THE EFFECT OF FINITE HEAT CONTENT AND THERMAL DIFFUSION ON THE GROWTH OF A SEA-ICE COVER

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Abstract. The practical analysis of the growth of a sea-ice cover is discussed with initial reference to the classical work of Stefan, whose basic equation connecting surface temperature with the growth of a uniform ice cover of negligible specific heat and hence infinite diffusivity is extended to cover "real" cases. The separate effects of a finite heat content and thermal diffusivity are derived theoretically and semi-empirically respectively, and combined in a more general ice-growth equation which is then tested in the analysis of annual sea-ice growth on Hudson Bay.

Résumé. L'analyse pratique de la croissance d'une couche de glace de mer est discutée. L'équation classique de Stefan qui donne la croissance en fonction de la température de la surface d'une couche de glace de chaleur spécifique négligeable et, par conséquent, de diffusivité thermique infinie, est modifiée pour couvrir les cas pratiques. Les influences de la chaleur spécifique sont déduites théoriquement et celles de la diffusivité thermale d'une façon semi-empirique. Elles sont enfin combinées dans une equation plus généralisee, qui est ensuite vérifiée dans l'étude de la croissance de la glace de mer annelle faite a la baie de Hudson.

Zusammenfassung. Die praktische Untersuchung des Wachstums einer Meer-Eisdecke wird unter anfänglichem Bezug auf die klassische Arbeit Stefan's besprochen, dessen Grundgleichung den Zusammenhang zwischen Oberflächentemperatur und Eisbildung im Fall vernachlässigbarer spezifischer Wärme und daher unendlicher Wärmediffusion darstellt. Der Anwendungsbereich dieser Gleichung wird auf "wirkliche" Fälle erweitert. Die Einflüsse einer endlichen spezifischen Wärme und Wärmediffusion werden getrennt theoretisch beziehungsweise halb-empirisch hergeleitet und dann zu einer allgemeinen Eisbildungsgleichung verbunden, die durch eine Untersuchung des Wachstums jährlichen Meer-Eises in der Hudson Bay geprüft wird.

1. Introduction

Theoretical studies of heat transmission through floating covers of even pure ice are considerably complicated by the continuous formation of ice at the ice–water interface. In the absence of warm water currents below, as long as a temperature gradient exists at the lower boundary, the ice must grow. The three equations of basic importance in specifying the thermal conditions existing in an ice cover of uniform thermal conductivity k, specific heat c, density ρ, and thickness h at time t are

$$\frac{\partial\theta}{\partial t} = \frac{k}{\rho c}\frac{\partial^2\theta}{\partial x^2}, \tag{1.1}$$

$$\mathcal{J} = k\frac{\partial\theta}{\partial x}, \tag{1.2}$$

$$\frac{dh}{dt} = \frac{k}{L\rho}\left(\frac{\partial\theta}{\partial x}\right)_{x=h} = \frac{1}{L\rho}\,\mathcal{J}_{x=h} \tag{1.3}$$

where θ is the temperature, $\mathcal{J}$ is the heat flux at a depth $0 \leqslant x \leqslant h$, L is the latent heat of formation, and dh/dt is the rate of advance of the lower boundary.

Numerous works, e.g. Kolesnikov (1958), which have appeared since Stefan's (1891) classical solutions to the ice-growth problem, have, on the whole, resulted in unwieldy expressions requiring many unsatisfactory approximations to permit practical applications. A clear quantitative (and qualitative) picture of thermal occurrences in a floating ice cover, especially if non-uniform, is best given by electrical analogue methods. This technique has been discussed by Schwerdtfeger (1964), in a design which involves the usual electrical analogues for equations (1.1) and (1.2), and a novel automatic switching device to simulate freezing and hence to provide an adjustable electrical analogue for latent heat and equation (1.3).

