Canad. J. Math. 2025, pp. 1–32
http://dx.doi.org/10.4153/S0008414X24001184

Minimum non-chromatic-choosable graphs with given chromatic number

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Abstract. A graph G is called chromatic-choosable if $\chi(G) = ch(G)$. A natural problem is to determine the minimum number of vertices in a non-chromatic-choosable graph with given chromatic number. It was conjectured by Ohba, and proved by Noel, Reed, and Wu that k-chromatic graphs G with $|V(G)| \leq 2k+1$ are chromatic-choosable. This upper bound on $|V(G)|$ is tight. It is known that if k is even, then $G = K_{3\star(k/2+1),1\star(k/2-1)}$ and $G = K_{4,2\star(k-1)}$ are non-chromatic-choosable k-chromatic graphs with $|V(G)| = 2k+2$. Some subgraphs of these two graphs are also non-chromatic-choosable. The main result of this paper is that all other k-chromatic graphs G with $|V(G)| = 2k+2$ are chromatic-choosable. In particular, if $\chi(G)$ is odd and $|V(G)| \leq 2\chi(G)+2$, then G is chromatic-choosable, which was conjectured by Noel.

1 Introduction

A *proper coloring* of a graph G is a mapping $\phi : V(G) \to \mathbb{N}$ such that $\phi(u) \neq \phi(v)$ for every edge uv of $E(G)$. A *k-coloring* of G is a proper coloring of G using colors from $[k] = \{1, 2, \ldots, k\}$. We say G is *k-colorable* if there is a k-coloring of G. The *chromatic number* $\chi(G)$ of G is the minimum k such that G is k-colorable.

List coloring is a natural generalization of classical graph coloring, introduced independently by Erdős–Rubin–Taylor [4] and Vizing [24] in 1970s. A *list assignment* of G is a mapping L which assigns to each vertex v a set $L(v)$ of permissible colors. An *L-coloring* of G is a proper coloring ϕ of G with $\phi(v) \in L(v)$ for each vertex v. We say that G is *L-colorable* if there exists an L-coloring of G, and G is *k-choosable* if G is L-colorable for any list assignment L of G with $|L(v)| \geq k$ for each vertex v. More generally, for a function $g : V(G) \to \mathbb{N}$, we say G is *g-choosable* if G is L-colorable for every list assignment L with $|L(v)| \geq g(v)$ for all $v \in V(G)$. The *choice number* $\mathrm{ch}(G)$ of G is the minimum k for which G is k-choosable.

A k-coloring of a graph G is a special case of list coloring, where each vertex v has the same list $L(v) = \{1, 2, \ldots, k\}$. So k-choosable implies k-colorable. At first glance, one might expect the reverse inequality to hold as well. The smaller intersection between lists would make it easier to assign distinct colors to adjacent vertices. However, the reverse inequality is far from true. It was observed in [4] and [24] that

Received by the editors September 3, 2023; revised September 11, 2024; accepted December 16, 2024.
Published online on Cambridge Core December 27, 2024.
This work was supported by the NSFC (Grant Nos. 12371359 and U20A2068).
AMS Subject Classification: **05C15**.
Keywords: Chromatic-choosable graphs, Ohba conjecture, Noel conjecture, near acceptable L-coloring, extremal graphs.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

for any integer k, there are bipartite graphs that are not k-choosable. So the difference $ch(G) - \chi(G)$ can be arbitrarily large.

A graph G is called *chromatic-choosable* if $\chi(G) = ch(G)$. Chromatic-choosable graphs have been studied a lot in the literature, and are related to some other difficult problems. For example, the famous Dinitz problem (see e.g., [25]) asks the following question:

Given an $n \times n$ array of n-sets, is it always possible to choose one from each set, keeping the chosen elements distinct in every row, and distinct in every column?

This problem can be equivalently stated as whether the line graph of $K_{n,n}$ is chromatic-choosable? This problem was solved by Galvin [5], who proved a more general result: the line graph of any bipartite multigraph is chromatic-choosable. On the other hand, Galvin's result is a special case of a more general conjecture—the list coloring conjecture: line graphs of all multigraphs are chromatic-choosable. The list coloring conjecture was posed independently by many different researchers: Albertson and Collins, Bollobás and Harris, Gupta, and Vizing (see [1, 7, 10]). It has attracted a lot of attention and remains open in general.

Ohba conjecture is another well-known conjecture about chromatic-choosable graphs. It was proved in [18] that for any graph G, $ch(G \vee K_n) = \chi(G \vee K_n)$ for sufficiently large n, where $G \vee H$ is the join of G and H, i.e., the graph obtained from the disjoint union of G and H by adding edges connecting every vertex of G to every vertex of H. This means that graphs G with $|V(G)|$ "close" to $\chi(G)$ are chromatic-choosable. A natural problem is how close should be $|V(G)|$ and $\chi(G)$ to ensure that G be chromatic-choosable. Equivalently, what is the minimum number of vertices in a non-k-choosable k-chromatic graph?

We denote by $K_{k_1 \star n_1, k_2 \star n_2, \dots, k_q \star n_q}$ the complete multi-partite graph with n_i parts of size k_i, for $i = 1, 2, \dots, q$. If $n_j = 1$, then the number n_j is omitted from the notation. It was proved in [3] that if k is an even integer, then $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ are not k-choosable. These two graphs are k-chromatic graphs with $2k+2$ vertices. Ohba [18] conjectured that for any positive integer k, k-chromatic graphs with at most $2k+1$ vertices are k-choosable. This conjecture has attracted considerable attention, and many partial results were proved before it was finally confirmed by Noel, Reed and Wu [17].

One approach has been to prove variants of Ohba's conjecture in which $|V(G)| \leq 2k+1$ is replaced by $|V(G)| \leq \Phi(\chi(G))$ for some function Φ with $\Phi(k) < 2k+1$. Ohba [18] proved such a variant with $\Phi(k) = k + \sqrt{k}$, and Reed and Sudakov [21] improved the result to $\Phi(k) = \frac{5}{3}k - \frac{4}{3}$. By using a sophisticated probabilistic method, Reed and Sudakov [20] proved that Ohba's conjecture is asymptotically true: if $|V(G)| \leq (2 - o(1))\chi(G)$, then G is chromatic-choosable.

Another approach has been to show the conjecture holds for special families of graphs. He, Li, Shen, and Zheng [22] proved Ohba's conjecture for graphs G with independence number $\alpha(G) \leq 3$, by extending a result of Ohba [19] who proved that if $|V(G)| \leq 2\chi(G)$ and $\alpha(G) \leq 3$, then G is chromatic-choosable. Kostochka, Stiebitz, and Woodall [13] improved this result and showed that Ohba conjecture holds for graphs G with $\alpha(G) \leq 5$. Also Ohba's conjecture were verified for some particular complete multipartite graphs in [9, 22, 23].

In 2015, Ohba's conjecture was finally confirmed by Noel, Reed, and Wu [17].

Theorem 1.1 (Noel–Reed–Wu Theorem) *Every k-colorable graph with at most $2k+1$ vertices is k-choosable.*

Nevertheless, this is not the end of the story. More problems related to Ohba's conjecture are posed and studied. One problem is what would be the choice number of k-chromatic graphs G with $|V(G)|$ slightly bigger than $2k+1$. This question was addressed in [16]. Another related problem is the online version of Ohba's conjecture, which was posed in [8], and has been studied in a few papers [2, 12, 14]. Some partial cases are verified and the conjecture remains open in general.

This paper explores the tightness of Ohba's conjecture. Although Ohba's conjecture is tight, $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ for even k are the only known k-chormatic graphs with $2k+2$ vertices that are not k-choosable. In particular, Ohba's conjecture was not known to be tight for odd integer k.

Noel [15] conjectured if k is odd, then all k-chromatic graphs with $2k+2$ vertices are k-choosable.

Observe that for a k-chromatic graph G, by adding edges between vertices of distinct color classes, the resulting graph has the same chromatic number, and whose choice number is not decreased. Therefore in the study of minimum non-chromatic choosable graphs, it suffices to consider complete multipartite graphs.

The main result of this paper is that $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ for even k are the only non-k-choosable complete k-partite graphs with $2k+2$ vertices.

Theorem 1.2 *Assume $G=(V,E)$ is a complete k-partite graph with $|V|\le 2k+2$, and $G\neq K_{4,2\star(k-1)}, K_{3\star(k/2+1),1\star(k/2-1)}$ when k is even, and L is a k-list assignment of G. Then G is L-colorable.*

As a consequence, Noel's conjecture is confirmed.

Corollary 1.3 *If k is odd, then every k-chromatic graph with at most $2k+2$ vertices is chromatic-choosable.*

For a positive integer k, let

$$\beta(k)=\min\{|V(G)|:\chi(G)=k<ch(G)\}.$$

For an odd integer k, it can be checked that $K_{5,2\star(k-1)}$ is not k-choosable. Thus we have the following corollary.

Corollary 1.4 *For the function β defined above,*

$$\beta(k)=\begin{cases}2k+2, & \text{if } k \text{ is even},\\ 2k+3, & \text{if } k \text{ is odd}.\end{cases}$$

Here is a brief outline of the proof of Theorem 1.2.

Assume G is a complete k-partite graph with $2k+2$ vertices, $G\neq K_{4,2\star(k-1)}, K_{3\star(k+1)/2,1\star(k-1)/2}$ when k is even, and L is a k-list assignment of G. Let $C_L=\bigcup_{v\in V}L(v)$. The first step is to construct a family $\mathcal{S}$ of independent sets that form a partition of $V(G)$. Let $G/\mathcal{S}$ be the graph obtained from G by identifying each independent set $S\in\mathcal{S}$ into a single vertex v_S. Let $L_{\mathcal{S}}$ be the list assignment of $G/\mathcal{S}$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

defined as $L_{\mathcal{S}}(v_S) = \bigcap_{u \in S} L(u)$. Build a bipartite graph $B_{\mathcal{S}}$ with partite sets $V(G/\mathcal{S})$ and C_L, with $\{v_S, c\}$ be an edge if $c \in L_{\mathcal{S}}(v_S)$. If $B_{\mathcal{S}}$ has a matching M that covers $V(G/\mathcal{S})$, then M defines an L-coloring of G, with each $S \in \mathcal{S}$ be colored with the color matched to v_S in M.

Assume that there is no such a matching M, and hence by Hall's theorem, there exists a subset $X_{\mathcal{S}}$ of $V(G/\mathcal{S})$ such that $|Y_{\mathcal{S}}| < |X_{\mathcal{S}}|$, where $Y_{\mathcal{S}} = N_{B_{\mathcal{S}}}(X_{\mathcal{S}})$. By analysing the lists $L(v)$ and independent sets S in $\mathcal{S}$, the inequality $|Y_{\mathcal{S}}| < |X_{\mathcal{S}}|$ may lead to a series of inequalities and eventually lead to a contradiction (which means that no such $X_{\mathcal{S}}$ exists and hence the desired matching M exists).

Assume no contradiction is derived, and $X_{\mathcal{S}}$ and $Y_{\mathcal{S}}$ do exist. We choose $X_{\mathcal{S}}$ so that $|X_{\mathcal{S}}| - |Y_{\mathcal{S}}|$ is maximum. By Hall's theorem, this implies that there is a matching M' in $B_{\mathcal{S}} - (X_{\mathcal{S}} \cup Y_{\mathcal{S}})$ that covers $V(G/\mathcal{S}) - X_{\mathcal{S}}$.

Definition 1.1 A *partial L-coloring* of G is an L-coloring of an induced subgraph $G[X]$ of G. Given an L-coloring ϕ of $G[X]$, L^{ϕ} is the list assignment of $G - X$ defined as $L^{\phi}(v) = L(v) - \phi(N_G(v) \cap X)$ for $v \in V(G - X)$. An L-coloring ϕ of $G[X]$ is a *good partial L-coloring* of G if the pair $(G - X, L^{\phi})$ satisfies the condition of Theorem 1.2.

The matching M' constructed above defines a *partial L-coloring* ψ *of* G that colors vertices in $\bigcup_{S \in V(G/\mathcal{S}) - X_{\mathcal{S}}} S$. One nice property of this partial coloring ψ is that if $\{v\} \in X_{\mathcal{S}}$ is a singleton part of $\mathcal{S}$, then $L^{\psi}(v) = L(v)$ (as $L(v) \subseteq Y_{\mathcal{S}}$). In other words some neighbours of v may have been colored, and yet v still has the same set of permissible colors.

By using this property, we want to extend ψ to a good partial L-coloring ϕ of G, that colors a subset X of G. If this can be done, then $G - X$ has an L^{ϕ}-coloring θ, and the union $\phi \cup \theta$ would be an L-coloring of G.

For the plan above to work, the choice of the partition $\mathcal{S}$ of $V(G)$ in the first step is crucial. Indeed, Theorem 1.2 is equivalent to saying that there is a choice of $\mathcal{S}$ such that $B_{\mathcal{S}}$ has a matching M that covers $V(G/\mathcal{S})$. We usually start with a proper coloring f of G, which is not necessarily an L-coloring, but "close" to an L-coloring, and let $\mathcal{S}$ be the color classes of f. In particular, the coloring f uses colors from C_L, and if $f(v) = c \notin L(v)$, then $f^{-1}(c) = \{v\}$ and c is contained in many lists. The concept of "near acceptable" L-coloring is defined to capture the required properties needed for the plan above to work. Near acceptable L-coloring was first used in [17]. The definitions of near acceptable L-colorings for the proofs of Noel–Reed–Wu theorem and Theorem 1.2 are slightly different. The slight difference makes it more difficult to construct a near acceptable L-coloring of G for the proof of Theorem 1.2, while the proof of Noel–Reed–Wu theorem is already complicated. For the proof of Theorem 1.2, before constructing a near acceptable L-coloring of G, a pseudo-L-coloring of G is constructed as an intermediate step. In many cases, we need to repeatedly modify a pseudo L-coloring until we obtain a near acceptable L-coloring.

In Section 2, we prove a sufficient condition for a complete multipartite graph G with all parts of size at most 3 to be g-choosable for a given function $g : V(G) \to \mathbb{N}$. This will be used in later proofs. In Section 3, we fix some notation and present some basic properties of a minimum counterexample. In Section 4, we prove Theorem 1.2 for complete k-partite graphs with most parts of size at most 3. These graphs are special as there is little difference between these graphs and the critical graphs $K_{4,2\star(k-1)}$ and

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$K_{3\star(k/2+1),1(k/2-1)}$ (for even k). In Section 5, we introduce the concept of pseudo-L-coloring of G and prove some properties of such colorings. In Section 6, we define the concept of near-acceptable L-coloring and show that the existence of a near-acceptable L-coloring of G implies the existence of a proper L-coloring of G. Some sufficient conditions for the existence of near-acceptable L-colorings of G are presented in Sections 7 and 8. A final contradiction is derived in Section 9.

