BULL. AUSTRAL. MATH. SOC. 73D15, 73G05, 76Q05
VOL. 8 (1973), 75-92.

Interchange of modal properties in the propagation of harmonic waves in heat-conducting materials

P. Chadwick

A study is made of the secular equation governing the propagation of plane harmonic waves of small amplitude in a continuum which is able to conduct heat. This equation defines an algebraic function, called the modal function, whose regular branches specify the slownesses of the possible modes of harmonic wave propagation as functions of the frequency. At each extreme of the frequency range one mode is diffusive in type and the others wave-like, and we suppose here that there is a single wave-like mode which produces changes of temperature. In this case the mode which is diffusive in type at low frequencies is wave-like or diffusive in the high-frequency limit according as the thermoelastic coupling constant (a dimensionless measure of the strength of thermo-mechanical interaction in the continuum) does or does not exceed unity. This property is shown to have a simple interpretation in terms of a Riemann surface of the modal function. The results obtained are quite general, referring to principal longitudinal waves in a homogeneously deformed isotropic heat-conducting elastic material, to dilatational disturbances of a stress-free configuration of such a material, and to acoustic waves in a heat-conducting inviscid fluid.

Received 18 September 1972. This paper was written during the author's tenure of a Visiting Professorship in the University of Queensland. He is grateful to the Department of Mathematics for its generous hospitality and to Dr A.F. Johnson for helpful correspondence on the work presented here.

1. Introduction

We examine in this paper some aspects of the behaviour of plane progressive harmonic waves of small amplitude travelling through a homogeneous continuum which permits the transfer of energy by thermal conduction as well as through the rate of working of the stress. The dependence upon position and time in such a wave motion is expressed by the exponential factor

$$\exp\{i\omega(s\mathbf{x}\cdot\mathbf{n}-t)\} \tag{1}$$

in which ω (assumed real) is the angular frequency and s (in general complex) the slowness of the wave, and $\mathbf{x}$, $\mathbf{n}$, t are in turn the position vector of a representative point of the continuum in its undisturbed state, a unit vector specifying the direction of propagation, and the time. The slowness is determined by a polynomial relation, often referred to as the *secular equation*, in which the coefficients are complex-valued functions of ω, and the roots of the secular equation correspond to possible *modes* (or *branches*) of harmonic wave propagation in the prescribed direction.

When the slowness is given the representation

$$s = v^{-1} + i\omega^{-1}q \ , \tag{2}$$

where v and q are real, the wave-form (1) becomes

$$\exp(-q\mathbf{x}\cdot\mathbf{n})\exp\{i\omega(v^{-1}\mathbf{x}\cdot\mathbf{n}-t)\} \tag{3}$$

from which it is apparent that v is the speed of propagation and q the attenuation coefficient of the mode with slowness s. This mode progresses in the direction of the wave normal $\mathbf{n}$ and gives rise to distributions of displacement and temperature which remain uniformly bounded as $\mathbf{x}\cdot\mathbf{n} \to \infty$ if and only if

$$v > 0 \quad \text{and} \quad q \geq 0 \ . \tag{4}$$

In the low-frequency limit $\omega \to 0$ a mode for which v approaches a positive value $\hat{v}$ and $q = O(\omega^2)$ is said to be *wave-like* since the limiting form of the exponential (3) then satisfies a linear wave equation (with characteristic wave speed $\hat{v}$); a mode for which there is a constant $\hat{\eta}$ such that $v \sim \hat{\eta}\omega^{\frac{1}{2}}$, $q \sim \hat{\eta}^{-1}\omega^{\frac{1}{2}}$ is referred to as being *diffusive in type* as $\omega \to 0$ since the limiting form of (3) then satisfies a linear

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

diffusion equation (with diffusivity $\frac{1}{2}\hat{\eta}^2$). Modes which are wave-like and diffusive in type in the high-frequency limit are likewise characterized by the respective properties $v \to \tilde{v}$, $q = O(1)$ and $v \sim \tilde{\eta}\omega^{\frac{1}{2}}$, $q \sim \tilde{\eta}^{-1}\omega^{\frac{1}{2}}$ as $\omega \to \infty$.

When heat conduction is the only dissipative process operating in the continuum one mode of harmonic wave propagation is diffusive in type and the others wave-like at each extreme of the frequency range for all choices of wave normal. In published investigations of body waves traversing a heat-conducting elastic material which is stress-free in its undisturbed state consideration has been restricted to circumstances in which the character (wave-like or diffusive) of each of the possible modes is the same when $\omega \to \infty$ as when $\omega \to 0$ [2, pp. 280-297, 3]. But recent work on the propagation of sound in a heat-conducting inviscid fluid [6] and in mixtures of such fluids [4] indicates the possibility that two modes, one wave-like and the other diffusive in type in the low-frequency limit, may undergo an exchange of properties as ω increases, the mode which is wave-like when $\omega \to 0$ being diffusive and the other mode wave-like in the high-frequency limit. In Sections 3 and 4 we give a complete analysis of this interchange effect, choosing for simplicity the case in which there is a single wave-like mode giving rise to a non-uniform temperature distribution. This situation is of basic significance in the theories of small amplitude wave propagation in isotropic elastic solids and inviscid fluids and the formulation of the appropriate secular equation presented in Section 2 is sufficiently general to allow results pertinent to these theories to emerge as special cases.

2. The secular equation

We consider in the first place a homogeneous elastic body which possesses a reference configuration R in which the density and temperature are uniform and the material is isotropic in its mechanical and thermal response. When disturbed by the passage of small amplitude waves the body is assumed to occupy an equilibrium configuration B which is reached from R by a homogeneous deformation together with a uniform temperature change. We denote by a_1, a_2, a_3 the principal stretches measured from the reference configuration R , by $\sigma_1, \sigma_2, \sigma_3$ the

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

principal Cauchy stresses and by T the temperature.