When an analogue computer of the above type is not available or when short calculations are required, it is still desirable to have suitable analytical procedures at hand. Stefan (1891) derived a simple solution for ice of negligible specific heat, which although infrequently acknowledged, is often used. It states that the ice thicknesses h_1 and h_2 before and after a time interval t_2-t_1 during which the surface temperature is given by θ_0, are given by

$$h_2^2-h_1^2 = \frac{2k}{L\rho}\int_{t_1}^{t_2} \theta_0\, dt. \tag{1.4}$$

Stefan himself discussed the two serious limitations of this equation which both result from neglecting the specific heat of the ice. These are that no allowance is made for the variable energy content of an ice cover nor for the time taken for a change in surface temperature to modify the temperature gradient at the ice–water interface. The assumption of zero specific heat of course implies infinite thermal diffusivity, $K = k/\rho c$, under which conditions equation (1.1) shows all temperature changes to occur instantaneously throughout the ice cover, which in turn results in a constant uniform temperature gradient. Observations reported by Barnes (1928, p. 28) to support his simple, Stefan-type equation actually show that the effect of finite thermal diffusivity is important even in the history of ice whose thickness is only of the order of a millimetre. Barnes made rapid measurements on St. Lawrence River ice formed in areas cleared by icebreakers and ferries. Invariably, the theoretically calculated ice thickness exceeded the observed value and, although no comment was given, this error progressively increased for greater ice thicknesses.

As a result of the high values for the specific heat of sea ice (Schwerdtfeger, 1963[b]) especially at temperatures near to the freezing point, Stefan's growth solution for ice of constant specific heat c:

$$h_2^2\left(1+\frac{c\theta_2}{3L}\right)-h_1^2\left(1+\frac{c\theta_1}{3L}\right) = \frac{2k}{L\rho}\int_{t_1}^{t_2}\theta_0\, dt \tag{1.5}$$

where θ_1 and θ_2 are the ice surface temperatures at times t_1 and t_2 respectively, and the other symbols are as in equation (1.4), is not sufficiently accurate in general. In the original paper, equation (1.5) is merely quoted, and appears to be partly empirical.

Because Stefan's general equation is inapplicable to ice in general, particularly sea ice to which most work on floating ice is directed, this paper will discuss the energy content of an inhomogeneous ice cover and the rate of transfer of heat through this medium, with a view of deriving a two-stage correction to Stefan's basic equation (1.4).

2. Changes of Heat Content of a Growing Ice Cover

Figure 1 shows the temperature distribution in an ice cover initially of thickness h, before and after an additional thin layer of thickness Δh has been formed, assuming the idealized case of continued uniformity of temperature gradient, which would occur only in the case of very slow growth. It is seen that the temperature of the ice at a distance x below the surface is given by:

$$\theta = (\theta_0-\theta_F)\frac{h-x}{h}+\theta_F = \theta_0-(\theta_0-\theta_F)\frac{x}{h} \tag{2.1}$$

where θ_0 is the ice surface temperature, and θ_F, the freezing temperature, that of the ice–water interface. The change in temperature of the ice at depth x after freezing of the layer Δh is

$$\Delta\theta = (\theta_0-\theta_F)\frac{x}{h}-(\theta_0-\theta_F)\frac{x}{h+\Delta h} \approx (\theta_0-\theta_F)\frac{x\,\Delta h}{h^2}. \tag{2.2}$$

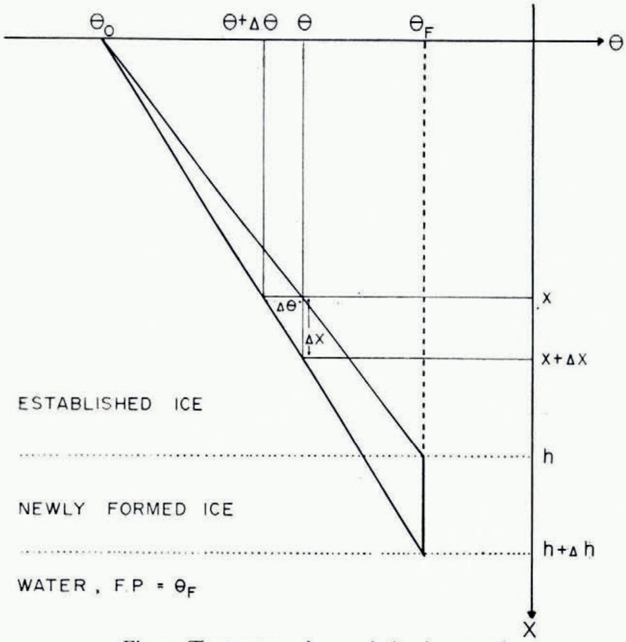

Fig. 1. Temperature changes during ice growth

The change in heat content of a volume Δx, in a cylinder of unit cross-sectional area, at a depth x is thus given by:

$$\Delta Q_{\rm i} = (\theta_0 - \theta_{\rm F}) \frac{x\,\Delta h}{h^2} \Delta x \rho c_{{\rm s}\,x} \tag{2.3}$$

where $c_{{\rm s}\,x}$ is the specific heat of the sea ice at a depth x. The specific heat of sea ice, or any ice frozen from water containing soluble impurities, was shown by the author (Schwerdtfeger, 1963[b]) to be given by

$$c_{\rm s} = -\frac{\sigma}{\alpha\theta^2} L_{\rm i} + \frac{\sigma}{\alpha\theta}(c_{\rm w} - c_{\rm i}) + c_{\rm i} \tag{2.4}$$

where θ and σ are the temperature and salinity of the ice, α is a constant relating temperature to the equilibrium salinity of the concentrated mother solution, $L_{\rm i}$ and $c_{\rm i}$ are the latent and specific heats of pure ice and $c_{\rm w}$ is the specific heat of water. Substitution of equation (2.1) in (2.4) gives the specific heat $c_{{\rm s}\,x}$ as a function of x, so that $\Delta Q_{\rm i}$ can be integrated through the ice cover to give the total change in heat content for unit increase in ice thickness,

$$\begin{aligned} Q_{\rm i} &= \overline{\rho c_{\rm s} \Delta\theta} \\ &= -\int_0^h (\theta_0 - \theta_{\rm F}) \frac{x}{h^2} \frac{\rho\sigma L_{\rm i}}{\alpha} \frac{dx}{[\theta_0 - (\theta_0 - \theta_{\rm F})x/h]^2} \\ &\quad + \int_0^h (\theta_0 - \theta_{\rm F}) \frac{x}{h^2} \frac{\rho\sigma(c_{\rm w} - c_{\rm i})}{\alpha} \frac{dx}{[\theta_0 - (\theta_0 - \theta_{\rm F})x/h]} + \int_0^h (\theta_0 - \theta_{\rm F}) \frac{x}{h^2} \rho c_{\rm i} dx. \end{aligned}$$

The integration leads to

$$\begin{aligned} Q_{\rm i} &= \overline{\rho c_{\rm s} \Delta\theta} \\ &= \rho\left[\left(\frac{L_{\rm i} + (c_{\rm w} - c_{\rm i})\theta_0}{(\theta_0 - \theta_{\rm F})\alpha} \sigma \ln\frac{\theta_0}{\theta_{\rm F}}\right) - \left(\frac{L_{\rm i} + (c_{\rm w} - c_{\rm i})\theta_{\rm F}}{\theta_{\rm F}\alpha}\right) + \frac{(\theta_0 - \theta_{\rm F})c_{\rm i}}{2}\right]. \end{aligned} \tag{2.5}$$

The term $\overline{c_s\Delta\theta}$ is a convenient notation for the mean value of the product of the specific heat and temperature change throughout the ice cover.

At the same time, the heat involved in the freezing of new ice only, is simply given by

$$Q_F = -L_s\rho \tag{2.6}$$

where L_s is the latent heat of sea ice. The author has previously shown (1963[b]) that:

$$L_s = (1-\sigma-\sigma/s)L_i \tag{2.7}$$

where s is the salt content of the parent sea-water. Hence

$$Q_F = -\rho(1-\sigma-\sigma/s)L_i. \tag{2.8}$$

It can immediately be seen that as $\theta_0 \to \theta_F$, $Q_i/Q_F \to 0$, and as $\theta_0 \to -\infty$, $Q_i/Q_F \to +\infty$ substantiating Stefan's observation that equation (1.4) holds most accurately for ice surface temperatures near to the freezing point.