2 Graphs with all parts of sizes at most 3

This section proves the following lemma, which gives a sufficient condition for $g : V(G) \to \mathbb{N}$, so that G is g-choosable when all parts of G have size at most 3. This lemma is analog to [14, Lemma 4], where a sufficient condition for G to be on-line g-choosable was given. The sufficient condition below is almost the same as that in [14, Lemma 5], except that for two vertices u, v in a 3-part of G, the upper bounds for the sum $g(u) + g(v)$ in the two lemmas are different, and which is needed in later applications.

Lemma 2.1 *Let G be a complete multipartite graph with parts of size at most 3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be a partition of the parts of G into classes such that $\mathcal{A}$ and $\mathcal{D}$ contain only parts of size 1, $\mathcal{B}$ contains all parts of size 2 and $\mathcal{C}$ contains all parts of size 3. Let k_1, k_2, k_3, d denote the cardinalities of classes $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively. Suppose that classes $\mathcal{A}$ and $\mathcal{D}$ are ordered, i.e., $\mathcal{A} = (A_1, \ldots, A_{k_1})$ and $\mathcal{D} = (D_1, \ldots, D_d)$. If $g : V(G) \to \mathbb{N}$ is a function for which the following hold:*

(a-1) $\quad g(v) \geq k_2 + k_3 + i, \qquad$ *for all* $1 \leq i \leq k_1$ *and* $v \in A_i$

(b-1) $\quad g(v) \geq k_2 + k_3, \qquad$ *for all* $v \in B \in \mathcal{B}$

(b-2) $\quad g(u) + g(v) \geq 3k_3 + 2k_2 + k_1 + d, \qquad$ *for all* $u, v \in B \in \mathcal{B}$

(c-1) $\quad g(v) \geq k_2 + k_3, \qquad$ *for all* $v \in C \in \mathcal{C}$

(c-2) $\quad g(u) + g(v) \geq 2k_3 + 2k_2 + k_1, \qquad$ *for all* $u, v \in C \in \mathcal{C}$

(c-3) $\quad \sum_{v \in C} g(v) \geq 4k_3 + 3k_2 + 2k_1 + d - 1, \qquad$ *for all* $C \in \mathcal{C}$

(d-1) $\quad g(v) \geq 2k_3 + k_2 + k_1 + i, \qquad$ *for all* $1 \leq i \leq d$ *and* $v \in D_i$

then G is g-choosable.

Proof Assume the parts of G are partitioned into $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and g is a function satisfying the inequalities (a-1)–(d-1), and L is a list assignment with $|L(v)| = g(v)$. We shall color an independent set S of G with a color $c \in \bigcap_{v \in S} L(v)$. Let $G' = G - S$ and L' be the list assignment of G' defined as $L'(x) = L(x) - \{c\}$ for $x \in V(G')$ and $g'(v) = |L'(v)|$. We shall verify that the pair (G', f') satisfies the condition of Lemma 2.1, and hence G' is L'-colorable by induction hypothesis (if $|V(G)| = 1$, then the result is trivial). Together with the coloring of S with color c, we obtain an L-coloring of G.

In the following, we describe the choice of the independent set S. The color c is always an arbitrary color in $\bigcap_{v \in S} L(v)$. We describe briefly how to verify the fact that (G', g') satisfies the condition of Lemma 2.1 (the proof of Lemma 5 of [14] is similar, and contains more detailed explanations). The partition $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$ of the parts of

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G' and the ordering of parts in $\mathcal{A}'$ and $\mathcal{D}'$ are inherited from the partition and the ordering of the parts of G, except that one part may have some vertices colored and remaining vertices form a part in another class. When a part from $\mathcal{B}$ or $\mathcal{C}$ has some vertices colored and the remaining vertex form a part in $\mathcal{A}'$ or $\mathcal{D}'$, we also need to put it in a correct order. Denote by k_1', k_2', k_3', d' the cardinalities of $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$, respectively. To verify the inequalities, it suffices to show that with g replaced by g', k_i replaced by k_i' and d replaced by d', the amount reduced on the left hand side is no more than the amount reduced on the right hand side.

The choice of S is determined in 8 cases. For $2 \leq i \leq 8$, Case i is considered only if all cases j with $j \leq i-1$ do not apply.

(1) If there exists $C \in \mathcal{B} \cup \mathcal{C}$ for which $\bigcap_{v \in C} L(v) \neq \emptyset$, then $S = C$.
Verification: For (a-1),(b-1),(c-1), (d-1), the left hand side is reduced by at most 1 (i.e., $g'(v) \geq g(v) - 1$), and the right hand side is reduced by at least 1. (For example, consider (a-1): $k_2' + k_3' + i = k_2 + k_3 + i - 1$). For (b-2), (c-2), the left hand side is reduced by at most 2 (i.e., $g'(u) + g'(v) \geq g(u) + g(v) - 2$), and the right hand side is reduced by at least 2. For (c-3), the left hand side is reduced by at most 3 (i.e., $\sum_{v \in C} g'(v) \geq \sum_{v \in C} g(v) - 3$), and the right hand side is reduced by at least 3.

(2) If there exist $C = \{u, v, w\} \in \mathcal{C}$ with $g(u) + g(v) = 2k_3 + 2k_2 + k_1$, and $L(u) \cap L(v) \neq \emptyset$, then $S = \{u, v\}$.
Verification: The part $\{w\}$ of G' is the last member of $\mathcal{D}'$. Thus $k_3' = k_3 - 1$ and $d' = d + 1$. Note that $g'(w) = g(w) \geq 4k_3 + 3k_2 + 2k_1 + d - 1 - (2k_3 + 2k_2 + k_1) = 2k_3 + k_2 + k_1 + d - 1 = 2k_3' + k_2' + k_1' + d'$. The other inequalities are verified as in Case 1.

(3) If there exists $C = \{v, u, w\} \in \mathcal{C}$, $g(v) = k_2 + k_3$, $L(v) \cap L(u) \neq \emptyset$, then $S = \{u, v\}$.
Verification: The part $\{w\}$ of G' is the last member of $\mathcal{A}'$. Thus $k_3' = k_3 - 1$ and $k_1' = k_1 + 1$. Note that $g'(w) = g(w) \geq 2k_3 + 2k_2 + k_1 - (k_3 + k_2) = k_3 + k_2 + k_1 = k_3' + k_2' + k_1'$. For $u, v \in C \in \mathcal{C}$, either $g(u) + g(v) \geq 2k_3 + 2k_2 + k_1 + 1$ or $g'(u) + g'(v) \geq g(u) + g(v) - 1$ (as Case 2 does not apply). Hence (c-2) holds for (G', g'). As Case 1 does not apply, the left hand side of (c-3) reduces by at most 2, and the right hand side is reduced by 2. Hence (c-3) holds for (G', g') as Case 1 does not apply. The other inequalities are verified as in Case 1.

(4) If there exists $C = \{v, u, w\} \in \mathcal{C}$, $g(v) = k_2 + k_3$, $L(v) \cap (L(u) \cup L(w)) = \emptyset$, then $S = \{v\}$.
Verification: In the remaining graph $G' = G - v$, the two vertices u, w are identified into a single vertex u^* with $L'(u^*) = L(u) \cap L(w)$. The set $\{u^*\}$is the last member of $\mathcal{A}'$. So $k_3' = k_3 - 1$, $k_1' = k_1 + 1$. Note that

$$g(u) + g(w) \geq (4k_3 + 3k_2 + 2k_1 + d - 1) - (k_3 + k_2) = 3k_3 + 2k_2 + 2k_1 + d - 1.$$

On the other hand the total number of colors is at most $|V| - 1 = 3k_3 + 2k_2 + k_1 + d - 1$. As $L(v)$ is disjoint with $L(u) \cup L(w)$, we have $|L(u) \cup L(w)| \leq 2k_3 + k_2 + k_1 + d - 1$. Hence

$$|L'(u^*)| = |L(u) \cap L(w)| \geq k_3 + k_2 + k_1 = k_3' + k_2' + k_1'.$$

Note that for $C \in \mathcal{C}$, $\sum_{v \in C} g'(v) \geq \sum_{v \in C} g(v) - 2$, as Case 1 does not apply. Hence (c-3) holds for (G', g'). The other inequalities are verified as in Case 3.

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(5) If there exists $B = \{u, v\} \in \mathcal{B}$, $g(v) = k_2 + k_3$, then $S = \{v\}$.
Verification: The part $\{u\}$ of G' is the last member of $\mathcal{D}'$. Thus $k_2' = k_2 - 1$ and $d' = d + 1$. Note that $g'(u) = g(u) \geq 3k_3 + 2k_2 + k_1 + d - (k_3 + k_2) = 2k_3 + k_2 + k_1 + d = 2k_3' + k_2' + k_1' + d'$. For $B' = \{x, y\} \in \mathcal{B}$, since Case 1 does not apply, $g'(x) + g'(y) \geq g(x) + g(y) - 1$. So (b-2) holds for (G', g'). The other inequalities are verified as in Case 4.

(6) If $k_1 \neq 0$ and $A_1 = \{v\}$, then $S = \{v\}$.
Verification: In this case, $k_1' = k_1 - 1$. As Cases 2,3,4 do not apply, (b-1), (c-1), and (c-2) were not tight for g, and hence they hold for (G', g'). Also for (a-1), the index of each member reduces by 1, and hence the right hand side reduces by 1, so it holds for (G', g'). The other inequalities are verified as in Case 5.

(7) Assume $k_3 \neq 0$ and $C = \{u, v, w\} \in \mathcal{C}$. As $|C_L| \leq |V| - 1 = 3k_3 + 2k_2 + k_1 + d - 1$, So $g(u) + g(v) + g(w) \geq 4k_3 + 3k_2 + 2k_1 + d - 1 > |C_L|$ and there is a color c which appears in two of the three color sets $L(u)$, $L(v)$, $L(w)$, say $c \in L(u) \cap L(v)$. Let $S = \{u, v\}$.
Verification: Let $\{w\}$be the only member of $\mathcal{A}'$. Then $k_3' = k_3 - 1$ and $k_1' = k_1 + 1$, $g'(w) = g(w) \geq k_2 + k_3 = k_2' + k_3' + 1 = k_2' + k_3' + k_1'$. The other inequalities are verified as in Case 6.

(8) If $d > 0$ and $D_1 = \{v\}$, then $S = \{v\}$.
Verification: In this case, $k_3 = k_1 = 0$ and $d' = d - 1$. (b-1) is not tight for g (as Case 5 does not apply), and hence holds for (G', g'). (b-2) holds for (G', g') as the left-hand size reduces by at most 1, and the right hand side reduces by 1. For other member of $\mathcal{D}'$, its index is recued by 1, and hence (d-1) holds for (G', g'). Note that $k_1, k_3 = 0$, so the other inequalities are vacant.

Assume all the cases above do not apply. Then $G = K_{2\star k_2}$, i.e., G consists of k_2 parts of size 2, and $g(v) \geq k_2$ for each vertex v. It is well-known [4] that in this case, G is g-choosable. ∎

3 Some notation and basic properties for a minimum counterexample

By a counterexample of Theorem 1.2, we mean a pair (G, L) such that G is a complete multipartite graph and L is a list assignment of G that satisfy the condition of Theorem 1.2, and G is not L-colorable. We say (G, L) is a minimal counterexample to Theorem 1.2 if (G, L) is a counterexample to Theorem 1.2 with

(1) $|V(G)|$ minimum,
(2) subject to (1), with $|C_L|$ minimum (recall that $C_L = \bigcup_{v \in V} L(v)$),

It is well-known [11] that $|C_L| < |V(G)|$. Let

(3.1) $$\lambda = |V| - |C_L| > 0.$$

In the remainder of this paper, we assume that (G, L) is a minimum counterexample to Theorem 1.2. Assume G is a complete k-partite graph. By Noel–Reed–Wu theorem, we know that k-chromatic graphs with at most $2k + 1$ vertices are k-choosable and hence G has exactly $2k + 2$ vertices, and

(3.2) $$|C_L| \leq 2k + 1.$$

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A part of G of size i (respectively, at least i or at most i) is called a *i-part* (respectively, *i^+-part*, or *i^--part*). Let

$$T = \{v : \{v\} \text{ is a singleton part of } G\}.$$

Let p_i, p_i^+ and p_i^- be the number of i-parts, i^+-parts and i^--parts, respectively.

For a subset X of $V(G)$, let

$$L(X) = \bigcup_{v \in X} L(v).$$

For three vertices x, y, z of G, let

$$L(x \vee y) = L(x) \cup L(y), L(x \wedge y) = L(x) \cap L(y),$$

$$L((x \wedge y) \vee z) = (L(x) \cap L(y)) \cup L(z).$$

For $c \in C_L$ and $C' \subseteq C_L$, let

$$L^{-1}(c) = \{v : c \in L(v)\},\ L^{-1}(C') = \bigcup_{c \in C'} L^{-1}(c).$$

For a part P of G and integer i, let

$$C_{P,i} = \{c \in C : |L^{-1}(c) \cap P| = i\},$$
$$\Lambda_{P,i} = \max\{|\bigcap_{v \in S} L(v)| : S \subseteq P, |S| = i\}.$$

.

Assume $\mathcal{S}$ is a partition of $V(G)$ into a family of independent sets. Each $S \in \mathcal{S}$ is called an $\mathcal{S}$ *part*. Recall that $G/\mathcal{S}$ is the graph obtained from G by identifying each part $S \in \mathcal{S}$ into a single vertex v_S, and $L_{\mathcal{S}}$ is the list assignment of $G/\mathcal{S}$ defined as $L_{\mathcal{S}}(v_S) = \bigcap_{v \in S} L(v)$. If $S = \{v\} \in \mathcal{S}$ consists of a single vertex of G, then we denote v_S by v. In this case, $L_{\mathcal{S}}(v) = L(v)$. For the partitions $\mathcal{S}$ constructed in this paper, most parts of $\mathcal{S}$ are singletons. To define $\mathcal{S}$, it suffices to list its non-singleton parts.

Recall that $B_{\mathcal{S}}$ is the bipartite graph with partite sets $V(G/\mathcal{S})$ and C_L, in which $\{v_S, c\}$ is an edge if and only if $c \in L_{\mathcal{S}}(v_S)$. A matching M in $B_{\mathcal{S}}$ covering $V(G/\mathcal{S})$ induces an $L_{\mathcal{S}}$-coloring of $G/\mathcal{S}$, which in turn induces an L-coloring of G. Since G is not L-colorable, no such matching M exists. By Hall's theorem, there is a subset $X_{\mathcal{S}}$ of $V(G/\mathcal{S})$ such that $|X_{\mathcal{S}}| > |N_{B_{\mathcal{S}}}(X_{\mathcal{S}})|$.

