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SIMPLE DIVISIBLE MODULES OVER INTEGRAL DOMAINS

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ABSTRACT. An R-module is a simple divisible module if it is a non-zero divisible module that has no proper non-zero divisible submodules. We study simple divisible modules and their endomorphism rings, give some examples and determine all simple divisible modules over some classes of rings.

Introduction. Let R be a commutative integral domain with identity. An R-module A is *divisible* if $A = rA$ for each non-zero $r \in R$. We say that a divisible R-module A is *simple divisible* if $A \neq 0$ and the only divisible submodules of A are 0 and A.

In this paper we study some properties of simple divisible modules, their endomorphism rings and the completion of R in the D-topology, where D is a simple divisible R-module. If a simple divisible module D is not a torsion module, then D is isomorphic to Q, the field of fractions of R. If D is a torsion module, then D is a module over the completion H of R in the R-adic topology and the annihilator $\mathrm{Ann}_H D$ of D in H is a closed prime ideal of H. We study the behavior of simple divisible modules under the action of the projective class group of R and with respect to restriction of scalars. Finally we consider when simple divisible modules can be realized as quotients (i.e., homomorphic images) of Q. Some examples are given and all the simple divisible modules over some classes of rings are determined.

Simple divisible R-modules have been introduced and studied for the first time by Matlis [8]. His definition is lightly different from ours, because in [8] simple divisible R-modules are required to be torsion, the ring R can contain zero-divisors, and moreover only simple divisible modules over rings of Krull dimension one are considered.

If A is an R-module, we denote the endomorphism ring of A by $\mathrm{End}_R(A)$. Moreover, if B is a subset of A and S is a subset of R, we denote the annihilator of B in R by $\mathrm{Ann}_R B$ and the annihilator of S in A by $\mathrm{Ann}_A S$. Therefore $\mathrm{Ann}_R B = \{r \in R | rB = 0\}$ and $\mathrm{Ann}_A S = \{x \in A | Sx = 0\}$.

1. **Simple divisible modules.** Throughout this paper R is a commutative integral

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domain with 1 and Q is its field of fractions. *We always assume that* $R \neq Q$.

We say that an R-module D is a *simple divisible module* if it is a non-zero divisible module that has no proper non-zero divisible submodules. Clearly, a non-zero divisible module D is simple divisible if and only if every non-zero homomorphism of a divisible module A into D is onto. In particular every non-zero endomorphism of a simple divisible module D is an epimorphism. This remark immediately yields our first lemma.

LEMMA 1. *The endomorphism ring* $\mathrm{End}_R(D)$ *of a simple divisible R-module D is an integral domain (not necessarily commutative).*

In particular the center $Z(\mathrm{End}_R(D))$ of $\mathrm{End}_R(D)$ is a commutative integral domain containing R.

For any integral domain R, the R-module Q is simple divisible. As the next proposition shows, this is the unique simple divisible module that is not a torsion R-module and whose endomorphism ring is a division ring. Probably because of this exceptional behavior Q has been excluded in Matlis' definition of simple divisible module [8, p. 46].

PROPOSITION 2. *Let D be a simple divisible R-module. Then either* $D \cong Q$ *or D is a torsion R-module. In this second case the integral domain* $\mathrm{End}_R(D)$ *is not a division ring.*

PROOF. If D is simple divisible and is not torsion, then its torsion submodule $t(D)$ is a proper divisible submodule of D. Therefore $t(D) = 0$ and D is a torsion-free divisible R-module. It follows that D is a vector space over Q, and it must be one-dimensional because it is simple divisible. This shows that $D \cong Q$ and proves the first part of the statement. For the second part suppose that D is a simple divisible R-module and that $\mathrm{End}_R(D)$ is a division ring. Then for every non-zero $r \in R$ the multiplication by r is a non-zero endomorphism of D because $rD = D$. Since $\mathrm{End}_R(D)$ is a division ring, this endomorphism must be invertible. In particular $\mathrm{Ann}_D r = 0$. This proves that D must be torsion-free in this case. □

Let R be an integral domain and A an R-module. The *R-adic topology* on A is defined by letting the submodules rA, where r is a non-zero element of R, be a subbase for the open neighborhoods of 0 in A. If H denotes the completion of R in the R-adic topology, H is a commutative ring isomorphic to $\mathrm{End}_R(Q/R)$ [7, Th. 10]. The topology of H as the completion of R coincides with the R-adic topology on H and every torsion R-module has a unique structure as an H-module [7, Th. 8 and 11]. In particular every torsion simple divisible R-module is canonically an H-module.

