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NOTE ON U-CLOSED SEMIGROUP RINGS

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Let D be an integral domain with quotient field K. If $\alpha^2 - \alpha \in D$ and $\alpha^3 - \alpha^2 \in D$ imply $\alpha \in D$ for all elements α of K, then D is called a u-closed domain. A submonoid S of a torsion-free Abelian group is called a grading monoid. We consider the semigroup ring $D[S]$ of a grading monoid S over a domain D. The main aim of this note is to determine conditions for $D[S]$ to be u-closed. We shall show the following Theorem: $D[S]$ is u-closed if and only if D is u-closed.

Let D be a domain with quotient field K. (The quotient field is denoted by q( ).) Let n be a natural number. If $\alpha^n \in D$ implies $\alpha \in D$ for all $\alpha \in K$, then D is called n-root closed. If D is n-root closed for every natural number n, then D is called root closed. If $\alpha^2 \in D$ and $\alpha^3 \in D$ imply $\alpha \in D$ for all $\alpha \in K$, then D is called seminormal. If $\alpha^2 - \alpha \in D$ and $\alpha^3 - \alpha^2 \in D$ imply $\alpha \in D$ for all $\alpha \in K$, then D is called u-closed. If $\alpha^2 - a\alpha \in D$ and $\alpha^3 - a\alpha^2 \in D$ imply $\alpha \in D$ for all $a \in D$ and $\alpha \in K$, then D is called t-closed.

A submonoid S of a torsion-free Abelian (additive) group is called a grading monoid (or a g-monoid). We consider the semigroup ring $D[S]$ of S over D. D.F. Anderson determined both conditions for $D[S]$ to be n-root closed and conditions for $D[S]$ to be root closed [1]. D.D. Anderson and D.F. Anderson determined conditions for $D[S]$ to be seminormal [2]. Throughout the paper, S denotes a g-monoid. The main aim of this note is to determine conditions for $D[S]$ to be u-closed.

Let G be the quotient group of S, that is, $G = \{s_1 - s_2 \mid s_1, s_2 \in S\}$. (The quotient group is denoted by q( ).) Let n be a natural number. If $n\alpha \in S$ implies $\alpha \in S$ for all $\alpha \in G$, then S is called n-root closed. If S is n-root closed for every n, then S is called integrally closed. Let T be a g-monoid with submonoid S. For $\alpha \in T$, if $n\alpha \in S$ for some natural number n, then α is called integral over S. The set of integral elements of T over S is called the integral closure of S in T. The integral closure $\overline{S}$ of S in q(S) is called the integral closure of S. S is integrally closed if and only if $\overline{S} = S$.

THEOREM 1. [1]

(1) *$D[S]$ is n-root closed if and only if D is n-root closed and S is n-root closed.*

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(2) *$D[S]$ is root closed if and only if D is root closed and S is integrally closed.*

If $2\alpha \in S$ and $3\alpha \in S$ imply $\alpha \in S$ for all $\alpha \in \mathrm{q}(S)$, then S is called seminormal.

THEOREM 2. [2] *$D[S]$ is seminormal if and only if D is seminormal and S is seminormal.*

LEMMA 1.

(1) *If $D[S]$ is t-closed, then D is t-closed.*

(2) *If $D[S]$ is u-closed, then D is u-closed.*

The proof of Lemma 1 is straightforward.

PROPOSITION 1. (See [5]).

(1) *If D is t-closed, then $D[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is t-closed, where $X_1, \ldots, X_n$ are indeterminates.*

(2) *If D is u-closed, then $D[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is u-closed.*

PROPOSITION 2. *Let G be a torsion-free Abelian group. Then $D[G]$ is t-closed if and only if D is t-closed.*

PROOF: The necessity follows from Lemma 1.

The sufficiency: Assume that $F^2 - fF \in D[G]$ and $F^3 - fF^2 \in D[G]$ for elements $f \in D[G]$ and $F \in \mathrm{q}\big(D[G]\big)$. There exists a finitely generated subgroup H of G such that $f \in D[H]$, $F \in \mathrm{q}\big(D[H]\big)$, $F^2 - fF \in D[H]$ and $F^3 - fF^2 \in D[H]$. $D[H]$ is isomorphic to $D[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ for some n. By Proposition 1, we have $F \in D[H]$. ∎

PROPOSITION 3. *Let G be a torsion-free Abelian group. Then $D[G]$ is u-closed if and only if D is u-closed.*

The proof of Proposition 3 is similar to that of Proposition 2.