We denote by $X_{\mathcal{S}}$ a subset of $V(G/\mathcal{S})$ for which $|X_{\mathcal{S}}| - |N_{B_{\mathcal{S}}}(X_{\mathcal{S}})|$ is maximum. Let

$$Y_{\mathcal{S}} = N_{B_{\mathcal{S}}}(X_{\mathcal{S}}) = \bigcup_{v_S \in X_{\mathcal{S}}} L_{\mathcal{S}}(v_S).$$

The choice of $X_{\mathcal{S}}$ implies that there is a matching $M_{\mathcal{S}}$ in $B_{\mathcal{S}} - (X_{\mathcal{S}} \cup Y_{\mathcal{S}})$ that covers all vertices in $V(G/\mathcal{S}) - X_{\mathcal{S}}$. The matching $M_{\mathcal{S}}$ defines a partial coloring $\psi_{\mathcal{S}}$ of $G[\bigcup_{S \in \mathcal{S} - X_{\mathcal{S}}} S]$ with colors from $C_L - Y_{\mathcal{S}}$.

These notation will be used throughout the whole paper.

Observation 3.1 *The following easy facts will be used often in the argument.*

(1) *There is an injective mapping $\phi : C_L \to V$ such that $c \in L(\phi(c))$.*
(2) *If f is a proper coloring of G, then there is a surjective proper coloring $g : V \to C_L$ such that for every vertex v, $g(v) \in L(v)$ or $g(v) = f(v)$.*

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(3) *No two vertices in the same part of G have the same list, and no color is contained only in the lists of vertices in a same part.*
(4) $G \neq K_{4,2\star(k-1)}$ *for any k and* $|T| \geq 1$.

Proof (1) is well-known ([17, Corollary 1.8]) and also easy to verify (use the minimality of $|C_L|$).

(2) was proved in [17, Proposition 1.13].

(3) If u, v are in the same part and $L(u) = L(v)$, then By Noel–Reed–Wu theorem, there is a proper L-coloring f of $G - u$, which extends to a proper L-coloring of G by letting $f(u) = f(v)$.

If there is a color c such that $L^{-1}(c) \subseteq P_i$ for some part P_i of G, then by Noel–ReedWu theorem, $G - L^{-1}(c)$ has an L-coloring f, which extends to an L-coloring of G by coloring vertices in $L^{-1}(c)$ with color c.

(4) It was proved in [3] that $K_{4,2\star(k-1)}$ is not k-choosable if and only if k is even. By our assumption, $G \neq K_{4,2\star(k-1)}$ for even k. Thus $G \neq K_{4,2\star(k-1)}$ for any k. It was proved in [6] that $G = K_{3\star 2,2\star(k-2)}$ is k-choosable. Using the fact that $|V(G)| = 2k + 2$, it is easy to see that $|T| \geq 1$. ∎

Lemma 3.2 *If P is a* 2^+*-part of G, then* $\bigcap_{v \in P} L(v) = \emptyset$. *Consequently for each color* $c \in C_L$, $|L^{-1}(c)| \leq k + p_1 + 2$.

Proof Assume the lemma is not true. We choose such a part P of maximum size, and color vertices in P by a common color c. Let $L'(v) = L(v) - \{c\}$ for $v \in V(G) - P$. If $|P| \geq 3$, then L' and $G - P$ satisfies the condition of Noel–Reed–Wu theorem and hence $G - P$ has an L'-coloring.

Assume $|P| = 2$. By (4) of Observation 3.1, $G - P \neq K_{4,2\star(k-2)}$. If $G - P \neq K_{3\star(q+1),1\star(q-1)}$, then by the minimality of G, $G - P$ has an L'-coloring. If $G - P = K_{3\star(q+1),1\star(q-1)}$, then since each 3-part P has at most two vertices v for which $c \in L(v)$, it is straightforward to verify that $G - P$ and L' satisfy the condition of Lemma 2.1. Hence $G - P$ has an L'-coloring.

For any color $c \in C$, each 2^+-part contains a vertex $v \notin L^{-1}(c)$. So

$$|L^{-1}(c)| \leq |V(G)| - p_2^+ = 2k + 2 - (k - p_1) = k + p_1 + 2.$$

This completes the proof of Lemma 3.2. ∎

4 Graphs with most parts of size at most 3

In this section, we consider complete k-partite graphs whose most parts are 3^--parts.

Let

$$\mathcal{G}_1 = \{K_{5,3\star(q-1),2\star(k-2q),1\star q} : k \geq 2q \geq 2\},$$
$$\mathcal{G}_2 = \{K_{4\star a,3\star(q-a),2\star b,1\star(k-q-b)} : a \leq 2, a \leq q, b \geq 0, q + b \leq k, a + 2q + b = k + 2.\}$$

Theorem 4.1 $G \notin \mathcal{G}_1 \cup \mathcal{G}_2$.

We may assume that $k \geq 8$, as for $k \leq 7$, we can check directly the graphs in $\mathcal{G}_1, \mathcal{G}_2$ are k-choosable.

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Assume $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. We order the parts of G as $P_1, P_2, \ldots, P_k$ so that

- if $G \in \mathcal{G}_1$, then P_1 is the 5-part and $P_2, P_3, \ldots, P_q$ are 3-parts with $\Lambda_{P_2,2} \ge \Lambda_{P_3,2} \ge \ldots \ge \Lambda_{P_q,2}$;
- if $G \in \mathcal{G}_2$, then the first a parts are the 4-parts of G, and $P_{a+1}, P_{a+2}, \ldots, P_q$ are 3-parts with $\Lambda_{P_2,2} \ge \Lambda_{P_3,2} \ge \ldots \ge \Lambda_{P_q,2}$. If $a = 2$, then order P_1, P_2 so that $\Lambda_{P_1,3} \ge \Lambda_{P_2,3}$.

Let

$$i_0 = \max\{j : \Lambda_{P_j,2} \ge j\}.$$

For a 3-part P of G, we have $3k \le \sum_{v \in P} |L(v)| \le |C_L| + |C_{P,2}| \le 2k + 1 + |C_{P,2}|$. So $|C_{P,2}| \ge k - 1$. As P has three 2-subsets, we have $\Lambda_{P,2} \ge (k-1)/3 \ge 2$.

Claim 4.2 *If $G \in \mathcal{G}_1$, then $C_{P_1,4} = \varnothing$ and $C_{P_1,3} \ne \varnothing$.*

Proof If $c \in C_{P_1,4}$, then we color vertices in $L^{-1}(c) \cap P_1$ with color c, and let $L'(v) = L(v) - \{c\}$ for $v \in G - (L^{-1}(c) \cap P_1)$. It is easy to verify that $G' = G - (L^{-1}(c) \cap P_1)$ and L' satisfy the condition of Lemma 2.1 (with $P_1 - L^{-1}(c)$ being the last part in $\mathcal{A}$, and with $\mathcal{D} = \varnothing$), and hence G' is L'-colorable, and G is L-colorable, a contradiction.

If $C_{P_1,3} = \varnothing$, then each color $c \in C_L$ is contained in $L(v)$ for at most two vertices $v \in P_1$. Hence $2(2k+1) \ge 2|C_L| \ge \sum_{v \in P_1} |L(v)| = 5k$, which implies that $k \le 2$, a contradiction. ∎

Claim 4.3 $G \ne K_{5,2\star(k-2),1}$.

Proof If $G = K_{5,2\star(k-2),1}$, then fix a 3-subset S_1 of P_1 with $\bigcap_{v \in S_1} L(v) \ne \varnothing$. Let $\mathcal{S}$ be the partition of $0V(G)$ with one non-singleton part S_1. Then $|V(G/\mathcal{S})| = 2k$ and hence $|X_\mathcal{S}| \le 2k$ and $|Y_\mathcal{S}| \le 2k - 1$. By Lemma 3.2, $|X_\mathcal{S} \cap P| \le 1$ for any 2-part P. So $|X_\mathcal{S}| \le k + 2$ and $|Y_\mathcal{S}| \le k + 1$. On the other hand, $|X_\mathcal{S}| \ge 2$ and hence $v \in X_\mathcal{S}$ for some vertex v with $|L_\mathcal{S}(v)| \ge k$ and hence $|Y_\mathcal{S}| \ge k$ and $|X_\mathcal{S}| \ge k+1$, and hence $|X_\mathcal{S} \cap P_1'| \ge 2$. This in turn implies that $|Y_\mathcal{S}| = k + 1$ and hence $|X_\mathcal{S}| = k + 2$. Then $P_1' \subseteq X_\mathcal{S}$ and $|Y_\mathcal{S}| \ge |L_\mathcal{S}(P_1')| \ge k + 2 = |X_\mathcal{S}|$ (by Claim 4.2), a contradiction. ∎

It follows from Observation 3.1 that $G \ne K_{4,2\star(k-1)}$ for any k. As $G \ne K_{5,2\star(k-2),1}$, G has at least two 3^+-parts. Therefore

$$i_0 \ge 2.$$

For $i = 1, 2, \ldots, i_0$, we shall choose a subset S_i of P_i of size 2 or 3, and let $\mathcal{S}$ be the partition of $V(G)$ with non-singleton parts $\{S_1, S_2, \ldots, S_{i_0}\}$. The rules for choosing the sets S_i will be given later.

For simplicity, in the graph $G/\mathcal{S}$, for $i = 1, 2, \ldots, i_0$, we denote v_{S_i} by z_i, and let

$$Z = \{z_1, z_2, \ldots, z_{i_0}\}.$$

We denote by P_i' the part of $G/\mathcal{S}$, where for $1 \le i \le i_0$, P_i' is obtained from the part P_i by identifying S_i into a new vertex z_i, and for $i_0 + 1 \le i \le k$, $P_i' = P_i$.

As $i_0 \ge 2$, we have $|V(G/\mathcal{S})| \le 2k$, and hence

$$|X_\mathcal{S}| \le 2k, \quad |Y_\mathcal{S}| \le 2k - 1. \tag{4.1}$$

We shall prove further upper and lower bounds for $|X_\mathcal{S}|$ and $|Y_\mathcal{S}|$ that eventually lead to a contradiction.

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The details are delicate and a little complicated, which is perhaps unavoidable, as $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ (for even integer k) are very close to graphs in $\mathcal{G}_1 \cup \mathcal{G}_2$, and they are not k-choosable. We divide the proofs for $G \notin \mathcal{G}_1$ and $G \notin \mathcal{G}_2$ into two subsections.

4.1 $G \notin \mathcal{G}_1$

Assume to the contrary that $G \in \mathcal{G}_1$.

The subsets S_i for $i = 1, 2, \ldots, i_0$ are chosen as follows:

(1) S_1 is a 3-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,3}$.

(2) For $2 \le i \le i_0$, S_i is a 2-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,2}$.

Assume for $i = 2, 3, \ldots, i_0$, $P_i = \{u_i, v_i, w_i\}$ and $S_i = \{u_i, v_i\}$.

Since $|P_1 - S_1| = 2$, by (3) of Observation 3.1, $|L(P_1 - S_1)| \ge k + 1$. As $(\bigcap_{v \in S_1} L(v)) \cap L(P_1 - S_1) = \varnothing$, we know that

$$|L_{\mathcal{S}}(P_1')| \ge k + 2. \tag{4.2}$$

It follows from the definition of $\mathcal{S}$ that for $i = 1, 2, \ldots, i_0$, $|L_{\mathcal{S}}(z_i)| \ge i_0$.

If $X_{\mathcal{S}} \subseteq Z$ and $z_i \in X_{\mathcal{S}}$ for some $i \le i_0$, then we have $|Y_{\mathcal{S}}| \ge |L_{\mathcal{S}}(z_i)| \ge i_0 \ge |X_{\mathcal{S}}|$, a contradiction. Thus $X_{\mathcal{S}} - Z \ne \varnothing$. Let $v \in X_{\mathcal{S}} - Z$. Then

$$|Y_{\mathcal{S}}| \ge |L_{\mathcal{S}}(v)| = |L(v)| \ge k, \quad |X_{\mathcal{S}}| \ge k + 1.$$

This implies that $|X_{\mathcal{S}} \cap P_i'| \ge 2$ for some i. As $|L_{\mathcal{S}}(A)| \ge k + 1$ for any 2-subset A of P_i' (for any i), we have

$$|Y_{\mathcal{S}}| \ge k + 1, \quad |X_{\mathcal{S}}| \ge k + 2. \tag{4.3}$$

Claim 4.4 $|Y_{\mathcal{S}}| \ge k + i_0$ *and hence* $|X_{\mathcal{S}}| \ge k + i_0 + 1$.

Proof If there is an index $i_0 + 1 \le i \le q$ such that $u, v \in X_{\mathcal{S}} \cap P_i'$, then $|Y_{\mathcal{S}}| \ge |L(u \vee v)| \ge 2k - |L(u \wedge v)| \ge 2k - i_0 > k + i_0$ (as $i_0 \le q - 1 < k/2$) and we are done.

Assume $|X_{\mathcal{S}} \cap P_i'| \le 1$ for any $i_0 + 1 \le i \le q$. If $\{z_i, w_i\} \subseteq X_{\mathcal{S}}$ for some $i \ge 2$, then $|Y_{\mathcal{S}}| \ge |L(w_i)| + |L_S(z_i)| + \ge k + i_0$, and we are done. Otherwise, $|X_{\mathcal{S}}| \ge k + 2$ (by (4.3)) implies that $P_1' \subseteq X_{\mathcal{S}}$ and $|X_{\mathcal{S}}| = k + 2$. By (4.2), $|Y_{\mathcal{S}}| \ge |L_{\mathcal{S}}(P_1')| \ge k + 2$, a contradiction. ∎

Claim 4.5 *If* $|Y_{\mathcal{S}}| = k + i_0$, *then* $\Lambda_{P_i,2} = i_0$ *for* $i = 2, 3, \ldots, i_0$ *and there is an index* $2 \le i \le i_0$ *such that* P_i *has a 2-subset* S *with* $|\bigcap_{v \in S} L(v)| \ge 2$ *and* $\bigcap_{v \in S} L(v) \cup L(P_i - S) \nsubseteq Y_{\mathcal{S}}$.

Proof Assume $|Y_{\mathcal{S}}| = k + i_0$. Then $|X_{\mathcal{S}}| \ge k + i_0 + 1$.

By the argument in the proof of Claim 4.4, for any index $i > i_0$, $|X_{\mathcal{S}} \cap P_i| \le 1$. This implies that $|X_{\mathcal{S}}| \le k + i_0 + 1$, and hence $|X_{\mathcal{S}}| = k + i_0 + 1$ and $P_i' \subseteq X_{\mathcal{S}}$ for $i = 1, 2, \ldots, i_0$. As $|L_{\mathcal{S}}(P_i')| \ge k + i_0$ for $2 \le i \le i_0$, we conclude that for $2 \le i \le i_0$, $Y_{\mathcal{S}} = L_{\mathcal{S}}(P_i')$ and $\Lambda_{P_i,2} = i_0$.

We shall find an index $2 \le i \le i_0$, a 2-subset S of P_i with $|\bigcap_{v \in S} L(v)| \ge 2$ and $\bigcap_{v \in S} L(v) \cup L(P_i - S) \nsubseteq Y_{\mathcal{S}}$.

Assume first that there is an index $2 \le i \le i_0$ such that $L(P_i) \nsubseteq Y_{\mathcal{S}}$.

As $L(w_i) \subseteq Y_{\mathcal{S}}$, we may assume that there is a color $c \in L(u_i) - Y_{\mathcal{S}}$. If $|L(v_i \wedge w_i)| \ge 2$, then let $S = \{v_i, w_i\}$, we are done.