Using values for the specific heat calculated by the author (1963[b]), Q_i/Q_F, the ratio of heat involved in cooling the ice cover to that connected with simultaneous freezing, may be

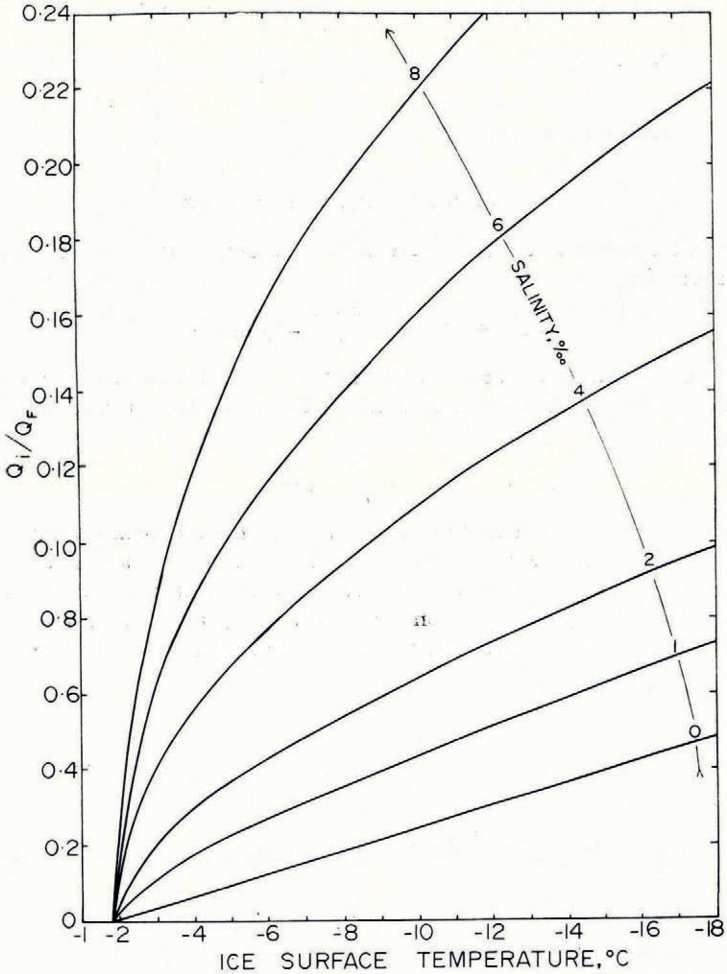

Fig. 2. Corrections to Stefan's simple ice-growth equation. Ratio of contributions to surface heat flux from ice-cover cooling Q_i to that for new ice forming Q_F as a function of ice surface temperature for various salinities

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calculated. The graph in Figure 2 shows these values as a function of salinity and temperature for ice frozen from sea-water having a freezing point of $-1\cdot8°$ C.

It is clear from Figure 2 that the latent heat due to freezing at the bottom of the ice cover is comparable in magnitude to the heat content change taking place in the cover as a whole. The first step towards an improved form of equation (1.4) must therefore be to replace the latent heat L by its "effective" value $L_s(1+Q_i/Q_F) = L_s+\overline{c_s\Delta\theta}$. It remains to correct for the time delay between changes in temperature gradients at the surface and at the ice–water interface; a method for this is given in the next section.

3. The Rate of Transfer of Energy through an Ice Cover

Solution of the thermal diffusion equation (1.1) shows that the time taken for a given degree of completion of a temperature change initiated by a sudden discontinuity in surface temperature is directly proportional to the square of the ice thickness, and inversely so to the diffusivity, i.e.

$$t_\theta \propto h^2/K. \tag{3.1}$$

The author has shown (Schwerdtfeger, unpublished) that a similar relation holds for an ice cover whose surface temperature is continually changing with time.