With each direction in the configuration H there are associated four modes of plane harmonic wave propagation, three of them wave-like and the other diffusive in type at each extreme of the frequency range. Each mode excites distributions of displacement and temperature change (both measured from H) which vary harmonically with place and time according to (1) (or (3)). A mode is said to be *uncoupled* if either the displacement or the temperature change vanishes identically and *coupled* if neither of these fields is zero, and the terms *longitudinal* and *transverse* apply respectively to modes for which the displacement and the wave normal are parallel and orthogonal.

We confine attention to principal waves, that is waves for which the direction of propagation is aligned with a principal axis of stress in H . Two of the modes are then uncoupled and transverse and the other two coupled and longitudinal. The uncoupled modes propagate without change of temperature and they are wave-like when $\omega \to 0$ and when $\omega \to \infty$. At each of these extremes one of the coupled modes is wave-like and the other diffusive in type but, as we shall see in Section 4, the character of a given mode may or may not be the same in both limits.

In general the secular equation governing small amplitude harmonic disturbances of the homogeneously deformed state H is a bi-quartic in the slowness, but in the case of a principal wave two factors, corresponding to the transverse modes, can be removed leaving a bi-quadratic equation expressible in the form

$$(5) \qquad w^2 - (1+\varepsilon-z)w - z = 0 \ ,$$

where

$$(6) \qquad w = (s\tilde{v})^{-2} \ , \quad z = i(\omega/\omega^*) \ ,$$

and

$$(7) \quad \rho\tilde{v}^2 = a_i(\partial\sigma_i/\partial a_i) \ , \quad \varepsilon = (T/\rho^2 c\tilde{v}^2)(\partial\sigma_i/\partial T)^2 \ , \quad \omega^* = \rho c\tilde{v}^2/k_i \ ,$$

the wave normal $\mathbf{n}$ being taken to lie along the principal axis of stress associated with the principal value σ_i . In equations (7), ρ is the density, c the specific heat at constant deformation and k_i the ith

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

principal value of the thermal conductivity tensor[1], each evaluated in H, and the values of a_i, T and the partial derivatives of σ_i also relate to this configuration. We refer to ε and ω^* in turn as the *thermoelastic coupling constant* and the *characteristic thermoelastic frequency* associated with the configuration H and the wave normal $\mathbf{n}$. Throughout the later analysis it is supposed that ε is strictly positive.[2]

Before embarking upon the discussion of equation (5) which is the main purpose of this paper we take note of the special cases mentioned in Section 1.

(a) Waves in a natural configuration of an isotropic heat-conducting elastic material. When R is a natural (that is, stress-free) configuration and H coincides with R the orientation of the principal axes of stress in H becomes arbitrary and, no matter what the direction of propagation, a small amplitude disturbance is a principal wave. It can be shown that, in this case,

$$\partial\sigma_i/\partial a_i = \lambda + 2\mu \;, \quad \partial\sigma_i/\partial T = -\alpha K \;, \quad k_i = k \;, \tag{8}$$

where λ, μ are the isothermal Lamé constants, α the volume coefficient of thermal expansion, $K = \lambda + \frac{2}{3}\mu$ the isothermal bulk modulus and k the thermal conductivity of the material, all evaluated in the natural configuration R. Equations (7) therefore become

$$\tilde{v}^2 = (\lambda+2\mu)/\rho_0 \;, \quad \varepsilon = \alpha^2 T_0 K^2/\rho_0^2 c_0 \tilde{v}^2 \;, \quad \omega^* = \rho_0 c_0 \tilde{v}^2/k \;, \tag{9}$$

the suffix 0 denoting evaluation in R [*cf.* 2, pp. 276–282].

(b) Waves in a heat-conducting inviscid fluid. A heat-conducting inviscid fluid may be regarded as a heat-conducting elastic material for which the Helmholtz free energy and the heat flux depend upon the

[1] For an isotropic heat-conducting elastic material at uniform temperature the thermal conductivity tensor is symmetric and coaxial with the Cauchy stress tensor.

[2] When $\varepsilon = 0$ the modes of harmonic wave propagation governed by equation (5) are uncoupled. We therefore exclude this case from further consideration, but it may be noted that the speeds of propagation and the attenuation coefficients of the modes over the entire frequency range are found by setting $\varepsilon = 0$ in equations (31) and discarding the remainder terms.

deformation through the density. It then follows that the stress is hydrostatic and hence, denoting the pressure by p ,

(10) $$a_i(\partial\sigma_i/\partial a_i) = \rho(\partial p/\partial\rho)$$

which is the isothermal bulk modulus. As in case (a) the principal axes of stress in H can be chosen arbitrarily which implies that a plane harmonic disturbance of H is necessarily a principal wave. Further, the transverse modes have infinite slowness and are therefore unable to propagate. The appropriate specializations of equations (7) are

(11) $$\tilde{v}^2 = K/\rho \ , \quad \varepsilon = \alpha^2 TK/\rho c = \gamma - 1 \ , \quad \omega^* = cK/k \ ,$$

where the symbols on the right-hand sides retain their previous meanings but are now evaluated in the undisturbed configuration H , and γ is the quotient of the specific heat at constant pressure by c , the specific heat at constant volume [*cf.* 7, pp. 58-61]. As would be expected in results referring to a fluid continuum, the formulae (11) for $\tilde{v}$, ε and ω^* no longer involve the reference configuration R .

3. The modal function and its regular branches

In the discussion of the secular equation (5) which now follows we regard z as a complex variable and write

(12) $$z = \zeta + i\chi \ , \quad \chi = \omega/\omega^* \ .$$

It must of course be borne in mind that, in view of equation $(6)_2$, the only part of the z-plane concerned in the calculation of modal properties is the positive imaginary axis, given by $\zeta = 0$, $\chi \geq 0$.