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Assume $|L(v_i \wedge w_i)| \leq 1$. This implies that $|L(v_i \vee w_i)| \geq 2k - 1 > k + i_0$. So there is a color $c' \in L(v_i) - Y_{\mathcal{S}}$. If $|L(u_i \wedge w_i)| \geq 2$, then let $S = \{u_i, w_i\}$, we are done. Assume $|L(u_i \wedge w_i)| \leq 1$. Hence

$$\begin{aligned} 2 + i_0 &\geq |L(u_i \wedge w_i)| + |L(v_i \wedge w_i)| + |L(u_i \wedge v_i)| \\ &= |C_{P_i,2}| \geq 3k - |L(P_i)| \geq 3k - (2k+1). \end{aligned}$$

This implies that $k - 3 \leq i_0 \leq q \leq k/2$, contrary to our assumption that $k \geq 8$.

Assume next that $L(P_i) = Y_{\mathcal{S}}$ for $2 \leq i \leq i_0$. As each color in $L(P_i)$ is contained in at most two lists of vertices of P_i, we have $2(k + i_0) \geq 3k$, i.e., $i_0 \geq k/2$. Hence $i_0 = k/2 = q$ and $G = K_{5,3\star(q-1),1\star q}$.

For each singleton part $\{v\}$ of G, we have $v \in X_{\mathcal{S}}$ and hence $L(v) \subseteq Y_{\mathcal{S}}$ for each singleton part $\{v\}$. Thus $L(\bigcup_{i=2}^{k} P_i) = Y_{\mathcal{S}}$.

Since $C_{P_1,4} = \emptyset$, we have $|L(P_1)| \geq 5k/3 > k + i_0 = |Y_{\mathcal{S}}|$. Let $c \in L(P_1) - Y_{\mathcal{S}}$. Then c is contained in the lists of vertices in P_1 only, in contradiction to Observation 3.1. ■

If $|Y_{\mathcal{S}}| = k + i_0$, then as $\Lambda_{P_i,2} = i_0$ for $2 \leq i \leq i_0$, we may assume that $S_2' = \{u_2, w_2\}$ is a 2-subset of P_2 for which $|\bigcap_{v \in S_2'} L(v)| \geq 2$ and $\bigcap_{v \in S_2'} L(v) \cup L(P_2 - S_2') \nsubseteq Y_{\mathcal{S}}$.

We let $\mathcal{S}'$ be the partition of $V(G)$ whose non-singleton parts are obtained from that of $\mathcal{S}$ by replacing S_2 with S_2', i.e., $\mathcal{S}' = \{S_1, S_2', S_3, \ldots, S_{i_0}\}$.

Instead of $G/\mathcal{S}$, we consider the graph $G/\mathcal{S}'$. We still have (4.3), i.e.,

$$|Y_{\mathcal{S}'}| \geq k + 1, \quad |X_{\mathcal{S}'}| \geq k + 2.$$

Then analog to the proof of Claim 4.4, we can show that

$$|Y_{\mathcal{S}'}| \geq k + i_0 + 1, \quad |X_{\mathcal{S}'}| \geq k + i_0 + 2.$$

Let $\mathcal{S}'' = \mathcal{S}$ if $|Y_{\mathcal{S}}| \geq k + i_0 + 1$, and $\mathcal{S}'' = \mathcal{S}'$ if $|Y_{\mathcal{S}}| = k + i_0$. Then

$$|Y_{\mathcal{S}''}| \geq k + i_0 + 1, \quad |X_{\mathcal{S}''}| \geq k + i_0 + 2.$$

For simplicity, we assume that $\mathcal{S}'' = \mathcal{S}$. Then $|X_{\mathcal{S}}| \geq k + i_0 + 2$ implies that $|X_{\mathcal{S}} \cap P_i| \geq 2$ for some $i \geq i_0 + 1$. Assume $\{u, v\} \subseteq X \cap P_i$ for some $i \geq i_0 + 1$. Then

$$|Y_{\mathcal{S}}| \geq |L(u \vee v)| = 2k - |L(u \wedge v)| \geq 2k - i_0. \tag{4.4}$$

Since $X_{\mathcal{S}}$ contains at most one vertex of any 2-part, we have

$$|X_{\mathcal{S}}| \leq k + 2q + 1 - i_0.$$

If for some $i \geq i_0 + 1$, $P_i = \{u_i, v_i, w_i\} \subseteq X_{\mathcal{S}}$, then

$$\begin{aligned} |Y_{\mathcal{S}}| \geq |L(P_i)| &= |L(u_i)| + |L(v_i)| + |L(w_i)| \\ &\quad - (|L(u_i \wedge v_i)| + |L(u_i) \cap L(w_i)| + |L(v_i) \cap L(w_i)|) \\ &\geq 3k - 3i_0. \end{aligned}$$

Hence $k + 2q + 1 - i_0 \geq |X_{\mathcal{S}}| \geq 3k - 3i_0 + 1$, which implies that $k \leq q + i_0 \leq 2q - 1$, in contrary
to $k \geq 2q$.

Thus $|X_{\mathcal{S}} \cap P_i'| \leq 2$ for $i \geq i_0 + 1$. This implies that $|X_{\mathcal{S}}| \leq k + q + 1$.

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On the other hand, $|Y_{\mathcal{S}}| \geq 2k - i_0$ (by (4.4)) implies that $|X_{\mathcal{S}}| \geq 2k - i_0 + 1$. Hence $k+q+1 \geq |X_{\mathcal{S}}| \geq 2k - i_0 + 1$, which implies that $k \leq i_0 + q \leq 2q - 1$, in contrary to $k \geq 2q$.

This completes the proof that $G \notin \mathcal{G}_1$.

4.2 $G \notin \mathcal{G}_2$

Assume to the contrary that $G \in \mathcal{G}_2$.

Claim 4.6 *Assume P is a 4-part of G and $\Lambda_{P,3} \leq 1$. Then $\Lambda_{P,2} \geq 2$. If $|\Lambda_{P,2}| \geq 3$, then for any 2-subset S of P with $|\bigcap_{v \in S} L(v)| = \Lambda_{P,2}$, for any $x \in P - S$,*

$$|\bigcap_{v \in S} L(v) \cup L(x)| \geq k + 2.$$

If $\Lambda_{P,2} = 2$, then there exists a 2-subset S of P such that $|\bigcap_{v \in S} L(v) \cap C_{P,2}| = 2$, and hence for any $x \in P - S$, $|\bigcap_{v \in S} L(v) \cup L(x)| \geq k + 2$.

Proof Assume P is a 4-part of G and $\Lambda_{P,3} \leq 1$. Assume $\Lambda_{P,2} \geq 3$ and S is a 2-subset of P with $|\bigcap_{v \in S} L(v)| = \Lambda_{P,2}$. Then for any $x \in P - S$, since $|\bigcap_{v \in S} L(v) \cap L(x)| \leq \Lambda_{P,3} \leq 1$, we have

$$|\bigcap_{v \in S} L(v) \cup L(x)| = |\bigcap_{v \in S} L(v)| + |L(x)| - |\bigcap_{v \in S} L(v) \cap L(x)| \geq \Lambda_{P,2} + k - 1 \geq k + 2.$$

Assume $\Lambda_{P,2} \leq 2$. As P has four 3-subsets, we have $|C_{P,3}| \leq 4$. As $\sum_{i=1}^{3} i|C_{P,i}| = \sum_{v \in P} |L(v)| \geq 4k$ and $\sum_{i=1}^{3} |C_{P,i}| \leq |C_L| \leq 2k + 1$, it follows that $|C_{P,2}| \geq 2k - 9 \geq 7$ (as $k \geq 8$). Since P has six 2-subsets, there exists a 2-subset S of P such that $|\bigcap_{v \in S} L(v) \cap C_{P,2}| \geq 2$. Hence $\Lambda_{P,2} \geq 2$ and therefore $\Lambda_{P,2} = 2$. Moreover, there exists a 2-subset S of P such that $|\bigcap_{v \in S} L(v) \cap C_{P,2}| = 2$. For any $x \in P - S$,

$$|\bigcap_{v \in S} L(v) \cup L(x)| \geq |\bigcap_{v \in S} L(v) \cap C_{P,2}| + |L(x)| \geq 2 + k. \qquad \blacksquare$$

Definition 4.1 For $i = 1, 2, \ldots, i_0$, we choose a subset S_i of P_i of size 2 or 3 as follows:

(1) For $a + 1 \leq i \leq i_0$, S_i is a 2-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,2}$.

(2) If $a = 1$ and $\Lambda_{P_1,3} > 0$, then let S_1 be a 3-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,3}$. Otherwise, let S_1 be a 2-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,2}$.

(3) Assume $a = 2$.
   - (i) If $\Lambda_{P_2,3} \geq 2$, then for $i = 1, 2$, let S_i be a 3-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,3}$.
   - (ii) If $\Lambda_{P_1,3} > 0$ and $\Lambda_{P_2,3} \leq 1$, then let S_1 be a 3-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,3}$, and let S_2 be a 2-subset of P_2 such that
     - (A) $|\bigcap_{v \in S_2} L(v)| = \Lambda_{P,2}$,
     - (B) $|\bigcap_{v \in S_2} L(v) \cup L(x)| \geq k + 2$ for any $x \in P_2 - S_2$,
     - (C) Subject to (A) and (B), $|L_{\mathcal{S}}(P_1') \bigcup L(P_2 - S_2)|$ is maximum.
   - (iii) If $\Lambda_{P_1,3} = 0$, then for $i = 1, 2$, let S_i be a 2-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,2}$, such that $|\bigcap_{v \in S_i} L(v) \cup L(x)| \geq k + 2$ for any $x \in P_i - S_i$ and subject to this condition, $|L_{\mathcal{S}}(P_1') \bigcup L_{\mathcal{S}}(P_2')|$ is maximum.

The existence of the 2-subset S in (ii) and (iii) has been proved in Claim 4.6.

It follows from the definition of $\mathcal{S}$ that for $i = 1, 2, \ldots, i_0$, $|L_{\mathcal{S}}(z_i)| \geq i$.

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The same argument as in the previous subsection shows that

$$|Y_{\mathcal{S}}| \geq k+1, \quad |X_{\mathcal{S}}| \geq k+2. \tag{4.5}$$

Claim 4.7 *If* $|P_i| = 4$, *then* $|X_{\mathcal{S}} \cap P'_i| \leq 2$.

Proof Assume $P_i = \{u_i, v_i, x_i, y_i\}$. Then $2 \leq |P'_i| \leq 3$. If $|P'_i| = 2$, then the conclusion is trivial.

Assume $|P'_i| = 3$ and assume to the contrary of the claim that $P'_i = \{z_i, x_i, y_i\} \subseteq X_{\mathcal{S}}$, where z_i is the identification of u_i and v_i. In this case, $L_{\mathcal{S}}(z_i) = L(u_i \wedge v_i)$ and $L_{\mathcal{S}}(x_i) = L(x_i)$, $L_{\mathcal{S}}(y_i) = L(y_i)$.

If $\Lambda_{P_i,3} = 0$, then $L_{\mathcal{S}}(z_i) \cap L(x_i \vee y_i) = \emptyset$. By the choice of S_i, $|L(x_i \wedge y_i)| \leq |L_{\mathcal{S}}(z_i)|$ and hence $|L(x_i \vee y_i)| \geq 2k - |L_{\mathcal{S}}(z_i)|$. Therefore $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_i)| + |L(x_i \vee y_i)| \geq 2k$, in contrary to (4.1).

If $\Lambda_{P_i,3} > 0$, then by the choice of S_i, we know that $i = a = 2$, $\Lambda_{P_2,3} = 1$ and $|S_1| = 3$, $|P'_1| = 2$. Therefore $|X_{\mathcal{S}}| \leq |V(G/\mathcal{S})| \leq 2k-1$, and $|Y_{\mathcal{S}}| \leq 2k-2$.

Assume $S_2 = \{u_2, v_2\}$. By the choice of S_2 (see Claim 4.6), $|L_{\mathcal{S}}(z_2)| \geq |L(x_i \wedge y_i)|$ and $|L_{\mathcal{S}}(z_i) \cap L(x_i \vee y_i)| \leq |L_{\mathcal{S}}(z_i) \cap L(x_i)| + |L_{\mathcal{S}}(z_i) \cap L(y_i)| \leq 2\Lambda_{P_i,3} = 2$. Hence $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_i)| + |L(x_i \vee y_i)| - 2 \geq 2k-2$. So $|X_{\mathcal{S}}| = 2k-1$ and $|Y_{\mathcal{S}}| = 2k-2$, and hence $X_{\mathcal{S}} = V(G/\mathcal{S})$. This implies that $i_0 = 2$.

By Lemma 3.2, G has no 2-part. Assume $P_3 = \{u_3, v_3, w_3\}$. Then since $\Lambda_{P_3,2} \leq 2$, and P_3 has three 2-subsets, we know that $|C_{P_3,2}| \leq 6$. Therefore

$$3k \leq |L(u_3)| + |L(v_3)| + |L(w_3)| = 2|C_{P_3,2}| + |C_{P_3,1}| \leq |C_L| + |C_{P_3,2}| \leq 2k+6,$$

a contradiction (as $k \geq 8$). ∎

Since $3p_3^+ + 2p_2 + p_1 \leq 2k+2 = 2(p_1 + p_2 + p_3^+) + 2$ and $G \neq K_{3\star(k/2+1),1\star(k/2-1)}$ (i.e., $k \neq 2q-2$), we have

$$G \in \{K_{4,3\star(q-1),1\star(q-1)}, K_{3\star q,2,1\star(q-2)}\} \text{ or } k \geq 2q.$$

Note that $X_{\mathcal{S}}$ contains at most one vertex of any 2-part. Combining with Claim 4.7, we have

$$|X_{\mathcal{S}}| \leq k + 2q - i_0.$$

Claim 4.8 *For any* $i \geq i_0 + 1$, $|X_{\mathcal{S}} \cap P_i| \leq 1$.

Proof If $i \geq q+1$, then P_i is 2^--part and hence $|P_i \cap X_{\mathcal{S}}| \leq 1$ (by Lemma 3.2 and (4.1)).

Assume $i_0 + 1 \leq i \leq q$.

First we prove that $|X_{\mathcal{S}} \cap P_i| \leq 2$. Assume to the contrary that $|X_{\mathcal{S}} \cap P_i| = 3$ for some $i \geq i_0 + 1$. Assume $P_i = \{u_i, v_i, w_i\}$. Then

$$\begin{aligned} |Y_{\mathcal{S}}| \geq |L(P_i)| &= |L(u_i)| + |L(v_i)| + |L(w_i)| \\ &\quad - (|L(u_i \wedge v_i)| + |L(u_i) \cap L(w_i)| + |L(v_i) \cap L(w_i)|) \\ &\geq 3k - 3i_0. \end{aligned}$$

Hence $k + 2q - i_0 \geq |X_{\mathcal{S}}| \geq 3k - 3i_0 + 1$, which implies that $2k + 1 \leq 2q + 2i_0 \leq 4q$. As $k \geq 2q-1$, we have $k = 2q-1$. Hence $q = i_0$, in contrary to $i_0 + 1 \leq i \leq q$.