THEOREM 3. *If D is a torsion simple divisible R-module, then* $\mathrm{Ann}_H D$ *is a closed prime ideal of H, which is neither an open subset nor a maximal ideal of H. Moreover* $R \cap \mathrm{Ann}_H D = 0$.

PROOF. Since D has a unique H-module structure extending that of R, there is

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a unique R-algebra homomorphism $\varphi : H \longrightarrow \mathrm{End}_R(D)$. But $\mathrm{End}_R(D)$ is an integral domain by Lemma 1, so that the kernel $\ker \varphi = \mathrm{Ann}_H D$ of φ is a prime ideal of H.

If $\mathrm{End}_R(D)$ is given the R-adic topology, then $\mathrm{End}_R(D)$ is Hausdorff, because if $f \in r\mathrm{End}_R(D)$ for every non-zero $r \in R$, the kernel of f must contain $\mathrm{Ann}_D r$, so that $\ker f$ must contain $\cup_{r\neq 0}\mathrm{Ann}_D r = D$, i.e., $f = 0$. It follows that $\varphi : H \longrightarrow \mathrm{End}_R(D)$ is a continuous homomorphism into a Hausdorff topological R-module, and hence its kernel $\ker \varphi = \mathrm{Ann}_H D$ is a closed ideal of H. Moreover $\mathrm{Ann}_H D$ is not an open subset of H, otherwise $\mathrm{Ann}_H D \supseteq rH$ for some non-zero $r \in R$, so that $rD = 0$, contradiction because every non-zero divisible module is faithful. Finally, $R \cap \mathrm{Ann}_H D = \mathrm{Ann}_R D = 0$ and $\mathrm{Ann}_H D$ is not a maximal ideal in H, otherwise D would be an $H/\mathrm{Ann}_H D$-module, that is, a vector space over $H/\mathrm{Ann}_H D$. In particular for each non-zero $r \in R$, the multiplication by r would be an automorphism of D. This shows that D would be a torsion-free R-module, contradiction. □

Let A be a fixed non-zero divisible module over an integral domain R. The *A-topology on R* is defined by taking the annihilators in R of the finitely generated subsets of A as a basis of neighborhoods of 0. For example, the Q/R-topology on R is exactly the R-adic topology. The ring R endowed with the A-topology is a Hausdorff topological ring [12, Prop. 1.5].

If A is a torsion divisible R-module the R-adic topology on R is finer than the A-topology. Note that the Q-topology is the discrete topology on R.

THEOREM 4. *Let D be a fixed simple divisible R-module. Then: (a) The completion $\tilde{R}$ of R in the D-topology is an integral domain; (b) D has a unique $\tilde{R}$-module structure extending that of R and is a simple divisible $\tilde{R}$-module; (c) $\tilde{R}$ is complete in its D-topology; (d) if D is torsion, $\tilde{R}$ is complete in its $\tilde{R}$-adic topology also.*

PROOF. (a) and (b). The center $Z(\mathrm{End}_R(D))$ of the ring $\mathrm{End}_R(D)$ can be endowed with the *finite topology* by taking the R-submodules $V(F) = \{f \in Z(\mathrm{End}_R(D)) | f(F) = 0\}$, where F ranges in the finite subsets of D, as a basis of neighborhoods of zero. Let $\psi : R \longrightarrow Z(\mathrm{End}_R(D))$ be the natural homomorphism. By [12, Prop. 1.5] ψ is a topological embedding and $Z(\mathrm{End}_R(D))$ is a complete topological R-module. Therefore there is a unique extension of ψ to a topological embedding $\tilde{\psi} : \tilde{R} \longrightarrow Z(\mathrm{End}_R(D))$ whose image is the closure of $\psi(R)$ in $Z(\mathrm{End}_R(D))$. In particular the ring $\tilde{R}$, isomorphic to a subring of $\mathrm{End}_R(D)$, is an integral domain and D has a unique $\tilde{R}$-module structure extending that of R. The $\tilde{R}$-module D is divisible because if $\tilde{r}$ is a non-zero element of $\tilde{R}$, the multiplication by $\tilde{r}$ is the non-zero R-endomorphism $\tilde{\psi}(\tilde{r})$ of D and therefore it is surjective. It follows easily that D is a simple divisible $\tilde{R}$-module.

(c) In order to prove that $\tilde{R}$ is complete in its D-topology it is sufficient to observe that its topology as the completion of R and its D-topology are one and the same, because both these topologies are induced on $\tilde{R}$ by the finite topology of $Z(\mathrm{End}_R(D))$.