Equation (5) defines a two-valued algebraic function $\underline{w}$, which we refer to as the *modal function*, in the domain M obtained by deleting from the z-plane the zeros of the discriminant of the left-hand side of (5). The discriminant is

(13) $$-z^2 - 2(1-\varepsilon)z - (1+\varepsilon)^2$$

and its zeros are the conjugate points $z = Z, \bar{Z}$, where

(14) $$Z = -(1-\varepsilon) + 2i\varepsilon^{\frac{1}{2}} \ .$$

At each point of M , $\underline{w}$ has two distinct values and the totality of such

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

values determines two regular branches of $\underline{w}$, w_1 and w_2, which can be given explicit representations by simply solving the quadratic equation (5):

(15) $$w_1(z) = \tfrac{1}{2}\{1+\varepsilon-z+d(z)\} \ , \quad w_2(z) = \tfrac{1}{2}\{1+\varepsilon-z-d(z)\} \ .$$

Here $d(z)$ is a square root of the negative of the discriminant (13) and we select the branch of $(z-Z)^{\frac{1}{2}}(z-\overline{Z})^{\frac{1}{2}}$ defined by

(16) $$d(z) = |z-Z|^{\frac{1}{2}}|z-\overline{Z}|^{\frac{1}{2}}\exp\{\tfrac{1}{2}i\arg(z-Z)+\tfrac{1}{2}i\arg(z-\overline{Z})\}$$

with

(17) $$-\tfrac{3}{2}\pi < \arg(z-Z) \le \tfrac{1}{2}\pi \ , \quad -\tfrac{1}{2}\pi \le \arg(z-\overline{Z}) < \tfrac{3}{2}\pi \ .$$

The singular points $z = Z, \overline{Z}$ of the modal function $\underline{w}$ are branch points of w_1 and w_2 and, in accordance with the definitions (17), each branch is regular in the domain obtained by cutting the z-plane from $z = Z$ to infinity and from $z = \overline{Z}$ to infinity in the manner shown in Figure 1. A two-sheeted Riemann surface for $\underline{w}$ may then be constructed by joining the domains of regularity of the branches w_1 and w_2, the left edges of the cuts on sheet 1 (the domain of regularity of w_1) being attached to the right edges of the cuts on sheet 2 and vice versa. A sketch of the resulting surface is given in Figure 2 and for ease of reference we apply the term *leaf* to the parts of the Riemann surface shown there as planes[3]; leaves 1 and 2 contain right half-planes of sheets 1 and 2 respectively.

The positions of the singular points in the z-plane are determined by the thermoelastic coupling constant ε through equation (14). As ε increases from zero they move along the upper and lower halves of the parabola given by $\chi^2 = 4(\zeta+1)$, crossing the imaginary axis when $\varepsilon = 1$ (see Figure 1).

[3] The crucial property distinguishing the leaves of the Riemann surface from the sheets is the inclusion in each leaf of a neighbourhood of the point at infinity.

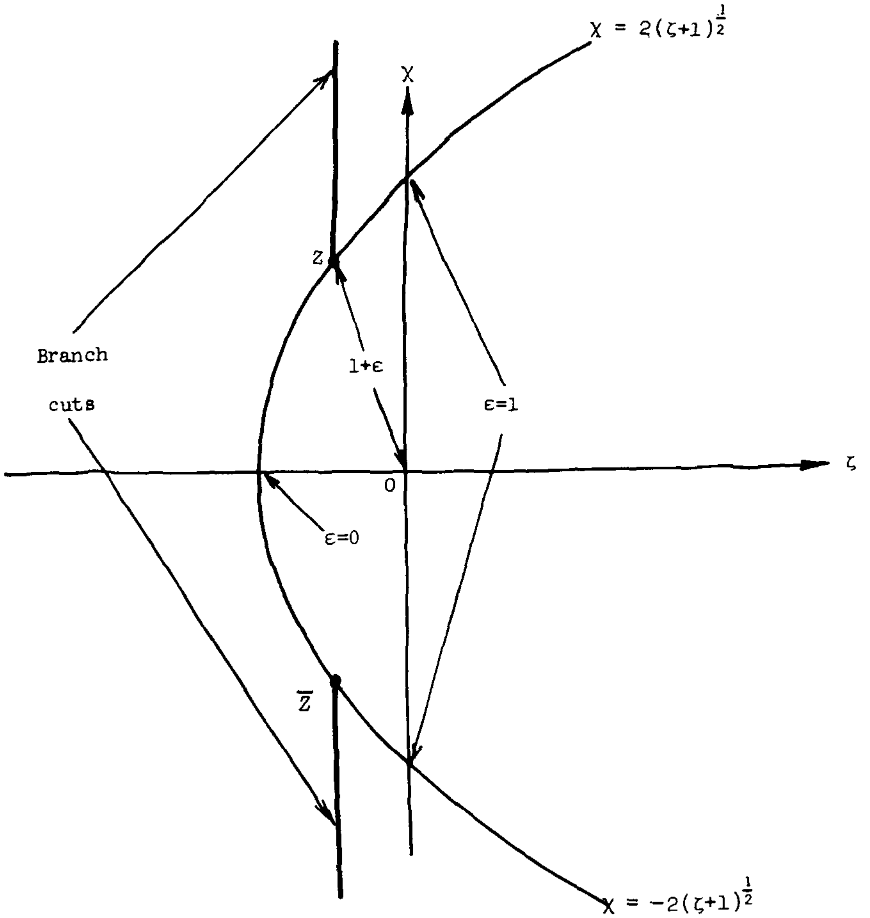

Figure 1. Positions in the complex z-plane of the branch points and branch cuts.