Since $|X_{\mathcal{S}} \cap P_i| \leq 2$ for all $i \geq i_0 + 1$, we know that $|X_{\mathcal{S}}| \leq k + q$ (by Claim 4.7).

If $|X_{\mathcal{S}} \cap P_i| = 2$ for some $q \geq i \geq i_0 + 1$, then $|Y_{\mathcal{S}}| \geq 2k - i_0$. Hence $k + q \geq |X_{\mathcal{S}}| \geq 2k - i_0 + 1$, which implies that $k = 2q - 1$ and $i_0 = q$, again in contrary to $i_0 + 1 \leq i \leq q$. ∎

It follows from Claims 4.7 and 4.8 that $|X_{\mathcal{S}}| \leq k + i_0$ and hence $|Y_{\mathcal{S}}| \leq k + i_0 - 1$.

Thus $|X_{\mathcal{S}} \cap P'_i| \leq 1$ for any $a + 1 \leq i \leq i_0$. Combining with Claim 4.8, we know that $|X_{\mathcal{S}} \cap P'_i| \leq 1$ for any $i \geq a + 1$. Since $|X_{\mathcal{S}}| \geq k + 2$ (by (4.5)), it follows from Claim 4.7 that $a = 2$ and $|X_{\mathcal{S}} \cap P'_i| = 2$ for $i = 1, 2$, and

$$|X_{\mathcal{S}}| = k + 2, |Y_{\mathcal{S}}| = k + 1 \text{ and } Y_{\mathcal{S}} = L_{\mathcal{S}}(X_{\mathcal{S}} \cap P'_1) = L_{\mathcal{S}}(X_{\mathcal{S}} \cap P'_2). \tag{4.6}$$

For $i = 1, 2$, assume $P_i = \{u_i, v_i, x_i, y_i\}$.

If $\Lambda_{P_2,3} \geq 2$, then $|S_2| = 3$, say $S_2 = \{u_2, v_2, x_2\}$. Then $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_2)| + |L(P_2 - S_2)| \geq k + 2$, a contradiction.

Assume $\Lambda_{P_2,3} \leq 1$. Then (ii) or (iii) holds, and $|S_2| = 2$, say $S_2 = \{u_2, v_2\}$. If $z_2 \in X_{\mathcal{S}}$, say $P'_2 \cap X_{\mathcal{S}} = \{z_2, x_2\}$, then $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_2) \cup L(x_2)| \geq k + 2$ (by Claim 4.6), contrary to (4.6).

Assume $z_2 \notin X_{\mathcal{S}}$. Then $x_2, y_2 \in X_{\mathcal{S}}$. Now $|L(x_2 \vee y_2)| \leq |Y_{\mathcal{S}}| = k + 1$ implies that $|L(x_2 \vee y_2)| = k + 1$ and $|L(x_2 \wedge y_2)| = k - 1$. This implies that $\Lambda_{P_2,2} = k - 1$ and hence $|L(u_2 \wedge v_2)| = k - 1$. As $k \geq 8$, i.e., $\Lambda_{P_2,2} = k - 1 \geq 7$, it follows from Claim 4.6 that $|L(x_2 \wedge y_2)| = \Lambda_{P,2} \geq 2$ and $|L(x_2 \wedge y_2) \cup L(v)| \geq k + 2$ for any $v \in P_2 - \{x_2, y_2\}$.

If (ii) holds, say $S_1 = \{u_1, v_1, x_1\}$, then $L(u_2 \vee v_2) = L_{\mathcal{S}}(z_1) \cup L(y_1)$. This implies that $L(x_2 \vee y_2) = L_{\mathcal{S}}(z_1) \cup L(y_1)$, for otherwise, by see (ii), we should have chosen $S_2 = \{x_2, y_2\}$. So $|L(P_2)| = k + 1$. Hence

$$\begin{aligned}
2k - 2 &= |L(x_2 \wedge y_2)| + |L(u_2 \wedge v_2)| \\
&= |L(x_2 \wedge y_2) \cap L(u_2 \wedge v_2)| + |L(x_2 \wedge y_2) \cup L(u_2 \wedge v_2)| \\
&\leq |L(x_2 \wedge y_2) \cap L(u_2 \wedge v_2)| + k + 1.
\end{aligned}$$

This implies that $L(x_2 \wedge y_2) \cap L(u_2 \wedge v_2) \neq \emptyset$, in contrary to Lemma 3.2.

Assume (iii) holds, and for $i = 1, 2$, $P_i = \{u_i, v_i, x_i, y_i\}$ and $S_i = \{u_i, v_i\}$. If $z_i \in X_{\mathcal{S}}$ for some $i = 1, 2$, then by Claim 4.6, $|L_{\mathcal{S}}(z_i)| \geq 2$ and hence $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(P'_i)| \geq k + 2$, contrary to (4.6).

Assume $z_1, z_2 \notin X_{\mathcal{S}}$. Then again by the choice of S_2, we have $L(u_2 \vee v_2) = L(x_1 \vee y_1) = L(x_2 \vee y_2)$, $|L(x_2 \wedge y_2)| = |L(u_2 \wedge v_2)| = k - 1$, and $|L(P_2)| = k + 1$. This leads to the same contradiction. This completes the proof of Theorem 4.1.

It was proved in [23] that $K_{6,2\star(k-3),1\star 2}$ is k-choosable. Combining with Theorem 4.1, we conclude that

$$p_1 \geq 3, p_3^+ \leq p_1 - 1, 3p_3^+ + 2p_2 + p_1 \leq |V| - 3. \tag{4.7}$$

5 Pseudo-L-coloring

As described in Section 1, our strategy for proving Theorem 1.2 is to partition $V(G)$ into a family $\mathcal{S}$ of independent sets, so that either there is a matching $M_{\mathcal{S}}$ in the bipartite graph $B_{\mathcal{S}}$ that covers $V(G/\mathcal{S})$ and hence produce an L-coloring of G, or using Hall's theorem to produce a good partial L-coloring of G that leads to an L-coloring of G by using induction. The partition $\mathcal{S}$ is obtained by constructing a proper coloring

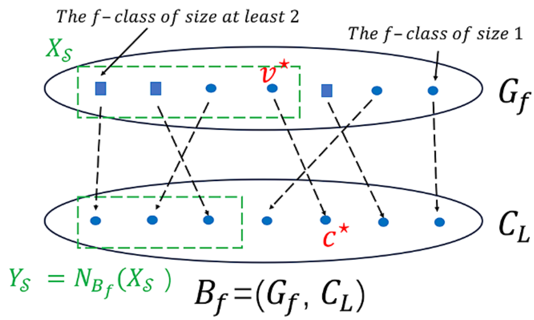

Figure 1: The bipartite graph B_f with partite sets G_f and C_L. Vertices in G_f are f-classes, some of them are singleton classes represented by solid circles, and other are 2^+-classes, represented by solid squares. The broken arrowed line indicate the coloring f. The edges of B_f are not drawn, and $Y_{\mathcal{S}} = N_{B_{\mathcal{S}}}(X_{\mathcal{S}})$. Vertex v^* is contained in $X_{\mathcal{S}}$ but $f(v^*) = c^* \notin Y_{\mathcal{S}}$. So $v^\star$ is a badly f-coloured vertex.

f of G, and the parts in $\mathcal{S}$ are the color classes of f. For this strategy to succeed, the coloring f needs to have some nice property. In this section, we define the concept of pseudo-L-coloring of G, and study properties of the partition $\mathcal{S}$ of $V(G)$ induced by such colorings.

Definition 5.1 A *pseudo L-coloring* of G is a proper coloring f of G such that $f(v) \in C_L$ for every vertex v, and if $f(v) = c \notin L(v)$, then $f^{-1}(c) = \{v\}$ is a singleton f-class.

In a pseudo L-coloring f of G, if $f(v) \notin L(v)$, then we say v is *badly f-colored* (or *badly colored* if f is clear from the context).

By Observation 3.1, if f is a pseudo-L-coloring of G, then there is a pseudo-L-coloring g of G such that $g(G) = C_L$ and for every badly g-colored vertex v of G, $g(v) = f(v)$. In the following, we may assume that all the pseudo-L-colorings f satisfy $f(G) = C_L$. However, when we construct a pseudo-L-coloring f of G, we do not need to verify that $f(G) = C_L$ (because if $f(G) \neq C_L$, then we change it to the pseudo-L-coloring g described above).

Definition 5.2 Assume f is a pseudo L-coloring of G. Let $\mathcal{S}_f$ be the family of f-classes, which is a partition of $V(G)$, i.e., $\mathcal{S}_f = \{f^{-1}(c) : c \in C_L\}$ where $f^{-1}(c)$ is the set of all vertices colored by c under f. We denote $G/\mathcal{S}_f$, $L_{\mathcal{S}_f}$, $B_{\mathcal{S}_f}$, $X_{\mathcal{S}_f}$ and $Y_{\mathcal{S}_f}$ by G_f, L_f, B_f, X_f, and by Y_f, respectively.

In the remainder of this section, assume f is a pseudo L-coloring of G. In the graph G_f, $f^{-1}(c)$ is identified into a single vertex. For simplicity, we denote this vertex by $f^{-1}(c)$. So $f^{-1}(c)$ is both a subset of $V(G)$ and a vertex of G_f. It will be clear from the context which one it is.

Since $|X_f| > |Y_f|$, there is a color class $f^{-1}(c) \in X_f$ such that $c \notin Y_f$. Hence $f^{-1}(c)$ is a singleton f-class $\{v\}$ and v is badly colored by f.

For a subset Q of $V(G_f)$, let $V(Q)$ be the subset of $V(G)$ defined as

$$V(Q) = \bigcup_{f^{-1}(c) \in Q} f^{-1}(c).$$

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Let ℓ be the number of f-classes $f^{-1}(c)$ of size $|f^{-1}(c)| \geq 2$. As $|V| > |C_L|$, $\ell \geq 1$. On the other hand, $\lambda = |V| - |C_L| \geq \ell$ and equality holds if and only if $f(G) = C_L$ and each f-class has size at most 2.

Recall that there is a matching $M_{\mathcal{S}_f}$ in $B_f - (X_f \cup Y_f)$ that covers all vertices in $V(G_f) - X_f$. The matching $M_{\mathcal{S}_f}$ defines a partial L-coloring of $G[\bigcup_{f^{-1}(c) \notin X_f} f^{-1}(c)]$ that colors vertices in $f^{-1}(c)$ with c', where $\{c', f^{-1}(c)\}$ is an edge in $M_{\mathcal{S}_f}$. We denote this partial L-coloring of G by ψ_f. The matching $M_{\mathcal{S}_f}$ maybe not unique. In this case, we let $M_{\mathcal{S}_f}$ be an arbitrary matching that covers $V(G_f) - X_f$.

We extend ψ_f to a partial L-coloring ϕ_f of G by coloring each f-classes $f^{-1}(c) \in X_f$ of size at least 2 by color c. By definition of pseudo-L-coloring, for such an f-class $f^{-1}(c)$, $c \in L_{\mathcal{S}}(f^{-1}(c))$. So ϕ_f is a proper L-coloring of G. Denote by X the set of vertices of G colored by ϕ_f. Note that only those f-classes $f^{-1}(c)$ of size at least 2 contained in X_f are colored by colors from Y_f. So

$$|\phi_f(X) \cap Y_f| \leq \ell.$$

If $G - X$ has an L^{ϕ_f}-coloring θ, then $\phi_f \cup \theta$ would be an L-coloring of G. Thus $G - X$ is not L^{ϕ_f}-colorable.

Lemma 5.1 *$V_f - X_f$ contains at most $\lambda - 1$ singletons of G. Moreover, if $V_f - X_f$ contains $\lambda - 1 \geq 1$ singletons of G, then $\ell = \lambda$ and the following hold:*

(1) *All f-classes have size 2 or 1, and there are exactly ℓ f-classes of size 2.*
(2) *All the ℓ f-classes of size 2 are contained in X_f.*
(3) *For each non-singleton part P of G, there is a singleton f-class $\{v\} \in X_f$ such that $v \in P$.*
(4) *If f has exactly one badly colored vertex, then $|Y_f| \geq k + 1$.*

Proof It follows from the definition of ϕ_f that for each vertex v of $G - X$, $\{v\} \in X_f$ is a singleton f-class, and $L(v) \subseteq Y_f$. As $|L^{\phi_f}(X) \cap Y_f| \leq \ell$,

$$|L^{\phi_f}(v)| \geq k - \ell, \forall v \in V(G - X).$$

If $G_f - X_f$ contains ℓ singletons of G, then

$$\chi(G - X) \leq k - \ell \text{ and } |V(G - X)| \leq 2k + 2 - 2\ell - \lambda \leq 2(k - \ell) + 1.$$

By Noel–Reed–Wu theorem, $G - X$ is L^{ϕ_f}-colorable, a contradiction.

So $G_f - X_f$ contains at most $\ell - 1$ singletons of G.

Assume $G_f - X_f$ contains $\lambda - 1$ singletons of G. Since $\ell \leq \lambda$, we have $\ell = \lambda$ and hence each f-class has size at most 2, and there are exactly ℓ f-classes of size 2, i.e., (1) holds. We shall prove that (2)–(4) hold.

(2): Assume to the contrary that there is an f-class of size 2 not in X_f. Then at most $\ell - 1$ f-classes are colored by colors from $Y_{\mathcal{S}}$. Hence

$$|L^{\phi_f}(v)| \geq k - (\ell - 1), \forall v \in V(G - X).$$

As

$$|V(G - X)| \leq 2k + 2 - 2\ell = 2(k - \ell) + 2 = 2(k - \ell + 1) \text{ and } \chi(G - X) \leq k - \ell + 1,$$

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$G - X$ with list assignment L^{ϕ_f} satisfy the condition of Noel–Reed–Wu theorem, and hence $G - X$ has an L^{ϕ}-coloring, a contradiction.

(3): If (3) does not hold, then there is a non-singleton part P of G such that all vertices of P are colord, i.e., P is a non-singleton part of G contained in X, and hence $\chi(G - X) \leq k - \lambda = k - \ell$. We still have $|V(G - X)| \leq 2k + 2 - 2\ell - (\lambda - 1) \leq 2(k - \ell) + 1$. Hence by Noel–Reed–Wu theorem, $G - X$ has an L^{ϕ_f}-coloring, a contradiction.

(4): Assume v^* is the only badly colored vertex. Then $\{v^*\}$ is an f-class of size 1 in X_f. This implies that $|Y_f| \geq |L(v^*)| \geq k$. Assume to the contrary that $|Y_f| = k$. This implies that for all singleton f-classes $\{v\} \in X_f$, $L(v) = Y_f$.