On considering sea ice, it is difficult to include the diffusivity changes in any practical application, as the time lags in temperature change are most conveniently determined from maxima or minima in the growth rate and temperature curves. This means that a significant change in the diffusivity occurs during a single interval of observation. However, the mean value of the diffusivity in the time interval between a minimum and the next maximum, or between a maximum and a minimum, in the ice temperature remains approximately constant, unless the temperatures averaged over the same times themselves show a marked change. Hence equation (3.1) is written:

$$t_\theta = \chi\bar{h}^2. \tag{3.2}$$

The general application of equation (3.2) requires an expression for the mean thickness $\bar{h}$ over the time interval t_θ. It is usually possible to write:

$$\bar{h} = \tfrac{1}{2}(2h_o+t_\theta dh/dt)$$

where h_o is the initial ice thickness* and dh/dt is the mean rate of increase during the time t_θ. Therefore

$$t_\theta \approx \frac{\chi h_o^2}{1-\chi h_o dh/dt}. \tag{3.4}$$

In order to obtain values for χ, the set of observations of temperature, heat flux and ice growth shown in Table I and Figures 3 and 4, were obtained on annual sea ice in Hudson Bay near Churchill, Manitoba, between January and May 1961. As well as showing rate of ice growth, the graph in Figure 3 also shows the heat flux at a depth of 20 cm. in the ice cover. The time axes in Figures 3 and 4 use single capital abbreviations for the months. Although the ice growth curve necessarily lacks detail, owing to the limited number of observations, the main features are visible and similar on both curves. The predominant minimum and maximum in the heat flux, occurring between 8 February and 9 February, and 22 February and 23 February respectively, are reproduced particularly clearly on the growth curve, but other minor trends also appear to be followed. The ratios of the two minima and the two maxima are approximately 80 and 70 cal. cm.$^{-3}$ respectively. As these values are reasonable approximations to the latent heat of formation plus cooling of the existing ice, it is reasonable to assume that the information from the graph is a suitable basis for further deduction.

Tables II and III below, summarize the method of calculating χ and time delays from the observations, reports on which have appeared by Schwerdtfeger and Pounder (1963) and Schwerdtfeger (1963[a]), the latter describing the heat-flux measurements.

* Strictly speaking, below the temperature probe or flux meter.

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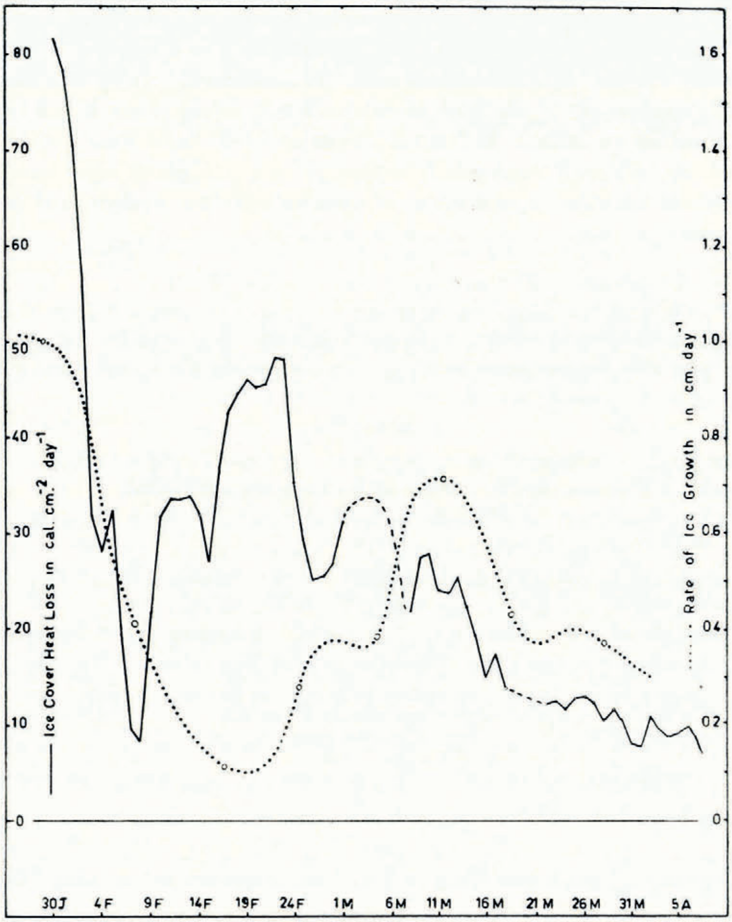