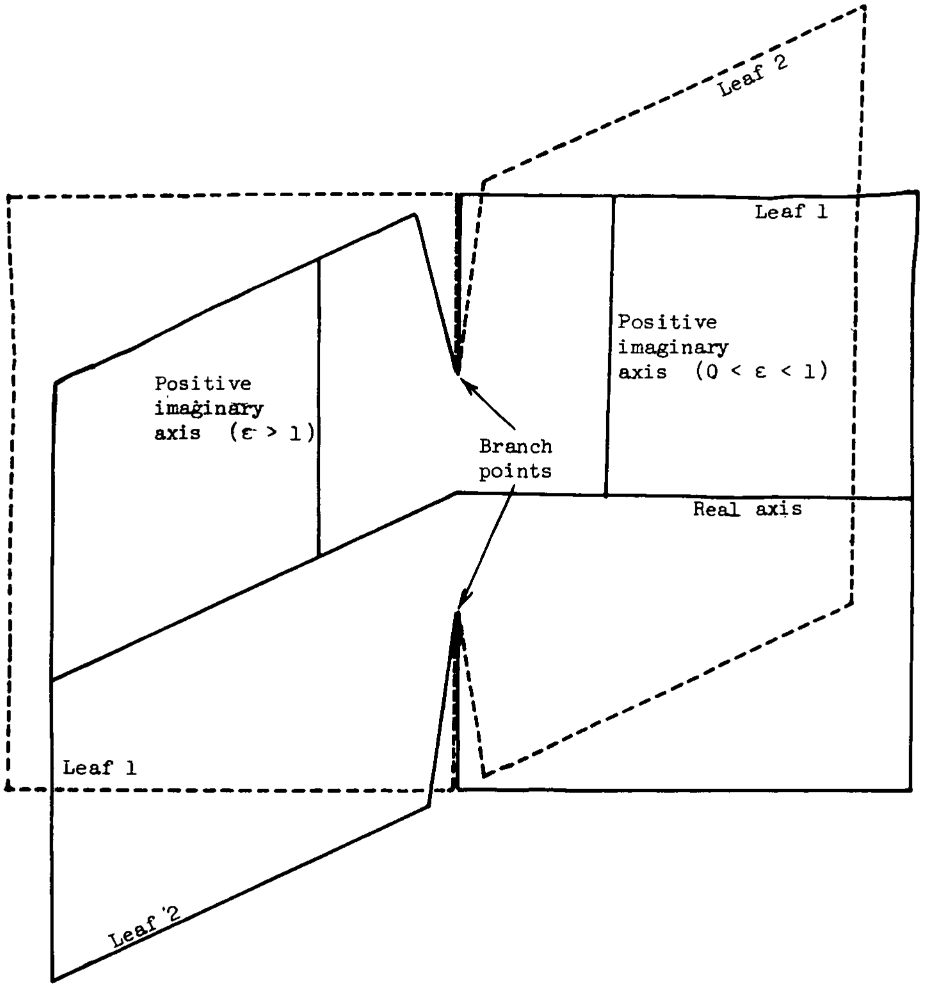

Figure 2. Schematic representation of a Riemann surface of the modal function $\underline{w}$. Sheets 1 and 2 (the domains of regularity of the branches w_1 and w_2) are indicated by solid and broken lines respectively.

4. *Properties of the modes of harmonic wave propagation determined by the regular branches of* $\underline{w}$

The regular branches of the modal function $\underline{w}$ are associated with the possible modes of wave propagation admitted by the secular equation (5) and we proceed in this section to derive from equations (15) properties of the two modes. The modes described by w_1 and w_2 are referred to as *mode* 1 and *mode* 2 respectively and we denote by v_1, v_2 and q_1, q_2 their respective speeds of propagation and attenuation coefficients.

Equations (2), (6) and $(12)_2$ together yield

$$\{w_\alpha(i\chi)\}^{-\frac{1}{2}} = \frac{\tilde{v}}{v_\alpha} + i\,\frac{\tilde{v}q_\alpha}{\omega^*\chi} \quad (\alpha = 1,\ 2)\ , \tag{18}$$

and we observe that if

$$\operatorname{Im} w_\alpha(i\chi) < 0 \quad (\alpha = 1,\ 2)\ , \tag{19}$$

the choice in equation (18) of the square root of $\{w_\alpha(i\chi)\}^{-1}$ with positive real part leads to the conditions (4) being satisfied, with $(4)_2$ holding as a strict inequality. At a given frequency, (19) are therefore sufficient conditions for the modes to propagate and to be asymptotically stable in the sense explained in Section 1.

(a) The complete solution. The evaluation of the regular branches w_1 and w_2 on the positive imaginary axis of the z-plane is facilitated by dividing the frequency range at $\chi = 1 + \varepsilon$ and introducing two pairs of angles, ϕ, Φ and ψ, Ψ , defined as follows.

For $0 \leqq \chi < 1 + \varepsilon$,

$$\sin\phi = \frac{\chi}{1+\varepsilon} \quad (0 \leqq \phi < \tfrac{1}{2}\pi)\ , \quad \tan\Phi\sec\Phi = \frac{1-\varepsilon}{1+\varepsilon}\tan\phi\sec\phi \quad (-\tfrac{1}{2}\pi < \Phi < \tfrac{1}{2}\pi)\ . \tag{20}$$

For $\chi > 1 + \varepsilon$,

$$\sin\psi = \frac{1+\varepsilon}{\chi} \quad (0 < \psi < \tfrac{1}{2}\pi)\ , \quad \tan\Psi\sec\Psi = \frac{1-\varepsilon}{1+\varepsilon}\tan\psi\sec\psi \quad (-\tfrac{1}{2}\pi < \Psi < \tfrac{1}{2}\pi)\ . \tag{21}$$