Assume $f^{-1}(c) \in X_f$ is an f-class of size at least 2, and P_i is the part of G containing $f^{-1}(c)$. As the size of $f^{-1}(c)$ is at least 2, P_i is not singleton-part and hence it follows from (3) that there is an f-class $\{v\} \in X_f$ such that $v \in P_i$. Thus, $L(v) = Y_f$, $c \in L(v)$ and we can color v with color c, and color v^* with $f(v)$. The resulting coloring is a pseudo L-coloring of G with no badly colored vertices, i.e., an L-coloring of G, a contradiction.

This completes the proof of Lemma 5.1. ■

Lemma 5.2 *Assume $\lambda \geq 2$ and G has at most $\lambda - 1$ singletons. Then $G_f - X_f$ contains at most $\lambda - 2$ singletons of G.*

Proof If G has at most $\lambda - 2$ singletons, then the conclusion is trivial. Assume G has exactly $\lambda - 1$ singletons (i.e., $p_1 = \lambda - 1$), and assume to the contrary that all the $\lambda - 1$ singletons of G are contained in $G_f - X_f$. By (3) of Lemma 5.1, for each of the $k - \lambda + 12^+$-parts P of G, X_f has a singleton f-class $\{v\}$ with $v \in P$. By Lemma 5.1, we have $\ell = \lambda$. By (2) of Lemma 5.1, all the ℓ f-classes of size 2 are contained in X_f. Thus

$$|V(X_f)| \geq 2\ell + k - \lambda + 1 = \lambda + k + 1. \tag{5.1}$$

If a 2-part P of G is contained in $V(X_f)$, then $L(P) \subseteq Y_f$. By Lemma 3.2,

$$2k \leq |L(P)| \leq |Y_f|.$$

This contradicts to the fact that $|Y_f| < |C_L| = |V| - \lambda \leq 2k$.

Thus for each 2-part P of G, $|P \cap V(G_f - X_f)| \geq 1$. (Note that a 2-part has no common color in the lists of its vertices, so P is not an f-class.) Hence

$$|V(G_f - X_f)| \geq \lambda - 1 + p_2. \tag{5.2}$$

As $p_1 = \lambda - 1$, it follows from (5.1) and (5.2) that

$$\begin{aligned} 2k + 2 = |V| = |V(X_f)| + |V(G_f - X_f)| &\geq (\lambda + k + 1) + (\lambda - 1 + p_2) \\ &= 2\lambda + k + p_2 = 2p_1 + 2 + k + p_2. \end{aligned}$$

So

$$p_3^+ + p_2 + p_1 = k \geq 2p_1 + p_2,$$

which implies that $p_3^+ \geq p_1$, in contrary to (4.7).

This completes the proof of Lemma 5.2. ■

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6 Near acceptable colorings

We have shown in the previous section that the partition $\mathcal{S}$ of $V(G)$ induced by a pseudo-L-coloring of G has some nice properties. However, for the proof of Theorem 1.2, one more restriction need to be added to a pseudo-L-coloring. In this section, we define the concept of near acceptable L-coloring of G, and prove that the partition $\mathcal{S}$ of G induced by a near acceptable L-coloring of G enables us to construct a proper L-coloring.

Definition 6.1 A color c is called *frequent* if one of the following holds:

(1) $|L^{-1}(c)| \geq k+2$.
(2) $|L^{-1}(c) \cap T| \geq \lambda$.
(3) $|T| = \lambda - 1 \geq 1$ and $T \subseteq L^{-1}(c)$.

Definition 6.2 A pseudo L-coloring f of G is *near acceptable* if each badly colored vertex is colored by a frequent color.

The concept of near acceptable L-coloring was first used in [17] for the proof of Noel–Reed–Wu theorem. For the proof of Theorem 1.2, as G has one more vertex, the definition of frequent colors is different from that in [17]. Thus the near acceptable L-coloring in this paper is different from the one in [17]. The difference makes it more difficult to find a near acceptable L-coloring of G. Nevertheless, we shall show that analog to [17], the existence of a near acceptable L-coloring of G implies the existence of an L-coloring of G.

Lemma 6.1 *G has no near acceptable L-coloring.*

Proof Assume to the contrary that f is a near acceptable L-coloring of G. Since $|X_f| > |Y_f|$, there is a color class $f^{-1}(c^*) \in X_f$ with $c^* \notin Y_f$. Hence $f^{-1}(c^*) = \{v^*\}$ is a badly colored singleton f-class.

Since $f^{-1}(c^*) = \{v^*\} \in X_f$, we have $L(v^*) \subseteq Y_f$, and hence

$$k \leq |L(v^*)| \leq |Y_f| < |X_f|.$$

On the other hand, $c^* \notin Y_f$ implies that for each $f^{-1}(c) \in X_f$, there exists $v \in f^{-1}(c)$, such that $c^* \notin L(v)$. Thus

$$|L^{-1}(c^*)| \leq 2k + 2 - |X_f| \leq k + 1.$$

So c^* is not a frequent color of Type (1).

By Lemma 5.1, $V_f - X_f$ contains at most $\lambda - 1$ singletons of G. Hence

$$|L^{-1}(c^*) \cap T| \leq \lambda - 1.$$

So c^* is not a frequent color of Type (2).

If $|T| = \lambda - 1 \geq 1$, then by Lemma 5.2, $|L^{-1}(c^*) \cap T| \leq |V(V_f - X_f) \cap T| \leq \lambda - 2$. Hence $T \nsubseteq L^{-1}(c^*)$. So c^* is not a frequent color of Type (3).

Therefore, c^* is not frequent, a contradiction. ∎

7 Upper bound on the number frequent colors

This section proves that there are at most $k-1$ frequent colors. Assume to the contrary that there is a set F of k frequent colors. We will construct a near acceptable coloring f of G in the following three steps:

(1) Construct a partial L-coloring f_1 of G using colors from $C_L - F$, that colors as many vertices as possible, and subject to this, the colored vertices are distributed among the parts of G as evenly as possible. Let V_1 be the set of vertices colored by f_1.
(2) Order the parts of G as $P_1, P_2, \dots, P_k$ so that $|P_i - V_1| \geq |P_{i+1} - V_1|$ for $i = 1, 2, \dots, k-1$. Color greedily in this order the vertices of $P_i - V_1$ by a common permissible color from F, until this process cannot be carried out any more. This partial L-coloring will be denoted by f_2. Let V_2 be the set of vertices colored by f_2.
(3) Extend $f_1 \cup f_2$ to a near acceptable L-coloring (for example, if for each remaining part P_i, $P_i - V_1$ contains at most one vertex, then we arbitrarily color that vertex by a remaining color from F to obtain a near acceptable L-coloring of G).

The difficult part is to prove that $f_1 \cup f_2$ can be extended to a near acceptable L-coloring. What we really proved is that if this cannot be done, then every part of G is a 3^--part, which is in contrary to Theorem 4.1.

In the proof, we often need to modify a partial L-coloring.

Definition 7.1 Assume f is a partial L-colorings of G. For distinct colors $c_1, c_2, \dots, c_t \in C_L$, and distinct indices $i_1, i_2, \dots, i_t \in \{1, 2, \dots, k\}$, we denote by

$$f(c_1 \to P_{i_1}, c_2 \to P_{i_2}, \dots, c_t \to P_{i_t})$$

the partial L coloring of G obtained from f by the following operation:

- First, for $j = 1, 2, \dots, t$, uncolor vertices in $f^{-1}(c_j)$ (it is allowed that $f^{-1}(c_j) = \emptyset$, i.e., c_j is not used by f).
- Second, for $j = 1, 2, \dots, t$, color vertices in $L^{-1}(c_j) \cap P_{i_j}$ by color c_j.

Now we are ready to prove the following lemma.

Lemma 7.1 *There are at most $k-1$ frequent colors.*

Proof Assume to the contrary that there is a set F of k frequent colors. A *valid* partial L-coloring f of G is a partial L-coloring of G using colors from $C_L - F$.

For a valid partial L-coloring f of G, for $i = 1, 2, \dots, k$, let

$$S_{f,i} = P_i \cap f^{-1}(C_L - F)$$

be the set of colored vertices in P_i. Let

$$\tau_1(f) = \sum_{i=1}^{k} |S_{f,i}|,$$

$$\tau_2(f) = \sum_{i=1}^{k} |S_{f_1,i}|^2.$$

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We choose a valid partial L-coloring f_1 of G such that

$$\tau(f_1) = (\tau_1(f_1), -\tau_2(f_1))$$

is lexicographically maximum, i.e., the number of colored vertices $\tau(f_1)$ is maximum, and subject to this, $\tau_2(f) = \sum_{i=1}^{k} |S_{f_1,i}|^2$ is minimum, which means that the colored vertices are distributed among the parts of G as evenly as possible.

Let $V_1 = f_1^{-1}(C_L - F) = \bigcup_{i=1}^{k} S_{f,i}$ be the set of vertices colored by f_1. By the maximality of $\tau_1(f_1)$, V_1 must have used all the colors in $C_L - F$, and hence $|C_L - F| \leq |V_1|$.

If $|V - V_1| \leq k$, then let $g : V - V_1 \to F$ be an arbitrary injective mapping. The union $f_1 \cup g$ is a near acceptable L-coloring of G, and we are done. Thus we may assume that

$$|V - V_1| \geq k + 1, \text{ and hence } |V_1| \leq k + 1. \tag{7.1}$$

For $i = 1, 2, \ldots, k$, let

$$R_{f_1,i} = P_i - S_{f,i}.$$

For a color $c \in C_L$, let

$$R_i(c) = |L^{-1}(c) \cap R_{f_1,i}|$$

be the number of vertices in $R_{f_1,i}$ that can be colored by c, and

$$R_i(C_L - F) = \sum_{c \in C_L - F} R_i(c)$$

be the total number of vertices in $R_{f_1,i}$ that can be colored by colors from $C_L - F$.

If $c \in C_L - F$, then

$$R_i(c) \leq |f_1^{-1}(c)|,$$

for otherwise, $f_1(c \to P_i)$ is a valid partial L-coloring of G which colors more vertices than f_1, in contrary to the choice of f_1.

Definition 7.2 A color $c \in C_L - F$ is said to be *movable to* P_i if $R_i(c) = |f_1^{-1}(c)|$. ■

Observation 7.2 *The following facts will be used frequently in the argument below.*

(1) *If $c \in C_L - F$ is movable to P_i, then $f_1(c \to P_i)$ is a valid partial L-coloring of G with $\tau_1(f_1(c \to P_i)) = \tau_1(f_1)$.*

(2) *$R_i(C_L - F) \leq |V_1 - P_i|$, and if $R_i(C_L - F) = |V_1 - P_i|$, then*

(P1) *every color $c \in C_L - F$ with $f_1^{-1}(c) \cap P_i = \varnothing$ is movable to P_i.*

(3) *If $f_1^{-1}(c)$ is a singleton f_1-class, then c is movable to P_i if and only if $c \in L(R_{f,i})$.*

(4) *For any choices of distinct colors $c_1, c_2, \ldots, c_t \in C_L$ and indices $i_1, i_2, \ldots, i_t$, $f_1(c_1 \to P_{i_1}, c_2 \to P_{i_2}, \ldots, c_t \to P_{i_t})$ is a partial L-coloring of G.*

Proof (1),(3), (4) are trivial.

(2): If $c \in C_L - F$ and $f^{-1}(c) \cap P_i \neq \varnothing$, then $R_i(c) = 0$, for otherwise, we can color vertices in $\{v \in R_{f_1,i} : c \in L(v)\}$ with color c. By the fact that $R_i(c) \leq |f_1^{-1}(c)|$, we have $R_i(c) \leq |f_1^{-1}(c) - P_i|$ for any color $c \in C_L - F$. Hence $R_i(C_L - F) = \sum_{c \in C_L - F} R_i(c) \leq |V_1 - P_i|$, and equality holds only if $R_i(c) = |f_1^{-1}(c)|$ for all $c \in C_L - F$ with $f_1^{-1}(c) \cap P_i = \varnothing$. ■

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Claim 7.3 *If* $|P_i| = 2$, *then* $S_{f_1,i} \neq \varnothing$.

Proof Assume to the contrary that $P_i = \{u, v\}$ and $S_{f_1,i} = \varnothing$. By Lemma 3.2, $L(u \wedge v) = \varnothing$. Hence $|C_L| \geq 2k$ and $|V_1| \geq |C_L - F| \geq k$. So there are at least $k f_1$-classes. As $|V_1| \leq k+1$ (see (7.1)), each f_1-class is a singleton, except at most one f_1-class is of size 2.

Since $S_{f_1,i} = \varnothing$, there is an index j_0 such that $|f_1(S_{f_1,j_0}))| \geq 2$. Assume $c_1, c_2 \in f_1(S_{f_1,j_0})$. At least one of $f_1^{-1}(c_1), f_1^{-1}(c_2)$ is a singleton f_1-class.

If $|C_L| = 2k$, then $L(u \vee v) = C_L$ and by (3) of Observation 7.2, one of c_1, c_2, say c_1, is movable to P_i and $f_1^{-1}(c_1)$ is a singleton f_1-class. If $|C_L| = 2k+1$, then there are $k+1 f_1$-classes, and hence each f_1-class is a singleton. So both $f_1^{-1}(c_1), f_1^{-1}(c_2)$ are singleton f_1-classes, and at least one of c_1, c_2 belongs to $L(R_{f_1,i})$ and hence is movable to P_i.

Assume $f_1^{-1}(c_1)$ is a singleton f_1-class and c_1 is movable to P_i.

Then $\tau_1(f_1(c_1 \to P_i)) = \tau_1(f_1), \tau_2(f_1(c_1 \to P_i)) < \tau_2(f_1)$. This is in contrary to our choice of f_1. ∎

By a re-ordering, if needed, we assume that

(R1) $$|R_{f_1,1}| \geq |R_{f_1,2}| \geq \ldots \geq |R_{f_1,k}|.$$

In the second step, starting from $i = 1$ to k, we do the following: If there is a color $c \in F$ such that $c \in \bigcap_{v \in R_{f_1,i}} L(v)$ and c is not used by $R_{f_1,j}$ for $j < i$, then we color $R_{f_1,i}$ with c. The step terminates when such a color does not exist.

Assume the second step stopped at $i_0 + 1$, and hence $R_{f_1,1}, \ldots, R_{f_1,i_0}$ are colored in the second step.

Note that in the ordering of $R_{f_1,1}, R_{f_1,2}, \ldots, R_{f_1,k}$, if some of the $R_{f_1,j}$'s has the same cardinality, then we can choose different ordering so that (R1) still holds. Also with a given ordering of $R_{f_1,1}, R_{f_1,2}, \ldots, R_{f_1,k}$, when we color all the vertices of $R_{f_1,i}$, there may be more than one choice of the colors. We assume that

(R2) Subject to (R1), the ordering of $R_{f_1,1}, R_{f_1,2}, \ldots, R_{f_1,k}$ and the coloring of the $R'_{f_1,i}$s are chosen so that i_0 is maximum.