Fig. 3. Ice surface heat flux and growth rate of sea ice in Hudson Bay near Churchill, Manitoba, January to April 1961

TABLE I. ICE THICKNESS AND RATE OF GROWTH AT BUTTON BAY

Day	*Thickness* cm.	*Mean growth rate* cm. day^{-1}	*Mean date for rate*
13 January	85·2		
		0·37	14·5 January
16 January	86·3		
		1·07	20·5 January
25 January	95·9		
		1·01	29·0 January
2 February	104·0		
		0·41	7·5 February
13 February	108·5		
		$0{\cdot}11_4$	16·5 February
20 February	109·3		
		0·28	24·5 February
1 March	111·8		
		$0{\cdot}38_6$	4·5 March
8 March	114·5		
		0·71	11·5 March
15 March	119·5		
		0·43	18·5 March
22 March	122·5		
		$0{\cdot}38_5$	28 March
4 April	127·5		

TABLE II

Date of flux maximum or minimum	*Date of corresponding ice growth maximum or minimum*	*Time difference in days, t_θ*	*Mean ice depth below flux meter during time, t_θ*	χ days cm.$^{-2}$
7·5 February	19 February	11·5	86	$(1\cdot 6\pm 0\cdot 3)\times 10^{-3}$
22·5 February	11 March	16·5	93·5	$(1\cdot 8\pm 0\cdot 3)\times 10^{-3}$

TABLE III. SUMMARY OF DATA ON EFFECT OF SURFACE FLUX ON GROWTH RATE

Date of flux measurement	*Depth of ice below meter at that time* h_o cm.	*Mean growth rate, dh/dt* cm.day^{-1}	t_θ *from eq.* (3·4) days	*Date for surface flux to be seen in growth rate*
30 January	81	0·6	12 ± 2	11 February
19 March	101	0·3	18 ± 3	6 April

In Table II we are thus led to a numerical value for what might be termed the "lag coefficient" $\chi = (1\cdot 7\pm 0\cdot 3)\times 10^{-3}$ days cm.$^{-2}$, which is applied in Table III.

When the surface temperature, rather than the heat flux at a depth of 20 cm., was compared with the rate of growth of ice, χ was found to be equal to $1\cdot 2\pm 0\cdot 2$, and the values for t_θ calculated by means of equation (3.4) were found to be 13 ± 2 and 18 ± 3 days respectively. The lower value of χ in this case is due to the higher value of the diffusivity when the ice above the flux meter influenced the mean value. The two alternatives for calculating t_θ are certainly in satisfactory accord.

The above conclusions are supported by the records showing ice temperature as a function of depth and time. In Figure 4 the ice temperatures in the Hudson Bay ice cover being discussed are shown at five points separated by 25 cm. along a vertical axis in the cover. It is seen that the time for similar characteristics became longer between the lower levels where the temperatures are higher and the mean diffusivity is lower. Similarly the time lag for transmission of a minimum in temperature was greater than for a maximum for the same reason. This is of course also why the mean value for χ was less for the entire cover than for the portion below the flux meter at the 20 cm. level. More extensive observations would permit the determination of the lag coefficient for an ice cover as a function of surface temperature. In general, of course, mean surface temperature and hence the lag coefficient tend to increase toward the end of the winter.

4. CONDUCTED SURFACE HEAT FLUX AND THE LATENT AND SPECIFIC HEATS OF SEA ICE

The same Hudson Bay observations provide empirical estimates for the latent and specific heats of sea ice for comparison with theoretical values such as that given by equation (2·7). The daily heat-flux totals at the 20 cm. level are shown in Figure 3. Table III shows that the flux measurements on 30 January and 19 March are reflected in terms of ice growth on 11 February and 6 April respectively. A graph of the ice thickness such as may be obtained from Table I, shows that the heat lost between the former dates caused the growth of 21·3 cm. of new ice and the cooling of an average of 97·8 cm. of established ice below the flux meter. The total flux of heat lost between 30 January and 19 May may be calculated as 1514·5 cal. cm.$^{-2}$. These quantities satisfy Stefan's simple ice growth equation with an effective latent heat $L = 77\cdot 1$ cal. cm.$^{-3}$, or for a density of 0·915 g. cm.$^{-3}$, $L = 77\cdot 6$ cal. g.$^{-1}$.