(d) By (a), (b) and (c) we can substitute $\tilde{R}$ with R, i.e., we can suppose that D is a simple divisible R-module and R is complete in the D-topology, and we have to show that R is complete in the R-adic topology. Now $\psi : R \longrightarrow Z(\mathrm{End}_R(D))$ is a topological embedding when R has the D-topology; since D is torsion, the R-adic topology is finer than the D-topology, so that ψ is a continuous mapping of R with the R-adic topology into $Z(\mathrm{End}_R(D))$ with the finite topology. Therefore ψ extends uniquely to $\varphi : H \longrightarrow Z(\mathrm{End}_R(D))$, because H is the completion of R (this φ is exactly the mapping that gives D its unique H-module structure). But R is complete in the D-topology, so that the image of ψ is closed in $Z(\mathrm{End}_R(D))$. It follows that $\psi(R) \supseteq \varphi(H)$, i.e., $H = R + \mathrm{Ann}_H D$. Since $R \cap \mathrm{Ann}_H D = 0$, it follows that $H = R \oplus \mathrm{Ann}_H D$ as an R-module. Therefore $\mathrm{Ann}_H D \cong H/R$, which is a divisible R-module [7, Th. 8]. But H is a reduced R-module, that is, the only divisible R-submodule of H is 0. It follows that $\mathrm{Ann}_H D = 0$, $H = R$ and R is complete in the R-adic topology. □

EXAMPLE 1. If R is a Dedekind domain, every divisible module is injective [9, Prop. 2.10]. Therefore in this case the simple divisible R-modules are exactly the indecomposable injective modules. As proved by Matlis [9, Corollary to Th. 2.32], the indecomposable injective R-modules are exactly the injective envelopes $E_R(R/P)$ where P is a prime ideal of R. Then $E_R(R/P) = Q$ if $P = 0$, and $E_R(R/P)$ is a module over the discrete valuation domain R_P if $P \neq 0$. In this case $E_R(R/P) \cong Q/R_P$. Hence over a Dedekind domain R the simple divisible modules up to isomorphism are exactly the R-modules Q and Q/R_P, where P is a non-zero prime ideal in R. The Q/R_P-topology on R is the usual P-adic topology and the endomorphism ring of Q/R_P is the completion of R in the P-adic topology.

EXAMPLE 2. Recall that a *uniserial* R-module is a module U with the property that if A and B are submodules of U then either $A \subseteq B$ or $B \subseteq A$. In particular a *valuation domain* is an integral domain R that is uniserial R-module.

LEMMA 5. *If U is a non-zero uniserial divisible module over an integral domain R, then U is a simple divisible R-module.*

PROOF. If U is not torsion, R must be a valuation domain because it is isomorphic to a submodule of U. In this case $U \cong Q$ and there is nothing to prove.

Suppose that U is torsion and let D be a proper divisible submodule of U. If x is an element of U not in D, then $Rx \supseteq D$ because U is uniserial. But U is torsion, so that $rx = 0$ for some $r \in R$, $r \neq 0$. Then $0 = r(Rx) \supseteq rD = D$. □

The endomorphism ring of a torsion uniserial divisible module U over an integral domain R has been studied by Shores and Lewis: it is a valuation domain and is the completion of R_P in the U-topology, where $P = \{r \in R | \mathrm{Ann}_D r \neq 0\}$ [11, Th. 3.3 and Cor. 3.8].

A uniserial divisible module over a valuation domain R is called *standard* if it is isomorphic to Q/I for an ideal I of R [4]. The existence of non-standard uniserial modules is one of the most challenging problems in the study of modules over valuation domains; it has been considered by Shelah [10], Fuchs [4], Franzen and Göbel [3], and Bazzoni and Salce [1]. (The results in these papers need particular set theoretic hypotheses.)

EXAMPLE 3. Recall that an integral domain R is said to be an h-local ring if each non-zero prime ideal of R is contained in only one maximal ideal of R and each non-zero element of R is contained in only a finite number of maximal ideals of R [7]. An integral domain is h-local if and only if every torsion module A is the direct sum of its localizations A_M where M ranges in the maximal ideals of R [7, Th. 22]. Therefore if D is a simple divisible module over an h-local domain R there exists a maximal ideal M of R such that $D = D_M$ is a simple divisible R_M-module. It could be proved via the module $D = Q/R$ over the ring R of Example 7 that this property doesn't hold for arbitrary integral domains, that is, the simple divisible R-modules are not necessarily modules over the localization of R at a maximal ideal.