We denote by f_2 the coloring constructed in the second step, and by V_2 the set of vertices colored in this step, and let $V_3 = V - V_1 - V_2$ be the set of uncolored vertices after the second step. Let F_1 be the frequent colors used in second step, and let $F_2 = F - F_1$. So $|F_1| = i_0$ and $|F_2| = k - i_0$. Note that it is possible that $i_0 = 0$ and $V_2 = \varnothing$.

If $|R_{f_1,i_0+1}| \leq 1$, then $|V_3| \leq k - i_0 = |F_2|$, and $f_1 \cup f_2$ can be extended to a near acceptable L-coloring of G by coloring V_3 injectively by F_2, and we are done.

Therefore the following hold:

(7.2) $$\begin{aligned} |R_{f_1,i_0+1}| &\geq 2, \\ |V_2| &\geq 2i_0, \\ |V_3| &\geq k - i_0 + 1, \\ |V_1| &= |V| - |V_2| - |V_3| \leq k - i_0 + 1. \end{aligned}$$

Observe that for each color $c \in F_2$,

$$R_{i_0+1}(c) \leq |R_{f_1,i_0+1}| - 1,$$

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and for each color $c \in F_1$,

$$R_{i_0+1}(c) \le |R_{f_1,i_0+1}|.$$

Hence

$$\begin{aligned} R_{i_0+1}(C_L - F) &= \sum_{c \in C_L - F} R_{i_0+1}(c) \\ &= \sum_{c \in C_L} R_{i_0+1}(c) - \sum_{c \in F_1 \cup F_2} R_{i_0+1}(c) \\ &\ge k|R_{f_1,i_0+1}| - (|R_{f_1,i_0+1}| - 1)(k - i_0) - |R_{f_1,i_0+1}| i_0 = k - i_0, \end{aligned} \tag{7.3}$$

and if the equality holds, then

$$\forall c \in F_2, R_{i_0+1}(c) = |R_{f_1,i_0+1}| - 1, \tag{7.4}$$

$$\forall c \in F_1, R_{i_0+1}(c) = |R_{f_1,i_0+1}|. \tag{7.5}$$

Combining (7.2) with (7.3) and by (2) of Observation 7.2, we have

$$k - i_0 + 1 \ge |V_1| \ge |V_1 - P_{i_0+1}| \ge R_{i_0+1}(C_L - F) \ge k - i_0. \tag{7.6}$$

So $|V_1| = k - i_0$ or $|V_1| = k - i_0 + 1$.

Case 1: $|V_1| = k - i_0$.

In this case,

$$\begin{aligned} &|V_1| = |V_1 - P_{i_0+1}| = R_{i_0+1}(C_L - F) = k - i_0, \\ &S_{f_1,i_0+1} = V_1 \cap P_{i_0+1} = \varnothing, \text{ and } P_{i_0+1} = R_{f_1,i_0+1}. \end{aligned} \tag{7.7}$$

So (P1) and (P2) holds for $i_0 + 1$. By Claim 7.3, $|P_{i_0+1}| = |R_{f_1,i_0+1}| \ge 3$. Hence $|V_2| \ge 3i_0$. This implies that

$$k - i_0 = |V_1| \le k - 2i_0 + 1,$$

and hence $i_0 \le 1$.

Case 1.1: $i_0 = 1$.

In this case, $|V_1| = k - 1$, $|V_2| \ge 3$ and $|V_3| \ge k$ (by (7.2), i.e., $|V_3| \ge k - i_0 + 1 = k$). Since $|V| = 2k + 2$, we conclude that $|V_2| = |R_{f_1,1}| = 3$ and $|V_3| = k$.

By (7.7), $R_2(C_L - F) = |V_1| = k - 1$. This implies that

$$\sum_{c \in F} R_2(c) = \sum_{c \in C} R_2(c) - \sum_{c \in C - F} R_2(c) = 3k - R_2(C_L - F) = 2k + 1.$$

Hence there is a color $c_1 \in F$ such that $R_2(C_L - F)(c_1) = |L^{-1}(c_1) \cap R_{f_1,2}| \ge 3 = |R_{f_1,1}| \ge |R_{f_1,2}|$. So $c_1 \in \bigcap_{v \in R_{f_1,2}} L(v)$. On the other hand, by (7.7), $P_2 = R_{f_1,2}$, and by Lemma 3.2, $\bigcap_{v \in P_2} L(v) = \varnothing$, a contradiction.

Case 1.2: $i_0 = 0$.

In this case,

$$|V_1| = R_1(C_L - F) = k, \quad |V_2| = 0, \quad |V_3| = k + 2. \tag{7.8}$$

Combining with $i_0 = 0$ and (P2), for each color $c \in F$, $R_1(c) = |R_{f_1,1}| - 1$.

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Claim 7.4 $|P_1| = |R_{f_1,1}| = 3$ *and* $R_1(c) = 2$ *for any color* $c \in F$.

Proof If $|R_{f_1,1}| \geq 4$, then for any color $c \in F$, $f_1(c \to P_1)$ can be extended to a near acceptable L-coloring of G by coloring the remaining $k-1$ vertices of V_3 injectively with the remaining $k-1$ colors of F (note that $|L^{-1}(c) \cap P_1| = |R_{f_1,1}| - 1 \geq 3$).

Thus $|P_1| = |R_{f_1,1}| = 3$ (cf. (7.7)). This implies that $R_1(c) = 2$ for any color $c \in F$. ■

If there is a color $c \in F$ such that $R_2(c) \geq 2$, then f_1 can be extended to a near acceptable L-coloring of G by coloring a 2-subset U_1 of of $R_{f_1,2}$ with a color $c \in \bigcap_{v \in U_1} L(v) \cap F$, coloring a 2-subset U_2 of $R_{f_1,1}$ by a color from $c' \in \bigcap_{v \in U_2} L(v) \cap (F - \{c\})$, and coloring the remaining $k-2$ vertices of V_3 injectively with the remaining $k-2$ colors of F.

Thus

$$R_2(c) \leq 1, \forall c \in F \text{ and } \sum_{c \in F} R_2(c) \leq k. \tag{7.9}$$

This implies that $|R_{f_1,2}| \leq 2$, for otherwise interchanging the roles of $R_{f_1,1}$ and $R_{f_1,2}$, we would have $R_2(c) = |R_{f_1,2}| - 1 \geq 2$ for all $c \in F$, in contrary to (7.9).

Claim 7.5 $|R_{f_1,i}| = 1$ *for* $i = 2, 3, \ldots, k$.

Proof Assume to the contrary that $|R_{f_1,2}| = 2$ (as $|R_{f_1,2}| \leq 2$), then by Observation 7.2, $R_2(C_L - F) \leq |V_1 - P_2| = |V_1| - |S_{f_1,2}| = k - |S_{f_1,2}|$ and

$$\sum_{c \in F} R_2(c) = \sum_{c \in C_L} R_2(c) - \sum_{c \in C_L - F} R_2(c) = 2k - R_2(C_L - F) \geq k + |S_{f_1,2}|. \tag{7.10}$$

Combining with (7.9), we have $|S_{f_1,2}| = 0$ and hence $R_{f_1,2} = P_2$, in contrary to Claim 7.3. By Claim 7.3, $|S_{f_1,2}| \geq 1$, in contrary to (7.9).

Therefore $|R_{f_1,2}| = 1$ and hence $|R_{f_1,i}| = 1$ for $i = 2, 3, \ldots, k$ (note that $|V_3| = k + 2$). ■

Claim 7.6 $|S_{f_1,j}| \leq 2$ *for* $j = 2, 3, \ldots, k$.

Proof If $|f_1(S_{f_1,j})| \geq 2$ for some j, say $c_1, c_2 \in f_1(S_{f_1,j})$, then $\tau_1(f_1(c_1 \to P_1)) = \tau_1(f_1)$ (as (P1) holds) and $\tau_2(f_1(c \to P_1)) < \tau_2(f_1)$, because

$$|S_{f_1,1}| = 0 \text{ (by 7.7)}, \ |S_{f_1(c_1 \to P_1),1}| = R_1(C_L - F)(c_1),$$

and

$$|S_{f_1(c_1 \to P_1),j}| = |S_{f_1,j}| - R_1(C_L - F)(c_1) > 0,$$

since $|f_1(S_{f_1,j})| \geq 2$. This is in contrary to our choice of f_1.

Hence for each $j \in \{2, 3, \ldots, k\}$, $|f_1(S_{f_1,j})| \leq 1$, and $|S_{f_1,j}| \leq |f_1^{-1}(c_j)|$ for some $c_j \in C_L - F$. As (P1) holds, $|f_1^{-1}(c_j)| = R_1(C_L - F)(c_j) \leq 2$. So $|S_{f_1,j}| \leq 2$. ■

Combining with Claims 7.4, 7.5, and 7.6, we have

$$|R_{f_1,1}| = 3, \ |S_{f_1,1}| = 0, \text{ and for } 2 \leq j \leq k, \ |R_{f_1,j}| = 1, \ |S_{f_1,j}| \leq 2.$$

So each part of G is 3^--part, in contrary to Theorem 4.1.

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Case 2: $|\mathbf{V_1}| = \mathbf{k} - \mathbf{i_0} + \mathbf{1}$.

If $P_{i_0+1} = R_{f_1,i_0+1}$, then by Claim 7.3, $|R_{f_1,i_0+1}| \geq 3$ and $|V_2| \geq 3i_0$. By (7.2), $|V_1| = |V| - |V_2| - |V_3| \leq 2k + 2 - 3i_0 - (k - i_0 + 1) = k - 2i_0 + 1$, and hence $i_0 = 0$. This implies that

$$|V_1| = k + 1,\ |V_2| = 0,\ |V_3| = k + 1.$$

By Observation 7.2, $R_1(C_L - F) \leq |V_1| = k + 1$, we conclude that

$$\sum_{c \in F} R_1(c) \geq \sum_{c \in C_L} R_1(c) - \sum_{c \in C_L - F} R_1(c) \geq 3k - R_1(C_L - F) \geq 2k - 1 \geq k + 1.$$

So there is a color $c \in F$ such that $R_1(c) \geq 2$. We can extend f_1 to a near acceptable L-coloring of G by coloring two vertices of $R_{f_1,1}$ with c, and the remaining $k - 1$ vertices of V_3 injectively with the remaining $k - 1$ colors of F.

Thus $P_{i_0+1} \neq R_{f_1,i_0+1}$, i.e., $S_{f_1,i_0+1} \neq \emptyset$.

As $S_{f_1,i_0+1} \neq \emptyset$, $|V_1 - P_{i_0+1}| = |V_1| - |S_{f_1,i_0+1}| < |V_1|$ and by (7.6), we have

$$|V_1 - P_{i_0+1}| = k - i_0 = R_{i_0+1}(C_L - F),\ |S_{f_1,i_0+1}| = 1. \tag{7.11}$$

So (P1) and (P2) holds for $i_0 + 1$.

Claim 7.7 *For each $1 \leq i \leq i_0 + 1$, $|R_{f_1,i}| = 2$ and for $j \geq i_0 + 2$, $|R_{f_1,i}| \leq 2$.*

Proof By (7.2), we have $|V_2| \geq 2i_0$, $|V_3| \geq k - i_0 + 1$. Since $|V_1| + |V_2| + |V_3| = 2k + 2$, we conclude that

$$|V_1| = k - i_0 + 1,\ |V_2| = 2i_0,\ |V_3| = k - i_0 + 1.$$

So $\forall j \leq i_0 + 1,\ |R_{f_1,j}| = 2,$ and $\forall j \geq i_0 + 2,\ |R_{f_1,j}| \leq 2.$ ∎

Claim 7.8 *For $1 \leq i \leq k$, if $|R_{f_1,i}| = 2$, then $|S_{f_1,i}| = 1$.*

Proof By Claim 7.7, $|R_{f_1,1}| = \ldots = |R_{f_1,i_0+1}|$. As (P2) holds, there are i_0 colors $c \in F_1 \subseteq F$ such that $R_{i_0+1}(c) = |R_{f_1,i_0+1}|$. Therefore, for any index j with $|R_{f_1,j}| = 2$, if we re-order the parts so that $R_{f_1,j}$ and R_{f_1,i_0+1} interchange positions (while the other parts stay at their position), (R1) and (R2) are satisfied. So the conclusions we have obtained for P_{i_0+1} hold for P_j. In particular, for any j with $|R_{f_1,j}| = 2$, we have $|S_{f_1,j}| = 1$. ∎

Claim 7.9 *$|S_{f_1,j}| \leq 2$ for all j.*

Proof As $(P1)$ holds for $i_0 + 1$, $|f_1^{-1}(c)| = R_{i_0+1}(c) \leq |R_{f_1,i_0+1}| = 2$ for any $c \in C_L - F$. If $|S_{f_1,j}| \geq 3$ for some j, then there is a color $c \in C_L - F$ for which the following holds:

- $|f_1^{-1}(c) \cap P_j| = 1$, or
- $|S_{f_1,j}| \geq 4$, and $|f_1^{-1}(c) \cap P_j| = 2$.

Let

$$f_1' = f_1(c \to P_{i_0+1}).$$

Then f_1' is a valid partial L-coloring of G with $\tau_1(f_1') = \tau_1(f_1)$ (as (P1) holds). By (7.11), $|S_{f_1,i_0}| = 1$. Thus either $|S_{f_1',j}| = |S_{f_1,j}| - 1 \geq 2$ and $|S_{f_1',i_0+1}| = 2$, or $|S_{f_1',j}| = |S_{f_1,j}| - 2 \geq 2$ and $|S_{f_1',i_0+1}| = 3$. Hence $\tau_2(f_1')) < \tau_2(f_1)$, in contrary to our choice of f_1. ∎

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It follows from Claims 7.8 and 7.9 that each part of G is 3^--part, in contrary to Theorem 4.1.

This completes the proof of Lemma 7.1.

8 Tighter upper bound for the number of frequent colors

In this section and the next section, we assume that (G, L) is a minimum counterexample to Theorem 1.2 with $\sum_{v \in V(G)} |L(v)|$ maximum.

This section proves that there are at most $k - p_1 - 1$ frequent colors. Assume to the contrary that there are $k - p_1$ frequent colors. We shall construct another k-list assignment L' of G that has k frequent colors. By Lemma 7.1, (G, L') is not a counterexample to Theorem 1.2. Hence there is an L'-coloring f of G. Using this coloring f, we construct a near-acceptable L-coloring of G, which contradicts Lemma 6.1.

Let F be the set of frequent colors, and $F' \subseteq F$ be the set of frequent colors of Type (1).

By Lemma 7.1, we may assume that $|F| \le k - 1$. If $\lambda = 1$, then for any $v \in T$, all colors in $L(v)$ are frequent of Type (2), a contradiction (note that $p_1 \ge 3$, so $T \ne \emptyset$). Thus $\lambda \ge 2$.

Lemma 8.1 $\lambda \le p_1 + 1$.