As χ increases from 0 to $1 + \varepsilon$, ϕ increases from 0 to $\frac{1}{2}\pi$, while Φ increases from 0 to $\frac{1}{2}\pi$ when $0 < \varepsilon < 1$ and decreases from 0

to $-\frac{1}{2}\pi$ when $\varepsilon > 1$; as χ increases from $1 + \varepsilon$, ψ decreases from $\frac{1}{2}\pi$ to 0 and Ψ decreases from $\frac{1}{2}\pi$ to 0 or increases from $-\frac{1}{2}\pi$ to 0 according as $0 < \varepsilon < 1$ or $\varepsilon > 1$. When $\varepsilon = 1$, $\Phi = \Psi = 0$, and we note the inequalities

$$(22)\qquad \begin{cases} 0 < \phi-\Phi < \pi \ , \quad 0 < \phi+\Phi < \pi \quad \text{for} \quad 0 < \chi < 1+\varepsilon \ , \\ 0 < \psi-\Psi < \pi \ , \quad 0 < \psi+\Psi < \pi \quad \text{for} \quad \chi > 1+\varepsilon \ . \end{cases}$$

Using the definitions (20) and (21) in conjunction with equations (15), (16) and the inequalities (17) we find that

$$(23)\qquad \left.\begin{aligned} w_1(i\chi) &= (1+\varepsilon)\sec\Phi\cos\tfrac{1}{2}(\phi-\Phi)\{\cos\tfrac{1}{2}(\phi+\Phi)-i\sin\tfrac{1}{2}(\phi-\Phi)\} \\ w_2(i\chi) &= (1+\varepsilon)\sec\Phi\sin\tfrac{1}{2}(\phi+\Phi)\{\sin\tfrac{1}{2}(\phi-\Phi)-i\cos\tfrac{1}{2}(\phi+\Phi)\} \end{aligned}\right\} \ 0 \le \chi < 1+\varepsilon \ ,$$

$$(24)\qquad \left.\begin{aligned} w_{1+\beta}(i\chi) &= (1+\varepsilon)\operatorname{cosec}\psi\sec\Psi\sin\tfrac{1}{2}(\psi+\Psi)\{\cos\tfrac{1}{2}(\psi+\Psi)-i\sin\tfrac{1}{2}(\psi-\Psi)\} \\ w_{2-\beta}(i\chi) &= (1+\varepsilon)\operatorname{cosec}\psi\sec\Psi\cos\tfrac{1}{2}(\psi-\Psi)\{\sin\tfrac{1}{2}(\psi-\Psi)-i\cos\tfrac{1}{2}(\psi+\Psi)\} \end{aligned}\right\} \ \chi > 1+\varepsilon \ ,$$

where

$$(25)\qquad \beta = \begin{cases} 0 & \text{when} \quad 0 < \varepsilon \le 1 \ , \\ 1 & \text{when} \quad \varepsilon > 1 \ . \end{cases}$$

When the dimensionless frequency χ has the transitional value $1 + \varepsilon$,

$$(26)\qquad \begin{cases} w_1\big(i(1+\varepsilon)\big) = \begin{cases} \frac{1}{2}(1+\varepsilon)\{(1+\delta)-i(1-\delta)\} & \text{when} \quad 0 < \varepsilon \le 1 \ , \\ \frac{1}{2}(1+\varepsilon)(1+\delta)(1-i) & \text{when} \quad \varepsilon > 1 \ , \end{cases} \\ w_2\big(i(1+\varepsilon)\big) = \begin{cases} \frac{1}{2}(1+\varepsilon)\{(1-\delta)-i(1+\delta)\} & \text{when} \quad 0 < \varepsilon \le 1 \ , \\ \frac{1}{2}(1+\varepsilon)(1-\delta)(1-i) & \text{when} \quad \varepsilon > 1 \ , \end{cases} \end{cases}$$

where

$$(27)\qquad \delta = \{|1-\varepsilon|/(1+\varepsilon)\}^{\frac{1}{2}} \quad (0 \le \delta < 1) \ ,$$

and it may readily be verified that equations (26) follow from (23) and (24) on proceeding to the limits $\chi \uparrow 1+\varepsilon$ and $\chi \downarrow 1+\varepsilon$ respectively.

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

Inspection of equations (23) to (27) shows that, by virtue of the inequalities (22), the sufficient conditions (19) are fulfilled for all $\chi > 0$. The secular equation (5) therefore entails the existence, at all positive frequencies, of two propagating modes which are asymptotically stable.

Expressions for the speeds of propagation and attenuation coefficients of the modes can be derived in a straightforward manner from equations (23), (24) and (26) with use of (18). The following formulae specify v_1, v_2 and q_1, q_2 as functions of χ over the entire frequency range.

For $0 \le \chi < 1+\varepsilon$,

$$(28)\quad \left\{\begin{aligned} v_1/\tilde{v} &= \left[2m(1+\varepsilon)\sec\Phi\cos\tfrac{1}{2}(\phi-\Phi)\{1+m^{-1}\cos\tfrac{1}{2}(\phi+\Phi)\}^{-1}\right]^{\frac{1}{2}}, \\ v_2/\tilde{v} &= \left[2m(1+\varepsilon)\sec\Phi\sin\tfrac{1}{2}(\phi+\Phi)\{1+m^{-1}\sin\tfrac{1}{2}(\phi-\Phi)\}^{-1}\right]^{\frac{1}{2}}, \\ q_1\tilde{v}/\omega^* &= \left[\tfrac{1}{2}m^{-1}(1+\varepsilon)\sin^2\phi\cos\Phi\sec\tfrac{1}{2}(\phi-\Phi)\{1-m^{-1}\cos\tfrac{1}{2}(\phi+\Phi)\}\right]^{\frac{1}{2}}, \\ q_2\tilde{v}/\omega^* &= \left[\tfrac{1}{2}m^{-1}(1+\varepsilon)\sin^2\phi\cos\Phi\operatorname{cosec}\tfrac{1}{2}(\phi+\Phi)\{1-m^{-1}\sin\tfrac{1}{2}(\phi-\Phi)\}\right]^{\frac{1}{2}}, \end{aligned}\right.$$

where $m = (1-\sin\phi\sin\Phi)^{\frac{1}{2}}$.