EXAMPLE 4. If an integral domain R is complete in the R-adic topology, then $\mathrm{Ann}_H D = 0$ for every torsion simple divisible R-module D. This is trivial because in this case $R = H$ and $\mathrm{Ann}_H D = \mathrm{Ann}_R D = 0$. We haven't been able to determine the prime ideals of H of the form $\mathrm{Ann}_H D$ with D a torsion simple divisible module over an arbitrary integral domain R. The following proposition gives a sufficient condition.

PROPOSITION 6. *Let R be an integral domain and H its R-adic completion. Let P be an ideal of H which is maximal with respect to the property $P \cap R = 0$. If P is not a maximal ideal in H, then there exists a torsion simple divisible R-module D such that* $\mathrm{Ann}_H D = P$.

PROOF. Note that P is a prime ideal of H because it is maximal with respect to the property $P \cap S = \emptyset$, where S is the multiplicatively closed subset $R\setminus\{0\}$ of H. Consider the ring $H \otimes Q = H_S$. Then $P \otimes Q = P_S$ is a maximal ideal of $H \otimes Q$, so that $H \otimes Q/P \otimes Q \cong (H/P) \otimes Q \cong (H/P)_S$ is a field. More precisely, $(H/P)_S$ is the field of fractions of H/P, which is not a field, because P is not a maximal ideal in H. Let V be a valuation subring of $(H/P)_S$ containing H/P and set $D = (H/P)_S/V$. Then D is a faithful V-module, so that $\mathrm{Ann}_{H/P} D = (H/P) \cap \mathrm{Ann}_V D = 0$, and therefore $\mathrm{Ann}_H D = P$. We must prove that D is a torsion simple divisible R-module. Since D is a homomorphic image of $(H/P)_S$, that is a field containing R (up to isomorphism), D is a divisible R-module. Moreover D is a homomorphic image of $(H/P)_S/(H/P)$ because $V \supseteq H/P$, and $(H/P)_S/(H/P)$ is a torsion R-module. Hence D is a torsion R-module. Let A be a proper divisible R-submodule of D. Since D is torsion, A is an H-submodule of D. But $\mathrm{Ann}_H D = P$, so that A is an H/P-submodule of D. Now D is a V-module, and if v is a non-zero element of V, there exist $h, h' \in H\setminus P$ such that $(h + P)v = h' + P$. By the maximality of P there exists a non-zero $r \in (hH + P) \cap R$, and thus $(h + P)A = (hH + P)A \supseteq rA = A$, that is, $(h + P)A = A$ and $vA = v(h + P)A = (h' + P)A \subseteq A$. Therefore A is a proper V-submodule of D. As in the proof of Lemma 5, there exists a non-zero $w \in V$ such that $wA = 0$ because D is a torsion uniserial V-module. Since V is contained in the field of fractions of H/P, there exists a non-zero $j + P \in H/P$ such that $(j + P)A = 0$. Then $j \in H\setminus P$, $(jH + P)A = 0$ and $jH + P$ is an ideal of H properly

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containing P. Again by the maximality of P there is a non-zero $s \in (jH + P) \cap R$. Then $A = sA \subseteq (jH + P)A = 0$. This proves that D is a simple divisible R-module. □

2. **Projective class group and restriction of scalars.** In this section we consider two functorial ways of constructing simple divisible modules: via the projective class group of R and via the restriction of scalars from an overring of R.

Recall that the *projective class group* $P(R)$ of R is the group of isomorphism classes of invertible R-modules with multiplication defined by the tensor product. (An R-module is *invertible* if and only if it is a rank one, finitely generated, projective R-module.)

PROPOSITION 7. *If D is a simple divisible R-module and P is an invertible R-module, then $D \otimes P$ is a simple divisible R-module. Moreover* $\mathrm{End}_R(D) \cong \mathrm{End}_R(D \otimes P)$ *and* $\mathrm{Ann}_H(D) = \mathrm{Ann}_H(D \otimes P)$.

PROOF. If A is a divisible R-module, $A \otimes_R P$ is also divisible. Moreover the functor $- \otimes_R P : R\text{-Mod} \longrightarrow R\text{-Mod}$ is an equivalence of categories because P is an invertible module. The first part of the proposition now follows from the fact that the equivalence $- \otimes_R P$ preserves monomorphisms and divisible modules.