Proof For $c \in C_L - F'$, by definition, $|L^{-1}(c)| \le k + 1$. By Lemma 3.2, for each $c \in F'$, $|L^{-1}(c)| \le k + p_1 + 2$. Therefore

$$k|V| \le \sum_{v \in V} |L(v)| = \sum_{c \in C_L} |L^{-1}(c)| \le |F'|(k + p_1 + 2) + |C_L - F'|(k + 1).$$

Hence

$$|F'| \ge \frac{k|V| - (k+1)|C_L|}{p_1 + 1} = \frac{k\lambda - |C_L|}{p_1 + 1}. \tag{8.1}$$

As $|F'| < k$, we have

$$|C_L| > k(\lambda - p_1 - 1). \tag{8.2}$$

Since $\lambda \ge 2$, we have $|C_L| \le 2k$. Plug this into (8.2), we have $\lambda \le p_1 + 2$.

If $\lambda = p_1 + 2$, then $|C_L| = |V| - \lambda = 2k + 2 - (p_1 + 2) = 2k - p_1 \le 2k - 3$ (as $p_1 \ge 3$). This implies that G has no 2-part (if $\{u, v\}$ is a 2-part of G, then $L(u) \cap L(v) = \emptyset$ and hence $|C_L| \ge 2k$). By (4.7), $2k - 1 = |V| - 3 \ge 3(k - p_1) + p_1$. Hence

$$p_1 \ge \frac{k+1}{2}. \tag{8.3}$$

By (8.1),

$$|F'| \ge \frac{k\lambda - |C_L|}{p_1 + 1} = \frac{k(p_1 + 2) - (2k - p_1)}{p_1 + 1} = \frac{(k+1)p_1}{p_1 + 1} = k - \frac{k - p_1}{p_1 + 1} > k - 1.$$

Hence $|F'| \ge k$, a contradiction. Thus $\lambda \le p_1 + 1$. ∎

Lemma 8.2 $F = \bigcap_{v \in T} L(v)$.

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Proof If $p_1 = \lambda - 1$, then each color in $\bigcap_{v\in T} L(v)$ is contained in at least $\lambda - 1$ singleton lists, and hence is a frequent color of Type (3).

If $p_1 \geq \lambda$, then each color in $\bigcap_{v\in T} L(v)$ is contained in at least λ singleton lists, and hence is a frequent color of Type (2).

In any case,

$$\bigcap_{v\in T} L(v) \subseteq F.$$

On the other hand, assume there is a frequent color $c \notin \bigcap_{v\in T} L(v)$, say $c \notin L(v)$ for some $v \in T$, then let L' be the list assignment of G defined as $L'(x) = L(x)$ for $x \neq v$ and $L'(v) = L(v) \cup \{c\}$. By our assumption that (G, L) is a minimum counterexample with $\sum_{v\in V(G)} |L(v)|$ maximum, G and L' is not a counterexample to Theorem 1.2. So G has an L'-coloring f. But then f is a near acceptable L-coloring of G, in contrary to Lemma 6.1. Therefore $F \subseteq \bigcap_{v\in T} L(v)$. ∎

Lemma 8.3 *There are at most $k - p_1 - 1$ frequent colors.*

Proof Assume to the contrary that $\{c_{p_1+1}, c_{p_1+2}, \ldots, c_k\}$ is a set of $k - p_1$ frequent colors.

Assume $T = \{v_1, v_2, \ldots, v_{p_1}\}$. We choose p_1 colors $c_1, c_2, \ldots, c_{p_1}$ so that for $i = 1, 2, \ldots, p_1$,

$$c_i \in L(v_i) - \{c_{p_1+1}, \ldots, c_k\} - \{c_1, \ldots, c_{i-1}\}.$$

As $|L(v_i)| \geq k$, the color c_i exists.

Let $C' = \{c_1, c_2, \ldots, c_k\}$ and define L' as follows:

$$L'(v) = \begin{cases} C' & \text{if } v \in T, \\ L(v) & \text{otherwise.} \end{cases}$$

By Lemma 8.1, $p_1 \geq \lambda - 1$. If $p_1 \geq \lambda$, then each color in C' is Type-2 frequent with respect to L'. If $p_1 = \lambda - 1$, then each color in C' is Type-3 frequent with respect to L'. By Lemma 7.1, (G, L') is not a minimum counterexample to Theorem 1.2. Since $C_{L'} \subseteq C_L$, we know that (G, L') is not a counterexample to Theorem 1.2. Hence G has an L'-coloring f.

Note that if $v \notin T$, then $f(v) \in L(v)$. We shall modify f to obtain a near acceptable L-coloring of G.

Let $T' = \{v_i : 1 \leq i \leq p_1, c_i \in f(T)\}$. As $|T - T'| = |f(T) - \{c_1, c_2, \ldots, c_{p_1}\}|$, there is a bijection $g : T - T' \to f(T) - \{c_1, c_2, \ldots, c_{p_1}\}$.

Let $f' : V \to C_L$ be defined as follows:

$$f'(v) = \begin{cases} f(v) & \text{if } v \notin T, \\ c_i & \text{if } v = v_i \in T', \\ g(v) & \text{if } v \in T - T'. \end{cases}$$

Then f' is a near acceptable L-coloring of G, in contradiction to Lemma 6.1. ∎

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9 Final contradiction

We shall find a subset X of T and a set F'' of $k - p_1$ colors so that for each $c \in F''$,

$$|L^{-1}(c) \cap X| \geq \lambda.$$

This would imply that all the $k - p_1$ colors in F'' are frequent (of Type (2)). This is in contrary to Lemma 7.1.

For any color $c \in C_L - F$, $|L^{-1}(c)| \leq k + 1$. Let

$$b = \min\{k + 1 - |L^{-1}(c)| : c \in C_L - F\}.$$

Lemma 9.1 *There is a subset X of T such that*

(1) $|X| \geq p_1 - \lambda + 1$.
(2) $|L(X)| \leq k + b$.

Moreover, if $b = 0$ *or* $p_1 = \lambda - 1$, *then* $|X| \geq p_1 - \lambda + 2$.

Proof Let $c' \in C_L - F$ be a color with $|L^{-1}(c')| = k + 1 - b$. By Lemma 8.2, there is a vertex $w \in T$ such that $c' \notin L(w)$. Define a list assignment L' as follows:

$$L'(v) = \begin{cases} L(v) \cup \{c'\} & v = w, \\ L(v) & \text{otherwise.} \end{cases}$$

By the maximality of $\sum_{v \in V(G)} |L(v)|$, G has an L'-coloring f. We must have $f(w) = c'$ and w is the only badly colored vertex, for otherwise f is a proper L-coloring of G.

Now f is a pseudo L-coloring of G. By Lemma 5.1, in the bipartite graph B_f, V_f has a subset X_f such that $|X_f| > |Y_f| = |N_{B_f}(X_f)|$, and $V_f - X_f$ contains at most $\lambda - 1$ singletons of G.

It is easy to see that $w \in X_f$ and $c' \notin Y_f$. Let

$$X = \{v \in T : \{v\} \text{ is an } f\text{-class in } X_f\}.$$

Then $|X| = |T| - |(V_f - X_f) \cap T| \geq p_1 - \lambda + 1$ and by Lemma 5.2, if $p_1 = \lambda - 1$, then $|X| = |T| - |(V_f - X_f) \cap T| \geq p_1 - \lambda + 2$.

Since each f-class in X_f contains a vertex v for which $c' \notin L(v)$, we have

$$|L(X)| \leq |Y_f| < |X_f| \leq |V| - |L^{-1}(c')| = k + 1 + b.$$

So $|L(X)| \leq k + b$.

It remains to prove that if $b = 0$, i.e., $|L^{-1}(c')| = k + 1$, then $|X| \geq p_1 - \lambda + 2$.

Assume to the contrary that $|L^{-1}(c')| = k + 1$ and $|X| = p_1 - \lambda + 1$. By Lemma 5.1, $|Y_f| \geq k + 1$ and hence $|X_f| \geq k + 2$, in contrary to $|X_f| \leq |V| - |L^{-1}(c')| = k + 1$.

This completes the proof of Lemma 9.1. ∎

We order the colors in $L(X)$ as $c_1, c_2, \dots, c_t$, so that

$$|L^{-1}(c_1) \cap X| \geq |L^{-1}(c_2) \cap X| \geq \dots \geq |L^{-1}(c_t) \cap X|,$$

where $t = |L(X)|$. Let $F'' = \{c_1, c_2, \dots, c_{k-p_1}\}$.

It suffices to show that

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \lambda,$$

and hence each color $c_i \in F''$ is a frequent of Type (2).

Let $Z = \{c_{k-p_1}, c_{k-p_1+1}, \ldots, c_t\}$. For each $v \in X$, $|L(v) \cap Z| \geq |L(v)| - (k - p_1 - 1) \geq p_1 + 1$. Hence

$$(9.1) \qquad |Z||L^{-1}(c_{k-p_1}) \cap X| \geq \sum_{i=k-p_1}^{t} |L^{-1}(c_i) \cap X| = \sum_{v \in X} |L(v) \cap Z| \geq |X|(p_1+1).$$

By Lemma 9.1,

$$|Z| = |L(X)| - (k - p_1 - 1) \leq p_1 + 1 + b.$$

Plugging this into (9.1), we have

$$(p_1 + 1 + b)|L^{-1}(c_{k-p_1}) \cap X| \geq |X|(p_1 + 1).$$

This implies that

$$(9.2) \qquad |L^{-1}(c_{k-p_1}) \cap X| \geq \frac{|X|(p_1+1)}{p_1+1+b}.$$

For each $c \in C_L - F$, $|L^{-1}(c)| \leq k + 1 - b$ (by definition of b). By Lemma 3.2, for $c \in F$, $|L^{-1}(c)| \leq k + p_1 + 2$. Hence

$$(9.3) \quad (2k+2)k \leq \sum_{v \in V} |L(v)| = \sum_{c \in C_L} |L^{-1}(c)| \leq |C_L - F|(k+1-b) + |F|(k+p_1+2).$$

Plugging $|C_L| = |V| - \lambda = 2k + 2 - \lambda$ and $|F| \leq k - p_1 - 1$ into (9.3), we have

$$(9.4) \quad (2k+2)k \leq (2k+2-\lambda-(k-p_1-1))(k+1-b) + (k-p_1-1)(k+p_1+2).$$

(Note that the coefficient of $|F|$ in the right hand side of (9.3) is positive.)

This implies

$$(9.5) \qquad b \leq \frac{(p_1+3-\lambda-k)(k+1) + (k-p_1-1)(k+p_1+2)}{k+p_1+3-\lambda}.$$

If $\lambda = 2$, then since $p_1 \geq 3$, by plugging $|X| \geq p_1 - \lambda + 1$ (see Lemma 9.1) into (9.2), we have

$$\begin{aligned} |L^{-1}(c_{k-p_1}) \cap X| &\geq \frac{(p_1-\lambda+1)(p_1+1)}{p_1+1+b} \geq \frac{(p_1-1)(p_1+1)}{p_1+1+\frac{(p_1+1)(k-p_1-1)}{k+p_1+1}} \\ &= \frac{(p_1-1)(k+p_1+1)}{2k} \geq \frac{2(k+p_1+1)}{2k} > 1. \end{aligned}$$

Since $|L^{-1}(c_{k-p_1}) \cap X|$ is an integer, $|L^{-1}(c_{k-p_1}) \cap X| \geq 2 = \lambda$ and we are done.

Therefore $\lambda \geq 3$ and $|C_L| \leq 2k - 1$. By Lemma 3.2, G has no 2-parts. By the same reason as (8.3), we have

$$p_1 \geq \frac{k+1}{2}.$$

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Combining (8.1) with Lemma 8.3, together with $p_1 \geq \frac{k+1}{2}$, we have

$$\frac{k-3}{2} \geq k - p_1 - 1 \geq |F'| \geq \frac{k\lambda - |C_L|}{p_1+1} = \frac{k\lambda - (2k+2-\lambda)}{p_1+1} = \frac{(k+1)\lambda - 2k - 2}{p_1+1}.$$

Hence

$$\lambda \leq \frac{\frac{(k-3)(p_1+1)}{2} + 2k + 2}{k+1} = \frac{p_1+1}{2} + 2 - \frac{2(p_1+1)}{k+1} < \frac{p_1+1}{2} + 1.$$

Since λ is an integer,

$$\lambda \leq \frac{p_1}{2} + 1. \tag{9.6}$$

Therefore

$$p_1 \geq 2\lambda - 2 \geq \lambda + 1.$$

Plugging this into (9.5), we have

$$\begin{aligned} b &\leq \frac{(p_1 + 3 - \lambda - k)(k+1) + (k - p_1 - 1)(k + p_1 + 2)}{k + p_1 + 3 - \lambda} \\ &\leq \frac{(p_1 + 3 - \lambda - k)(k+1) + (k - p_1 - 1)(k + p_1 + 2)}{k+4} \quad (\text{ as } p_1 \geq \lambda + 1) \\ &= \frac{(p_1+1)(k - p_1 - 1) + (k+1)(2-\lambda)}{k+4} \\ &\leq \frac{\frac{k-3}{2}(p_1+1) + (k+1)(2-\lambda)}{k+4} \quad (\text{by (8.3), i.e., } p_1 \geq \frac{k+1}{2}) \\ &= \frac{1}{2}(p_1 + 1 - 2\lambda) + \frac{2k + 2 + 3\lambda - \frac{7}{2}(p_1+1)}{k+4} \\ &\leq \frac{1}{2}(p_1 + 1 - 2\lambda) + \frac{k + 1/2}{k+4} \\ &< \frac{1}{2}(p_1 + 1 - 2\lambda) + 1. \end{aligned}$$

It follows from (9.6) that $p_1 \geq 2\lambda - 2$.
If $p_1 \in \{2\lambda - 2, 2\lambda - 1\}$, then $b = 0$. This implies that $|X| \geq p_1 - \lambda + 2$.
It follows from (9.2) that

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \frac{|X|(p_1+1)}{p_1+1+b} \geq \frac{(p_1 - \lambda + 2)(p_1+1)}{p_1+1} \geq \lambda.$$

If $p_1 \geq 2\lambda$, then

$$b \leq \frac{1}{2}(p_1 + 1 - 2\lambda) + \frac{1}{2} \leq \frac{1}{2}(p_1 + 1 - 2\lambda) + \frac{1}{2}(p_1 + 1 - 2\lambda) = p_1 + 1 - 2\lambda.$$

Hence

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \frac{(p_1 - \lambda + 1)(p_1+1)}{p_1+1+b} \geq \frac{(p_1 - \lambda + 1)(p_1+1)}{2(p_1 + 1 - \lambda)} = \frac{p_1+1}{2} \geq \lambda.$$

This completes the whole proof of Theorem 1.2.

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This paper characterizes all non-k-choosable complete k-partite graphs G with $2k+2$ vertices. If the number of vertices of G increases, and the chromatic number remains k, then the choice number of G may increase. It was proved in [16] that k-chromatic graphs with $n \geq 2k+1$ vertices have choice number at most $\lceil \frac{n+k-1}{3} \rceil$. It would be interesting to characterize graphs for which this upper bound on the choice number is sharp.

Acknowledgments We thank the referee for a careful reading of the manuscript and for many valuable comments that improved the presentation of this paper.

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https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press