For $\chi > 1+\varepsilon$,

$$(29)\quad \left\{\begin{aligned} v_{1+\beta}/\tilde{v} &= \left[2n(1+\varepsilon)\operatorname{cosec}\psi\sec\Psi\sin\tfrac{1}{2}(\psi+\Psi)\{1+n^{-1}\cos\tfrac{1}{2}(\psi+\Psi)\}^{-1}\right]^{\frac{1}{2}}, \\ v_{2-\beta}/\tilde{v} &= \left[2n(1+\varepsilon)\operatorname{cosec}\psi\sec\Psi\cos\tfrac{1}{2}(\psi-\Psi)\{1+n^{-1}\sin\tfrac{1}{2}(\psi-\Psi)\}^{-1}\right]^{\frac{1}{2}}, \\ q_{1+\beta}\tilde{v}/\omega^* &= \left[\tfrac{1}{2}n^{-1}(1+\varepsilon)\operatorname{cosec}\psi\cos\Psi\operatorname{cosec}\tfrac{1}{2}(\psi+\Psi)\{1-n^{-1}\cos\tfrac{1}{2}(\psi+\Psi)\}\right]^{\frac{1}{2}}, \\ q_{2-\beta}\tilde{v}/\omega^* &= \left[\tfrac{1}{2}n^{-1}(1+\varepsilon)\operatorname{cosec}\psi\cos\Psi\sec\tfrac{1}{2}(\psi-\Psi)\{1-n^{-1}\sin\tfrac{1}{2}(\psi-\Psi)\}\right]^{\frac{1}{2}}, \end{aligned}\right.$$

where $n = (1-\sin\psi\sin\Psi)^{\frac{1}{2}}$ and β is again given by (25).

For $\chi = 1+\varepsilon$,

$$
(30)\quad
\left.\begin{aligned}
v_1/\tilde{v} &= 2\{2(1+\varepsilon)^{-\frac{1}{2}}+1+\delta\}^{-\frac{1}{2}} , \\
v_2/\tilde{v} &= 2\{2(1+\varepsilon)^{-\frac{1}{2}}+1-\delta\}^{-\frac{1}{2}} , \\
q_1\tilde{v}/\omega^* &= \tfrac{1}{2}(1+\varepsilon)\{2(1+\varepsilon)^{-\frac{1}{2}}-1-\delta\}^{\frac{1}{2}} , \\
q_2\tilde{v}/\omega^* &= \tfrac{1}{2}(1+\varepsilon)\{2(1+\varepsilon)^{-\frac{1}{2}}-1+\delta\}^{\frac{1}{2}} ,
\end{aligned}\right\}\ \text{when}\quad 0 < \varepsilon \le 1 ,
$$

$$
\left.\begin{aligned}
v_1/\tilde{v} &= \{2(2^{\frac{1}{2}}-1)(1+\varepsilon)(1+\delta)\}^{\frac{1}{2}} , \\
v_2/\tilde{v} &= \{2(2^{\frac{1}{2}}-1)(1+\varepsilon)(1-\delta)\}^{\frac{1}{2}} , \\
q_1\tilde{v}/\omega^* &= \{\tfrac{1}{2}(2^{\frac{1}{2}}-1)(1+\varepsilon)(1+\delta)^{-1}\}^{\frac{1}{2}} , \\
q_2\tilde{v}/\omega^* &= \{\tfrac{1}{2}(2^{\frac{1}{2}}-1)(1+\varepsilon)(1-\delta)^{-1}\}^{\frac{1}{2}} ,
\end{aligned}\right\}\ \text{when}\quad \varepsilon > 1 .
$$

(b) The characters of modes 1 and 2. We now derive from equations (28) to (30) limiting forms of the speeds of propagation and attenuation coefficients of the modes appropriate to the lower and upper extremes of the frequency range. Making use of the definitions (20) and (21) we find that

$$
(31)\quad
\left.\begin{aligned}
v_1 &= \tilde{v}(1+\varepsilon)^{\frac{1}{2}}\{1+O(\chi^2)\} , \\
v_2 &= \tilde{v}\{2(1+\varepsilon)^{-1}\chi\}^{\frac{1}{2}}\{1+O(\chi)\} , \\
q_1 &= (\varepsilon\omega^*/2\tilde{v})(1+\varepsilon)^{-\frac{5}{2}}\chi^2\{1+O(\chi^2)\} , \\
q_2 &= (\omega^*/\tilde{v})\{\tfrac{1}{2}(1+\varepsilon)\chi\}^{\frac{1}{2}}\{1+O(\chi)\} ,
\end{aligned}\right\}\ \text{as}\quad \chi \to 0 ,
$$

(*cf.* [2, p. 285]) and

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

$$(32)\quad \left.\begin{aligned} v_{1+\beta} &= \tilde{v}\{1+O(\chi^{-2})\}\ , \\ v_{2-\beta} &= \tilde{v}(2\chi)^{\frac{1}{2}}\{1+O(\chi^{-1})\}\ , \\ q_{1+\beta} &= (\varepsilon\omega^*/2\tilde{v})\{1+O(\chi^{-2})\}\ , \\ q_{2-\beta} &= (\omega^*/\tilde{v})(\tfrac{1}{2}\chi)^{\frac{1}{2}}\{1+O(\chi^{-1})\}\ , \end{aligned}\right\}\ \text{as}\ \ \chi \to \infty\ .$$

From equations (31) mode 1 is seen to be wave-like and mode 2 diffusive in type in the low-frequency limit and equations (32), with (25), show that the modes preserve these characters in the high-frequency limit when $0 < \varepsilon \leq 1$. But when $\varepsilon > 1$, mode 1 is diffusive in type and mode 2 wave-like as $\chi \to \infty$, putting in evidence the interchange of modal properties to which we have alluded in Sections 1 and 2. The interchange effect is associated with the transfer of the branch points $z = Z, \bar{Z}$ from the left to the right half of the z-plane and it can be interpreted in the following way with regard to the Riemann surface of the modal function $\underline{w}$ depicted in Figure 2.