Moreover the endomorphism rings of D and $D \otimes P$, corresponding objects in the equivalence, are canonically isomorphic. Finally $\mathrm{Ann}_H D$ and $\mathrm{Ann}_H(D \otimes P)$ are the kernels of the unique R-algebra homomorphisms $H \longrightarrow \mathrm{End}_R(D)$ and $H \longrightarrow \mathrm{End}_R(D \otimes P)$ (proof of Proposition 3). Since $\mathrm{End}_R(D)$ and $\mathrm{End}_R(D \otimes P)$ are canonically isomorphic R-algebras, it follows that $\mathrm{Ann}_H D = \mathrm{Ann}_H(D \otimes P)$. □

By Proposition 7 the projective class group $P(R)$ acts on the set of the isomorphism classes of simple divisible R-modules.

If R is an integral domain and Q is its field of fractions, an *overring* of R is any ring S such that $R \subseteq S \subseteq Q$. If S is an overring of R, an ideal I of R is *contracted from* S if there exists an ideal J of S such that $I = R \cap J$ (or, equivalently, if $I = R \cap IS$).

Let S be an overring of an integral domain R, and let $\mathcal{D}_S$ ($\mathcal{D}_R$) be the full subcategory of S-Mod (R-Mod) whose objects are all the divisible R-modules (S-modules). Then the restriction of scalars induces a full and faithful functor $F : \mathcal{D}_S \longrightarrow \mathcal{D}_R$. In fact if A, B are divisible S-modules, then A, B are divisible R-modules a fortiori, and $\mathrm{Hom}_S(A, B) = \mathrm{Hom}_R(A, B)$: to prove this, note that if A, B are divisible S-modules and $f : A \longrightarrow B$ is R-linear, then f is S-linear, because if $a \in A$ and $s \in S$, then $s = x/y$ for some $x, y \in R$, $y \neq 0$, and $a = yb$ for some $b \in A$, so that $f(sa) = f(syb) = f(xb) = xf(b) = syf(b) = sf(yb) = sf(a)$.

THEOREM 8. *The restriction of scalars $F : \mathcal{D}_S \longrightarrow \mathcal{D}_R$ induces an isomorphism of categories between $\mathcal{D}_S$ and the full subcategory of $\mathcal{D}_R$ whose objects are the divisible R-modules D with the following property: for every $d \in D$ the ideal* $\mathrm{Ann}_R\, d$ *is contracted from S. An S-module A is a simple divisible S-module if and only if $F(A)$ is a simple divisible R-module.*

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Proof. If A is a divisible S-module, then $\operatorname{Ann}_R a = R \cap \operatorname{Ann}_S a$ is contracted from S for every $a \in A$.

Conversely, let D be a divisible R-module such that $\operatorname{Ann}_R d$ is contracted from S for every $d \in D$. Define an S-module structure on D in the following way: if $s \in S$ and $d \in D$, there exist $x, y \in R$, $y \neq 0$, such that $s = x/y$, and there exists $d' \in D$ such that $d = yd'$; set $sd = xd'$. This is a well-defined multiplication, because if also $s = x_1/y_1$ and $d = y_1 d_1'$, x_1, $y_1 \in R$, $y_1 \neq 0$, and $d_1' \in D$, then there exist d'', $d_1'' \in D$ such that $d' = y_1 d''$ and $d_1' = yd_1''$, so that $yy_1(d'' - d_1'') = y(y_1 d'') - y_1(yd_1'') = yd' - y_1 d_1' = 0$. Therefore yy_1 belongs to the ideal $\operatorname{Ann}_R(d'' - d_1'')$. This ideal is contracted from S, so that $xy_1 = (x/y)(yy_1) \in R \cap S\operatorname{Ann}_R(d'' - d_1'') = \operatorname{Ann}_R(d'' - d_1'')$. Therefore $xy_1(d'' - d_1'') = 0$. It follows that $xd' = xy_1 d'' = xy_1 d_1'' = syy_1 d_1'' = x_1 yd_1'' = x_1 d_1'$. This proves that the multiplication is well-defined. Now it is immediate to see that the S-module D is divisible, which proves the first part of the statement.

By what we have just shown, if A is a divisible S-module every R-submodule of A that is a divisible R-module is also an S-submodule and is divisible as an S-module. This immediately yields the second part of the statement. □

By Theorem 8 there is a one-to-one correspondence between the isomorphism classes of simple divisible S-modules and the isomorphism classes of the simple divisible R-modules D such that $\operatorname{Ann}_R d$ is contracted from S for every $d \in D$.