The torsion-free rank of $\mathrm{q}(S)$ is denoted by $\mathrm{tfr}\,(S)$.

LEMMA 2. *If $\mathrm{tfr}\,(S) < \infty$, then S is isomorphic to a submonoid of the g-monoid $\mathbf{R}$ of real numbers.*

PROOF: Let $G = \mathrm{q}(S)$, and let $\alpha_1, \ldots, \alpha_n$ be a maximal independent subset of G. Let $\beta_1, \ldots, \beta_n$ be a set of real numbers which is linearly independent over $\mathbf{Z}$. Let $\alpha \in G$. Then we have $n\alpha = \sum n_i\alpha_i$ for integers n and n_i with $n \neq 0$. We set $\sigma(\alpha) = (1/n)\sum n_i\beta_i$. Then σ is an isomorphism of G into $\mathbf{R}$. ∎

EXAMPLE 1. [4, Proposition 3.4] Let k be a field and S a submonoid of the rational numbers $\mathbf{Q}$. If $k[S]$ is seminormal, then $k[S]$ is integrally closed.

LEMMA 3. *For a domain D and a g-monoid S, the following conditions are equivalent:*

(1) *$D[S]$ is t-closed.*

(2) *For every finitely generated submonoid S_0 of S and the integral closure T of S_0 in S, $D[T]$ is t-closed.*

PROOF: (1) $\Longrightarrow$ (2): For elements $F \in \mathrm{q}\big(D[T]\big)$ and $f \in D[T]$, assume that $F^2 - fF \in D[T]$ and $F^3 - fF^2 \in D[T]$. By the assumption, we have $F \in D[S]$. The integral closure of $D[T]$ is $\overline{D}[\overline{T}]$, where $\overline{D}$ is the integral closure of D and $\overline{T}$ is the integral closure of T [**3**, Corollary 12.11]. It follows that $F \in D[T]$.

(2) $\Longrightarrow$ (1): Assume that $F^2 - fF \in D[S]$ and $F^3 - fF^2 \in D[S]$ for elements $f \in D[S]$ and $F \in \mathrm{q}\big(D[S]\big)$. There exists a finitely generated submonoid S_0 of S such that $f \in D[S_0]$, $F \in \mathrm{q}\big(D[S_0]\big)$, $F^2 - fF \in D[S_0]$ and $F^3 - fF^2 \in D[S_0]$. By the assumption, we have $F \in D[T]$ for the integral closure T of S_0 in S. ∎

Lemmas 2, 3 and Proposition 2 imply that conditions for $D[S]$ to be t-closed reduce to the case where D is a field, S is a submonoid of $\mathbf{R}$ with $S \subsetneqq \mathrm{q}(S)$ and $\mathrm{tfr}\,(S) < \infty$.

LEMMA 4. *For a domain D and a g-monoid S, the following conditions are equivalent:*

(1) *$D[S]$ is u-closed.*

(2) *For every finitely generated submonoid S_0 of S and the integral closure T of S_0 in S, $D[T]$ is u-closed.*

The proof of Lemma 4 is similar to that of Lemma 3.

Lemma 4 and Proposition 3 reduce conditions for $D[S]$ to be u-closed to the case where S is a submonoid of $\mathbf{R}$ with $S \subsetneqq \mathrm{q}(S)$ and $\mathrm{tfr}\,(S) < \infty$.

EXAMPLE 2. Assume that $\mathrm{tfr}\,(S) = 1$. Then $D[S]$ is t-closed if and only if D is t-closed and S is isomorphic to an integrally closed submonoid of $\mathbf{Q}$.

PROOF: By Lemma 2, we may assume that S is a submonoid of $\mathbf{Q}$.

The necessity: We see that $K[S]$ is t-closed, and hence seminormal, where $K = \mathrm{q}(D)$. By Example 1, $K[S]$ is integrally closed. Hence S is integrally closed.