When $0 < \varepsilon < 1$ the branch cuts lie in the left half of the z-plane and a path starting at the origin of sheet 1 and proceeding along the positive imaginary axis leads to the point at infinity of leaf 1. The implication is that mode 1 preserves its wave-like character throughout the frequency range and that mode 2 is similarly entirely diffusive in type. But when $\varepsilon > 1$ the branch cuts are situated in the right half of the z-plane and reference to Figure 2 shows that a description of the positive imaginary axis of sheet 1 from the origin now leads to the point at infinity of leaf 2, indicating that mode 1, which is wave-like as $\omega \to 0$, is diffusive in type in the high-frequency limit while mode 2 displays the converse pattern of behaviour.

The right-hand sides of equations (31) supply the leading terms of expansions of the speeds of propagation and attenuation coefficients in powers of χ . There series are obtained from Taylor expansions of the regular branches w_1 and w_2 about $z = 0$ which, as an element of the domain M , is an ordinary point of the modal function $\underline{w}$. Now a basic result in the theory of algebraic functions [1, pp. 24-29] asserts that the radius of convergence of such a Taylor series is the distance from the ordinary point on which it is centred to the nearest singularity of the algebraic function. In the present instance this distance is $|Z| = |\bar{Z}| = 1 + \varepsilon$. Expansions of modal properties in direct powers of the dimensionless frequency χ therefore have radius of convergence $1 + \varepsilon$.[4] We note in passing that the point $z = i(1+\varepsilon)$ (corresponding to $\chi = 1 + \varepsilon$) lies on the circle of convergence with centre $z = 0$; this explains the significance of the division of the frequency range made in subsection (a). The right-hand sides of equations (32) likewise contain the leading terms in developments of v_1, v_2 and q_1, q_2 in inverse powers of χ and, by further appeal to the basic theorem mentioned above, the radius of convergence of such series is found to be $(1+\varepsilon)^{-1}$.

(c) The critical case $\varepsilon = 1$. The nature of the interchange effect revealed by the foregoing analysis is further illuminated by an examination of the situation, arising when $\varepsilon = 1$, in which an exchange of modal characters is imminent. Results referring to this special case have been given by Dillon [5] in connection with small amplitude disturbances of a natural configuration of an isotropic heat-conducting elastic material and by McKinney and Oser [6] for acoustic wave propagation in a heat-conducting inviscid fluid, but the following simple closed-form solution, obtained by setting $\Phi = \Psi = 0$ in equations (28) and (29), appears to have passed unnoticed in previous work.

For $0 \le \chi \le 2$,

$$(33)\quad \begin{cases} v_1/\tilde{v} = (2\cos\tfrac{1}{2}\phi)^{\frac{1}{2}}\sec\tfrac{1}{4}\phi\ , & q_1\tilde{v}/\omega^* = 2(\sin\phi\sin\tfrac{1}{2}\phi)^{\frac{1}{2}}\sin\tfrac{1}{4}\phi\ , \\ v_2/\tilde{v} = (2\sin\tfrac{1}{2}\phi)^{\frac{1}{2}}\sec\tfrac{1}{4}(\pi-\phi)\ , & q_2\tilde{v}/\omega^* = 2(\sin\phi\cos\tfrac{1}{2}\phi)^{\frac{1}{2}}\sin\tfrac{1}{4}(\pi-\phi)\ , \end{cases}$$

with $\phi = \sin^{-1}(\tfrac{1}{2}\chi)$.

For $\chi \ge 2$,

[4] This result corrects an erroneous statement about the convergence properties of such power series made by the author in an earlier paper [2, p. 285].