Corollary 9. *If R is an integral domain, $V \subseteq Q$ is a valuation overring of R and I is an ideal of V, then Q/I is a simple divisible R-module.*

Proof. Lemma 5 and Theorem 8. □

3. **Quotients of Q.** If R is an arbitrary integral domain with field of fractions Q, an R-module is said to be *h-divisible* if it is a homomorphic image of a vector space over Q [7]. An R-module A contains a unique largest h-divisible submodule $h(A)$ that contains every h-divisible submodule of A. Given a simple divisible R-module D, its submodule $h(D)$ is divisible, so that either $h(D) = D$ or $h(D) = 0$. If $h(D) = D$, D must be a quotient of Q. If $h(D) = 0$, then D is *h-reduced*, that is, $\operatorname{Hom}_R(Q, D) = 0$. Therefore the simple divisible R-modules are naturally divided into two classes: the quotients of Q and the h-reduced simple divisible R-modules. For instance the injective simple divisible R-modules are quotients of Q, and the non-standard, uniserial, divisible modules over a valuation domain are h-reduced simple divisible modules (Example 2).

The action of the projective class group $P(R)$ on the isomorphism classes of simple divisible R-modules described in §2 can be made explicit for the quotients of Q: every invertible R-module is isomorphic to an R-submodule P of Q, so that when $A \subseteq Q$ and Q/A is a simple divisible R-module, then $Q/A \otimes_R P \cong Q/AP$.

The representation of the simple divisible R-modules as quotients of Q is particularly important when the projective dimension $p.\dim_R Q$ of the R-module Q is one.

For instance, if Q is countably generated as an R-module then $p.\dim_R Q = 1$. For an integral domain R, $p.\dim_R Q = 1$ if and only if every divisible R-module is h-divisible [4, Th. VI.1.3]. In particular if $p.\dim_R Q = 1$, every simple divisible R-module is a quotient of Q and every torsion simple divisible R-module is a quotient of $K = Q/R$. But K is a direct sum of countably generated R-modules when $p.\dim_R Q = 1$ [5], so that every simple divisible module over a ring R with $p.\dim_R Q = 1$ is either isomorphic to Q or countably generated.

Proposition 10. *Let R be an integral domain such that $p.\dim_R Q = 1$. If A is a proper R-submodule of Q that is complete in its R-adic topology, then Q/A is a simple divisible R-module and its endomorphism ring is isomorphic to a subring of Q.*

Proof. Since $p.\dim_R Q = 1$, every divisible R-module is h-divisible. Therefore in order to prove that Q/A has no proper non-zero divisible submodules it is sufficient to prove that every non-zero homomorphism $f: Q \longrightarrow Q/A$ is onto. Apply the functor $\mathrm{Hom}_R(Q, -)$ to the exact sequence $0 \longrightarrow A \longrightarrow Q \longrightarrow Q/A \longrightarrow 0$. Then the sequence $\mathrm{Hom}_R(Q,Q) \longrightarrow \mathrm{Hom}_R(Q,Q/A) \longrightarrow \mathrm{Ext}^1_R(Q,A)$ is exact. But A is complete in its R-adic topology, so that it is cotorsion [7, Th. 9], that is, $\mathrm{Ext}^1_R(Q,A) = 0$. It follows that every homomorphism $Q \longrightarrow Q/A$ factors through the canonical projection $\pi: Q \longrightarrow Q/A$. Since $\mathrm{Hom}_R(Q,Q) \cong Q$, there exists $q \in Q$, $q \neq 0$, such that $f(x) = qx + A$ for every $x \in Q$. In particular f is onto. This shows that Q/A is simple divisible. But since every homomorphism $Q \longrightarrow Q/A$ factors through π, it is easy to see that $\mathrm{End}_R(Q/A)$ is isomorphic to the subring $(A :_Q A) = \{q \in Q \mid qA \subseteq A\}$ of Q. □

Proposition 10 shows that there is a connection between simple divisible modules and completeness in the R-adic topology.

Corollary 11. *Let R be an integral domain.*

If Q/R is a simple divisible R-module, then the completion H of R in the R-adic topology is an integral domain.

If R is complete in the R-adic topology and $p.\dim_R Q = 1$, then Q/I is a simple divisible R-module for every ideal I of R.

Proof. The first part follows from Theorem 4 because the Q/R-topology on R coincides with the R-adic topology. For the second part, every non-zero ideal I of R is complete in the R-adic topology by [7, Theorems 9 and 14]. Therefore the result follows from Proposition 10. □

We have seen that if $p.\dim_R Q = 1$, all the simple divisible modules are quotients of Q, and conversely we have found some sufficient conditions for a fixed non-zero quotient of Q to be simple divisible (Proposition 10 and Corollary 11). Our next result shows that if all the non-zero quotients of Q are simple divisible R-modules, the set $\mathrm{Spec}(R)$ of the prime ideals of R ordered by inclusion must have a particular form.