The sufficiency: Then $K[S]$ is integrally closed. Assume that $F^2 - fF \in D[S]$ and $F^3 - fF^2 \in D[S]$ for elements $f \in D[S]$ and $F \in \mathrm{q}\big(D[S]\big)$. Then we have $F \in K[S]$. On the other hand, by Proposition 2, we have $F \in D[G]$ for $G = \mathrm{q}(S)$. Hence $F \in D[S]$. ∎

An element of $D[S]$ is denoted by $\sum_{finite} a_s X^s$ for elements $a_s \in D$ and $s \in S$ with X a symbol.

LEMMA 5. *Assume that D is u-closed with characteristic $p > 0$, and S a submonoid of $\mathbf{R}$. Then $D[S]$ is u-closed.*

PROOF: Assume that $F^2 - F \in D[S]$ and $F^3 - F^2 \in D[S]$ for some element $F \in \mathrm{q}\big(D[S]\big)$. We must show that $F \in D[S]$. We may assume that $F \neq 0$. By Proposition 3, we have $F \in D[G]$, where $G = \mathrm{q}(S)$. Put $F = a_1 X^{\alpha_1} + \cdots + a_l X^{\alpha_l}$, where each a_i is non-zero and $\alpha_1 < \cdots < \alpha_l$. Suppose that $\alpha_k \notin S$ for some k. For every natural number n larger than 1, we have $F^n - F \in D[S]$. Specifically, we have $F^{p^i} - F \in D[S]$ for every natural number i. Since $\alpha_k \notin S$, the coefficient of X^{α_k} in $F^{p^i} - F$ is zero. It follows that $\alpha_k = p^i \alpha_{l(i)}$ for some number $l(i)$. There exist natural numbers $i < j$ such that

$\alpha_{l(i)} = \alpha_{l(j)}$. Then we have $(p^j - p^i)\alpha_{l(i)} = 0$, and hence $\alpha_{l(i)} = 0$. It follows that $\alpha_k = 0$, and hence $\alpha_k \in S$, a contradiction. □

LEMMA 6. *Assume that D is of characteristic 0, p a prime number and $X_1, \ldots, X_l$ indeterminates. Let M be a monomial appearing in $(X_1 + \cdots + X_l)^p$. Then M is either of the form X_i^p for some i or $cX_{l(1)}^{e_1} \ldots X_{l(n)}^{e_n}$, where $n > 1$, each e_i is a natural number, c is a multiple of p and $l(i) \neq l(j)$ for $i \neq j$.*

The proof of Lemma 6 is elementary. The case of $l = 2$: we have $(X_1 + X_2)^p = \sum_i {}_pC_i X_1^{p-i} X_2^i$. And, if $1 < i < p$, then ${}_pC_i$ is a multiple of p.

LEMMA 7. *Assume that D is u-closed with characteristic 0. Then $D[S]$ is u-closed.*

PROOF: Assume that $F^2 - F \in D[S]$ and $F^3 - F^2 \in D[S]$ for some element $F \in \mathrm{q}\big(D[S]\big)$. We must show that $F \in D[S]$. We may assume that $F \neq 0$. By Proposition 3, we have $F \in D[G]$, where $G = \mathrm{q}(S)$. Put $F = a_1X^{\alpha_1} + \cdots + a_lX^{\alpha_l}$, where each a_i is non-zero and $\alpha_i \neq \alpha_j$ for $i \neq j$. Suppose that $\alpha_k \notin S$ for some k. Suppose that there exists an infinite number of prime numbers p such that $\alpha_k = p\alpha_{l(p)}$ for some number $l(p)$. Then there are prime numbers $p < q$ such that $l(p) = l(q)$. It follows that $\alpha_k = 0$, and hence $\alpha_k \in S$, a contradiction.