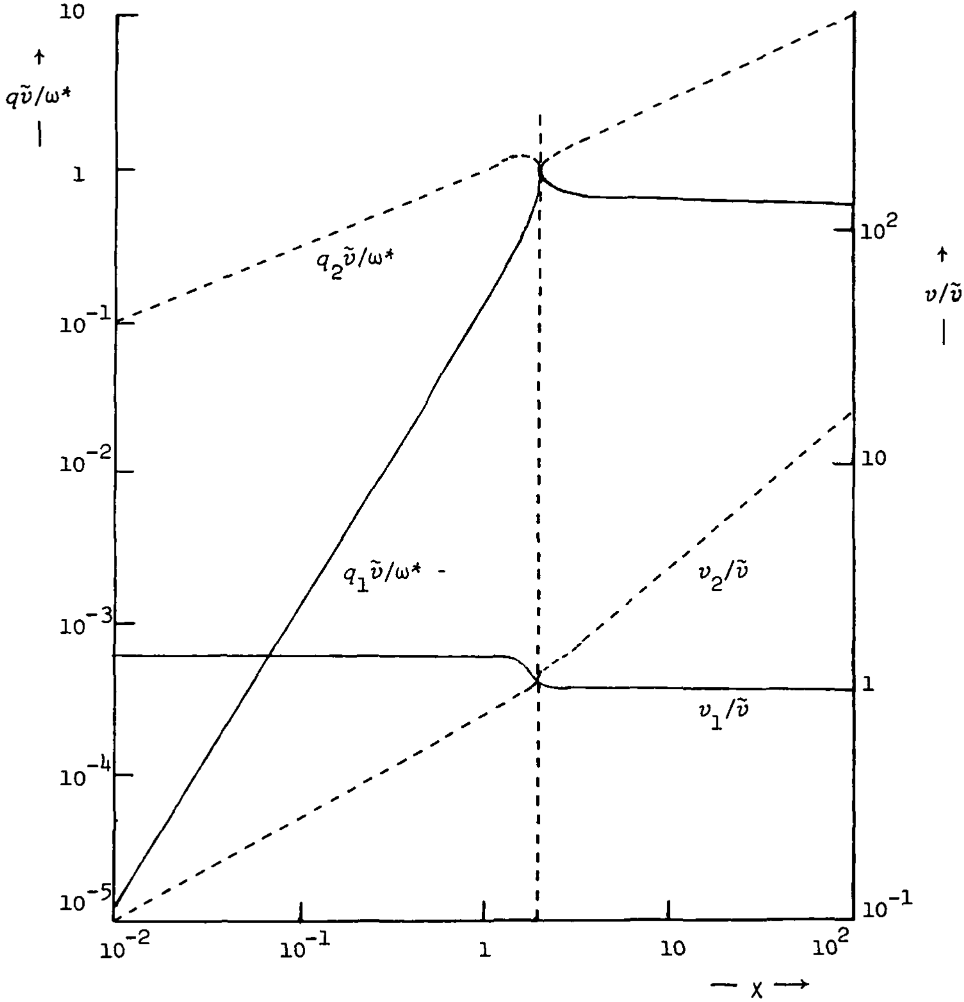

Figure 3. Graphical representation of the variation with frequency of the speeds of propagation and attenuation coefficients of the two modes in the critical case $\varepsilon = 1$. ——— mode 1, ----- mode 2.

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

$$(34)\quad \begin{cases} v_1/\tilde{v} = (\cos\tfrac{1}{2}\psi)^{-\frac{1}{2}}\sec\tfrac{1}{4}\psi\ , & q_1\tilde{v}/\omega^* = (\tfrac{1}{2}\sin\psi\sin\tfrac{1}{2}\psi)^{-\frac{1}{2}}\sin\tfrac{1}{4}\psi\ , \\ v_2/\tilde{v} = (\sin\tfrac{1}{2}\psi)^{-\frac{1}{2}}\sec\tfrac{1}{4}(\pi-\psi)\ , & q_2\tilde{v}/\omega^* = (\tfrac{1}{2}\sin\psi\cos\tfrac{1}{2}\psi)^{-\frac{1}{2}}\sin\tfrac{1}{4}(\pi-\psi)\ , \end{cases}$$

with $\psi = \sin^{-1}(2/\chi)$.

When $\chi = 2$ the values of $w_1(i\chi)$ and $w_2(i\chi)$ are equal ($i\chi$ then coinciding with a singular point of the modal function $\underline{w}$) and we find from equations (33) and (34) that

$$v_1/\tilde{v} = v_2/\tilde{v} = 2\left(2^{\frac{1}{2}}-1\right)^{\frac{1}{2}}\ ,\quad q_1\tilde{v}/\omega^* = q_2\tilde{v}/\omega^* = \left(2^{\frac{1}{2}}-1\right)^{\frac{1}{2}}\ .$$

In addition,

$$\frac{d}{d\chi}\left(\frac{v_1}{\tilde{v}}\right) \to -\infty\ ,\quad \frac{d}{d\chi}\left(\frac{v_2}{\tilde{v}}\right) \to \infty \quad \text{as} \quad \chi \to 2\ ,$$

$$\frac{d}{d\chi}\left(\frac{q_1\tilde{v}}{\omega^*}\right) \to \begin{cases} \infty & \text{as} \quad \chi \uparrow 2\ , \\ -\infty & \text{as} \quad \chi \downarrow 2\ , \end{cases} \qquad \frac{d}{d\chi}\left(\frac{q_2\tilde{v}}{\omega^*}\right) \to \begin{cases} -\infty & \text{as} \quad \chi \uparrow 2\ , \\ \infty & \text{as} \quad \chi \downarrow 2\ . \end{cases}$$

The graphs of the speeds of propagation therefore cross at $\chi = 2$ where they both have inflexions with vertical tangent, and the graphs of the attenuation coefficients also meet at $\chi = 2$ and each have cusps with vertical tangent at the point of intersection. These features are prominent in Figure 3 which shows the variations of $v_1/\tilde{v}$, $v_2/\tilde{v}$ and $q_1\tilde{v}/\omega^*$, $q_2\tilde{v}/\omega^*$ with χ on log-linear and log-log scales respectively. The incipient interchange of modal properties predicted by the analysis is clearly displayed by these curves.

References

[1] Gilbert Ames Bliss, *Algebraic functions* (Colloquium Publ. 16. Amer. Math. Soc., New York, 1933).

[2] P. Chadwick, "Thermoelasticity. The dynamical theory", *Progress in solid mechanics*, Vol. I, 265-328 (edited by I.N. Sneddon and R. Hill, North-Holland, Amsterdam, 1960).

https://doi.org/10.1017/S0004972700045482 Published online by Cambridge University Press

[3] P. Chadwick and L.T.C. Seet, "Wave propagation in a transversely isotropic heat-conducting elastic material", *Mathematika* 17 (1970), 255-274.

[4] R.E. Craine and A.F. Johnson, "Acoustic wave propagation in a binary mixture of inviscid fluids", *J. Sound Vib.* 20 (1972), 191-207.

[5] O.W. Dillon, Jr, "Thermoelasticity when the material coupling parameter equals unity", *J. Appl. Mech.* 32 (1965), 378-382.

[6] J.E. McKinney and H.J. Oser, "Acoustic propagation and stability within an inviscid, heat-conducting fluid", *J. Res. Nat. Bur. Standards Sect. B* 74 (1970), 67-84.

[7] R.N. Thurston, "Wave propagation in fluids and normal solids", *Physical Acoustics*, Vol. I, part A, 1-110 (edited by W.P. Mason. Academic Press, New York and London, 1964).

Department of Mathematics,
University of Queensland,
St Lucia,
Queensland,

and

School of Mathematics and Physics,
University of East Anglia,
Norwich,
England.