Proposition 12. *Let R be a domain such that all the non-zero quotients of Q are simple divisible. Then the following conditions hold: (a)* $\mathrm{Spec}(R)\setminus\{0\}$ *is directed* *downward, that is, if P_1, P_2 are non-zero prime ideals of R there exists a non-zero prime ideal P_3 such that $P_3 \subseteq P_1 \cap P_2$. In particular, R has at most one minimal non-zero prime ideal. (b) If R has a minimal non-zero prime ideal, then $p.\dim_R Q = 1$ and the simple divisible R-modules are exactly the non-zero quotients of Q (up to isomorphism).*

PROOF. (a) Suppose that there exist two non-zero prime ideals P_1 and P_2 of R such that $P_1 \cap P_2$ does not contain non-zero prime ideals of R. Let S be the complement of $P_1 \cup P_2$ in R and let R_S be the ring of fractions of R with respect to the multiplicatively closed subset S. Then R_S has exactly two maximal ideals and every non-zero prime ideal of R_S is contained in exactly one of these two maximal ideals. In particular R_S is a non-local, h-local domain, so that Q/R_S is a decomposable R_S-module [7, Th. 22]. Therefore Q/R_S is a decomposable R-module and in particular it is not simple divisible, contradiction. This proves (a).

(b) Suppose that there exists a prime ideal P of R minimal among the non-zero prime ideals of R. Then P is is unique by (a), and if $x \in P$ and $x \neq 0$, then x is contained in every non-zero prime ideal of R. In particular Q coincides with the ring of fractions of R with respect to the multiplicatively closed subset $\{x^n | n \in \mathbf{N}\}$. Therefore Q is a countably generated R-module, $p.\dim_R Q = 1$ and every simple divisible module is a quotient of Q. □

EXAMPLE 5. If R is a valuation domain, then $p.\dim_R Q = 1$ if and only if Q is a countably generated R-module [4, Th. IV.2.4]. In this case the simple divisible R-modules are exactly the non-zero quotients of Q. In fact every simple divisible R-module is h-divisible because $p.\dim Q = 1$, and conversely every quotient of Q is uniserial, hence simple divisible by Lemma 5.

The hypothesis $p.\dim_R Q = 1$ cannot be eliminated because of the possible existence of nonstandard uniserial R-modules (Example 2).

EXAMPLE 6. Matlis has proved that if R is a Noetherian integral domain, then all the non-zero quotients of Q are simple divisible if and only if the integral closure of R in Q is a discrete valuation ring that is a finitely generated R-module [6, Th. 2]. If these equivalent conditions hold, then R is a local domain of Krull dimension one [6, Th. 2] and the simple divisible R-modules are exactly the non-zero quotients of Q (up to isomorphism, Proposition 12).

For instance, if R is a complete, Noetherian, local domain of Krull dimension one, then the simple divisible R-modules are exactly the non-zero quotients of Q [6, p. 579]. Here "complete" can be understood both in the R-adic topology and in the M-adic topology (M the maximal ideal of R), because the two topologies coincide for a Noetherian local domain of dimension one.

EXAMPLE 7. We give an example of an integral domain R such that: (1) the simple divisible R-modules are exactly the non-zero quotients of Q (up to isomorphism); (2) R is complete in its R-adic topology and $p.\dim Q = 1$; (3) the projective class group

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of R can be any fixed abelian group (in particular R is not local); (4) if D is the simple divisible R-module Q/R, then the set of ideals $\{\mathrm{Ann}_R\, d | d \in D\}$ is not totally ordered under inclusion.

In order to construct such an R, recall that any abelian group can be realized as the projective class group of a Dedekind domain S [2]. Let K be the field of fractions of S and let V be a complete valuation domain (not a field) with residue field K. Suppose that Q, the field of fractions of V, is a countably generated V-module. Let R be the fiber product of S and V over K, that is, $R = \pi^{-1}(S)$, where $\pi : V \longrightarrow K$ is the canonical projection. We shall now show that the ring R has the required properties.