This experimental result may now be compared to theoretically calculated values. A salinity profile, reported by the author (1963[b]), indicated a mean salinity of 5‰ for the ice cover. Equation (2.7) shows the latent heat of formation of this ice from sea-water of 30‰ salinity to be 65·8 cal. g.$^{-1}$. Figure 2 indicates that 10·6 cal. g.$^{-1}$ were lost by cooling of the older established ice, whose average surface temperature was $-9\cdot 0^\circ$ C. These two thermal

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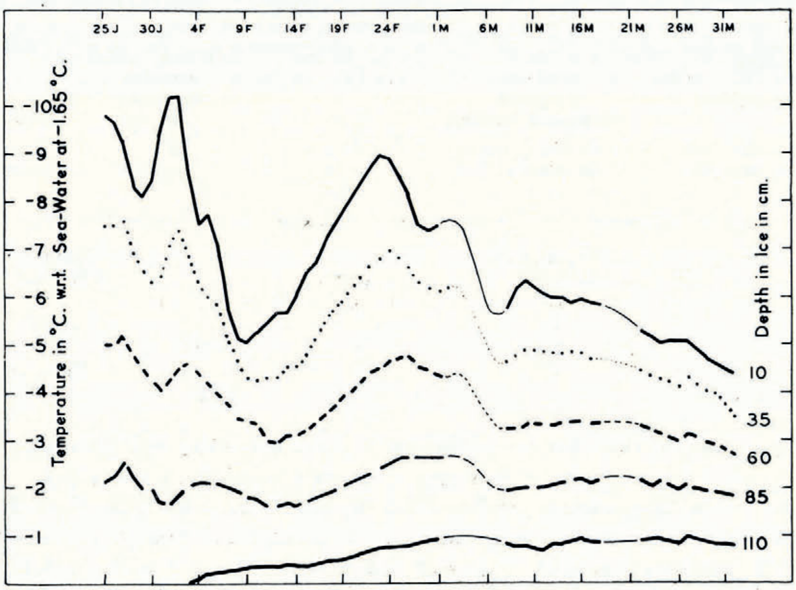

Fig. 4. Ice temperatures at various depths in degrees Centigrade below the sea-water temperature of $-1\cdot65^\circ$ *C. Data from Hudson Bay near Churchill, Manitoba, January to April 1961*

quantities total $76\cdot4$ cal. g.$^{-1}$, being in good agreement with the experimentally determined $77\cdot6$ cal. g.$^{-1}$. Assuming that the mean salinity of the cover was estimated to the nearest part per thousand, a little less than 5 per cent uncertainty is attached to the theoretically calculated quantity.

5. Ice Surface Temperature, Growth and Thermal Conductivity

The necessary modifications to enable a practical application of Stefan's simple ice growth equation ($1\cdot4$) have been separately tested in the preceding sections. The full procedure will now be illustrated for the same Hudson Bay data which provided some of the necessary empirical information. In calculating the freezing exposure, i.e. $\int_{t_1}^{t_2}\theta_0 dt$, it is considered that surface temperatures on 30 January and 19 March became effective in influencing the ice growth on 11 February and 6 April respectively.

The fully modified form of Stefan's equation (1.4) is

$$h_2^2 - h_1^2 = \frac{2k}{L_S + \overline{c\,\Delta\theta}} \sum_{t_1}^{t_2} (\theta_0 - \theta_F)\Delta t. \tag{5.1}$$

In the case of the data at hand

$$h_1 = 107\cdot7 \text{ cm. (11 February)}$$
$$h_2 = 128 \text{ cm. (6 April)}$$
$$\rho = 0\cdot915 \text{ g. cm.}^{-3}$$
$$L_S + \overline{c\,\Delta\theta} = 77\cdot0 \text{ cal.g.}^{-1} \text{ (from Section 4)}$$
$$\sum_{t_1}^{t_2} (\theta_0 - \theta_F) = 397\cdot7^\circ \text{ C. day} = 3\cdot44 \times 10^7\ {}^\circ\text{C. sec. (from Table IV)}$$
$$\Delta t = 1 \text{ day}$$
$$t_1 = 30 \text{ January}$$
$$t_2 = 19 \text{ March}$$

