réductifs*, Compositio Math. **108** (1997), 151–184.

Sch12 P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

Sch10 J. Schwermer, *Geometric cycles, arithmetic groups and their cohomology*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), 187–279.

Ser77 J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, vol. 42 (Springer, New York, 1977), translated from the second French edition by Leonard L. Scott.

Ser79 J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67 (Springer, New York, 1979), translated from the French by Marvin Jay Greenberg.

Ser97 J.-P. Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke* T_p, J. Amer. Math. Soc. **10** (1997), 75–102.

Ser07 J.-P. Serre, *Bounds for the orders of the finite subgroups of* $G(k)$, in *Group representation theory* (EPFL Press, Lausanne, 2007), 405–450.

Shi11 S. W. Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), 1645–1741.

Shi12 S. W. Shin, *Automorphic Plancherel density theorem*, Israel J. Math. **192** (2012), 83–120.

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press

ST12 S.-W. Shin and N. Templier, *Sato–Tate theorem for families and low-lying zeros of automorphic L-functions,* submitted, 129pp, with Appendix A by R. Kottwitz and Appendix B by R. Cluckers, J. Gordon and I. Halupczok. Preprint (2012), arXiv:1208.1945.

Tay04 R. Taylor, *Galois representations*, Ann. Fac. Sci. Toulouse **13** (2004), 73–119.

TY07 R. Taylor and T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), 467–493.

Vaa85 J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 183–216.

Vog93 D. A. Vogan Jr, *The local Langlands conjecture*, in *Representation theory of groups and algebras*, Contemporary Mathematics, vol. 145 (American Mathematical Society, Providence, RI, 1993), 305–379.

Wal85 J.-L. Waldspurger, *Quelques propriétés arithmétiques de certaines formes automorphes sur* GL(2), Compositio Math. **54** (1985), 121–171.

Wal97 J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), 153–236.

Wal03 J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques d'après Harish-Chandra*, J. Inst. Math. Jussieu **2** (2003), 235–333.

Wal06 J.-L. Waldspurger, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), 423–525.

Wal08 J.-L. Waldspurger, *L'endoscopie tordue n'est pas si tordue*, Mem. Amer. Math. Soc. **194** (2008), x+261.

Wal10 J.-L. Waldspurger, *Les facteurs de transfert pour les groupes classiques: un formulaire*, Manuscripta Math. **133** (2010), 41–82.

Yam79 T. Yamada, *A remark on Schur indices of p-groups*, Proc. Amer. Math. Soc. **76** (1979), 45.

Yu09 J.-K. Yu, *Bruhat–Tits theory and buildings*, in *Ottawa lectures on admissible representations of reductive p-adic groups*, Fields Institute Monographs, vol. 26 (American Mathematical Society, Providence, RI, 2009), 53–77.

Sug Woo Shin swshin@math.mit.edu
Department of Mathematics, Massachusetts Institute of Technology,
77 Massachusetts Avenue, Cambridge, MA 02139, USA

and

Korea Institute for Advanced Study, 85 Hoegiro,
Dongdaemun-gu, Seoul 130-722, Republic of Korea

Nicolas Templier templier@math.princeton.edu
Department of Mathematics, Princeton University,
Princeton, NJ 08544, USA

https://doi.org/10.1112/S0010437X14007428 Published online by Cambridge University Press