We may assume that $D \supset \mathbf{Z}$. We may assume that $a_1, \ldots, a_m$ is a transcendence basis of the field $\mathbf{Q}(a_1, \ldots, a_l)$ over $\mathbf{Q}$. There exists an element $\theta \in \mathbf{Q}(a_1, \ldots, a_l)$ which is integral over $\mathbf{Z}[a_1, \ldots, a_m]$ such that $\mathbf{Q}(a_1, \ldots, a_l) = \big(\mathbf{Q}(a_1, \ldots, a_m)\big)(\theta)$. Let n be the degree of θ over $\mathbf{Q}(a_1, \ldots, a_m)$. Then, there exists a non-zero element f of $\mathbf{Z}[a_1, \ldots, a_m]$ so that each a_i is of the form

$$\frac{f_{i0}}{f} + \frac{f_{i1}}{f}\theta + \cdots + \frac{f_{i,n-1}}{f}\theta^{n-1},$$

where every f_{ij} is an element of $\mathbf{Z}[a_1, \ldots, a_m]$. Therefore each element of $\mathbf{Z}[a_1, \ldots, a_l]$ is of the form

$$\frac{f_0}{f^d} + \frac{f_1}{f^d}\theta + \cdots + \frac{f_{n-1}}{f^d}\theta^{n-1}$$

with $f_i \in \mathbf{Z}[a_1, \ldots, a_m]$ and a natural number d. There exists a number M so that if p is a prime number larger than M, then $\alpha_k \neq p\alpha_i$ for each i and $f \notin p\mathbf{Z}[a_1, \ldots, a_m]$. Let p be a prime number larger than M. By Lemma 6, the coefficient of X^{α_k} in F^p is of the form pa for some element $a \in \mathbf{Z}[a_1, \ldots, a_l]$. Since $F^p - F \in D[S]$, we have $pa = a_k$.

Let $p_1 < p_2 < \cdots$ be the prime numbers larger than M. We show that there exists an element $b(n) \in \mathbf{Z}[a_1, \ldots, a_l]$ such that $p_1p_2 \ldots p_nb(n) = a_k$ for every n. For the proof, we rely on induction on n. Thus suppose that there exists an element $b(n)$ of $\mathbf{Z}[a_1, \ldots, a_l]$ such that $p_1p_2 \ldots p_nb(n) = a_k$. There exist integers l_1 and l_2 such that $l_1p_1p_2 \ldots p_n + l_2p_{n+1} = 1$. Then we have $l_1a_k + l_2p_{n+1}b(n) = l_1p_1p_2 \ldots p_nb(n) + l_2p_{n+1}b(n) = b(n)$. Since $p_{n+1}a' = a_k$ for some element a' of $\mathbf{Z}[a_1, \ldots, a_l]$, it follows that $p_{n+1}\big(l_1a' + l_2b(n)\big) = l_1p_{n+1}a' + l_2p_{n+1}b(n) = b(n)$.

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Therefore $p_1p_2\dots p_{n+1}\big(l_1a' + l_2b(n)\big) = a_k$.

We have an increasing chain of principal ideals of $\mathbf{Z}[a_1,\dots,a_l]$: $\big(b(1)\big) \subset \big(b(2)\big) \subset \cdots$. Since $\mathbf{Z}[a_1,\dots,a_l]$ is a Noetherian ring, we have $\big(b(h)\big) = \big(b(h+1)\big)$ for some h. Then it follows that

$$\frac{1}{p_{h+1}} \in \mathbf{Z}[a_1,\dots,a_l].$$

Hence we have

$$\frac{1}{p_{h+1}} = \frac{f_0}{f^d} + \frac{f_1}{f^d}\theta + \dots + \frac{f_{n-1}}{f^d}\theta^{n-1}$$

with $f_i \in \mathbf{Z}[a_1,\dots,a_m]$ and a natural number d. It follows that $f \in p_{h+1}\mathbf{Z}[a_1,\dots,a_m]$, a contradiction. ∎

Theorem 3. *$D[S]$ is u-closed if and only if D is u-closed.*

Proof: The necessity follows from Lemma 1.

The sufficiency: We may assume that S is a submonoid of $\mathbf{R}$. If D is of charatcteristic $p > 0$, Lemma 5 implies that $D[S]$ is u-closed. If D is of characteristic 0, Lemma 7 implies that $D[S]$ is u-closed. ∎

Question [4] What are conditions for $D[S]$ to be t-closed?

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