From these figures we obtain a thermal conductivity of $k = 4{\cdot}87 \times 10^{-3}$ cal. cm.$^{-1}$ sec.$^{-1}$ °C.$^{-1}$ for the sea-ice cover. The chief sources of error are in the effective latent heat term, some 5 per cent, and the uncertainty in the temperature transmission time, which leads to a possible 5 per cent error in the freezing exposure. The value of the thermal conductivity is thus only certain to within about 7 per cent.

It may be noted that the thermal conductivity of sea ice of 5‰ salinity and a temperature of $-5{\cdot}5°$ C. (the mean temperature of the cover) has been given as $4{\cdot}7 \times 10^{-3}$ c.g.s. units on theoretical grounds by the authoı (Schwerdtfeger, 1963[b]). This is well within the limits of the experimental determinations.

TABLE IV. ICE SURFACE TEMPERATURE

Date	*Temperature* 09.00 hr.	*Temperature* 21.00 hr.	*Date*	*Temperature* 09.00 hr.	*Temperature* 21.00 hr.	*Date*	*Temperature* 09.00 hr.	*Temperature* 21.00 hr.
30 January	11·5	11·4	24 February	11·0	10·8	21 March	5·5	
31 January	12·3	12·8	25	9·4	9·2	22	5·4	5·3
1 February	12·8	12·9	26	8·8	9·1	23	5·4	5·4
2	12·8	11·5	27	8·7	8·9	24	5·3	5·3
3	10·1	9·2	28 February	8·9	9·4	25	5·3	5·4
4	8·4	8·6	1 March	9·3	8·0	26	5·2	5·2
5	8·6	8·9	2	8·0		27	5·3	4·9
6	8·2	7·0	3	7·9		28	5·1	5·0
7	6·1	5·6	4	7·8		29	4·9	4·9
8	5·5	5·4	5	7·8		30	4·8	4·6
9	5·3	6·8	6	7·7		31 March	4·6	4·5
10	6·8	7·4	7	7·6		1 April	4·5	4·5
11	6·5	8·1	8	7·6	7·7	2	4·5	4·2
12	7·2	7·5	9	7·8	7·9	3	4·3	3·9
13	7·7	8·1	10	7·8	7·7	4	4·3	4·1
14	7·8	7·9	11	7·7	7·5	5	4·2	4·1
15	7·2	8·1	12	7·6	7·5	6	4·0	3·8
16	8·6	9·3	13	7·4	7·6	7	3·9	3·9
17	9·6	10·0	14	7·1	7·0	8	3·9	3·9
18	10·1	10·5	15	7·1	6·9	9	3·9	3·7
19	10·4	10·7	16	7·0	6·9	10	3·8	3·9
20	10·8	11·2	17	6·4	6·4	11	3·9	3·8
21	11·2	11·4	18	6·3	6·2	12	3·6	3·5
22	11·4	12·0	19	6·0	5·9	13	3·4	3·5
23 February	11·3	10·9	20 March	5·7		14 April	3·8	3·7

Temperatures in °C. below the freezing point of sea-water ($-1{\cdot}65$°C.).
The eight single values are estimated daily mean temperatures only.

6. CONCLUSION

The two examples given in Sections 4 and 5 show to what degree the thermal properties of an inhomogeneous ice cover can be accounted for by relatively simple techniques. It should be stressed that this discussion has taken no account of the fact that the heat content of upper and lower ice layers respond at different times to a change in surface temperature. Because of this, the modified Stefan equation (5.1) is more accurate when covering longer periods of time. Although it would be possible to calculate the time distribution of heat energy from different depths, as distinct from that of the actual freezing process, more accurate information on the temperature and specific-heat profiles than would normally be available, or calculable, would be required. Given such information there would be ample justification for the use of an analogue computer. In many practical cases however the modified Stefan equation (5.1) will provide an adequate answer.

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