Let $M = \pi^{-1}(0)$ denote the maximal ideal of V and $m \in M$ a fixed non-zero element. Then M is a prime ideal in R and Q is the field of fractions of R. If $\{q_n | n \in \mathbf{N}\}$ is a set of generators of Q as a V-module, then $\{m^{-1}q_n | n \in \mathbf{N}\}$ is a set of generators of Q as an R-module, because $Rm^{-1}q_n \supseteq Mm^{-1}q_n \supseteq Vq_n$ for every n. In particular $p.\dim_R Q = 1$.

Let $(I, \leqq)$ be a directed set and $C : I \longrightarrow R$ be a Cauchy net in R endowed with the R-adic topology. Let $\epsilon : R \longrightarrow V$ denote the inclusion mapping. Then it is easy to see that $\epsilon C : I \longrightarrow V$ is a Cauchy net in V with the V-adic topology (because every neighborhood vV of 0 in V with the V-adic topology contains the neighborhood mvR of 0 in R with the R-adic topology). But V is complete, and if ϵC converges to $v_0 \in V$, then $v_0 \in R = \pi^{-1}(S)$ (because there exists $i_0 \in I$ such that $\epsilon C(i_0) - v_0 \in mV$, so that $\pi(v_0) = \pi\epsilon C(i_0) \in S$). It is now easy to see that C converges to v_0 in R, and this proves that R is complete in the R-adic topology. Hence 2) holds.

Since $p.\dim_R Q = 1$, every simple divisible R-module is a non-zero quotient of Q. Conversely let A be a proper R-submodule of Q. Fix $q \in Q \backslash A$. Then $A \subseteq qV$, otherwise there exists $a \in A$, $a \notin qV$, so that $aV \supset qV$; since aM is the unique maximal V-submodule of aV, we have $aM \supseteq qV$, and in particular $q \in aM \subseteq aR \subseteq A$, contradiction. Therefore $A \subseteq qV \subseteq qm^{-1}M \subseteq qm^{-1}R$, that is, A is contained in a cyclic R-submodule of Q. It follows that Q/A is isomorphic to Q/I for some ideal I of R. By Corollary 11, $Q/A \cong Q/I$ is a simple divisible R-module. This concludes the proof of 1).

For the proof of 3) it is sufficient to note that the canonical group homomorphism $\tau : P(R) \longrightarrow P(S)$ given by $P \longmapsto P \otimes_R S \cong P/MP$ is an isomorphism. It is injective because if $P \in P(R)$ and $P \otimes_R S \cong S$, then $P \otimes_R S \cong P/MP$ is a cyclic R-module. If $x \in M$, then $1 - x$ is invertible in V, so that $(1 - x)v = 1$ for some $v \in V$. But then $S \ni \pi(1) = \pi((1 - x)v) = \pi(1 - x)\pi(v) = \pi(v)$, i.e., $v \in \pi^{-1}(S) = R$ and $1 - x$ is invertible in R. This proves that M is contained in the Jacobson radical of R. Now P is finitely generated, P/MP is cyclic and M is contained in the Jacobson radical, so that P is cyclic by the Nakayama Lemma. This proves that τ is injective.

In order to prove that τ is surjective, fix an invertible S-module P'. Then P' is isomorphic to an ideal of S, i.e., $P' \cong Ss_1 + \cdots + Ss_n$ with $s_i \in S$. Let $r_1, \ldots, r_n$ be representatives of $s_1, \ldots, s_n$ in R and let P be the ideal $Rr_1 + \cdots + Rr_n$ of R. Then $P \otimes_R V = PV$ is a finitely generated ideal of V, hence it is cyclic, and in particular projective. Moreover for each $s_i \neq 0$, the corresponding r_i is an invertible element in V, so that $Mr_i = M$. Hence $MP = M$, and in particular $P \otimes_R S \cong P/MP = Rr_1 + \cdots + Rr_n/M \cong Ss_1 + \cdots + Ss_n \cong P'$ is a projective ideal of S. Since $P \otimes_R V$ and $P \otimes_R S$ are projective V- and S-modules respectively, P is a projective R-module by [13, Th. 1.1]. This shows that P is invertible and τ is surjective. Note that we have proved that every finitely generated ideal of R is projective, that is, R is semihereditary.

Finally the set of all non-zero principal ideals of R is not totally ordered under inclusion because R is not a valuation domain. Now when $D = Q/R$ and $r \in R$ is non-zero, one has $\mathrm{Ann}_R(r^{-1} + R) = rR$, so that the set of ideals $\{\mathrm{Ann}_R d | d \in D\}$ contains the set of all non-zero principal ideals of R. One concludes that the set $\{\mathrm{Ann}_R\, d | d \in D\}$ is not totally ordered under inclusion.

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