Proceedings of the Edinburgh Mathematical Society (1980), 23, 61–67 ©

CENTRES FOR NEAR-RINGS: APPLICATIONS TO COMMUTATIVITY THEOREMS

by HOWARD E. BELL
(Received 15th September 1978)

1. Introduction

Let R be an arbitrary near-ring and define the multiplicative centre $Z(R)$ by

$$Z(R)=\{a\in R \mid ax=xa \text{ for all } x\in R\}.$$

In previous papers (**2, 3, 5**) we have established additive or multiplicative commutativity for various near-rings R in which selected elements were restricted to lie in $Z(R)$; the near-rings involved were usually distributively-generated (d-g) and were frequently assumed to have a multiplicative identity element as well.

In this paper we first prove a commutativity theorem involving $Z(R)$, without the assumption that R is d-g. We then introduce two other notions of centre, incorporating additive and multiplicative commutativity simultaneously, and use these in the formulation of commutativity theorems. Some of our results are for d-g near-rings, others for more general classes.

2. Definitions and terminology

Basic near-ring definitions are as in (**3**); in particular, we assume *left* distributivity, so that $x0=0$ for all $x\in R$. If $0x=0$ for all $x\in R$, we call R *zero-symmetric*; if $ab=0$ implies $ba=0$, we call R *zero-commutative*. The near-ring R will be called *periodic* if for each $x\in R$, there exist distinct positive integers $m=m(x)$ and $n=n(x)$ for which $x^m=x^n$.

As above, we denote the multiplicative centre by $Z(R)$, or simply Z. The additive group of R will be denoted by $(R,+)$ and its centre by $\S(R)$. The set of nilpotent elements of R will be written as N or $N(R)$, the set of distributive elements of R as D or $D(R)$; and for arbitrary subsets S of R, the right and two-sided annihilators of S will be written as $A_r(S)$ and $A(S)$. For arbitrary $x, y\in R$, the additive and multiplicative commutators $x+y-x-y$ and $xy-yx$ will be denoted, respectively, by (x, y) and $[x, y]$.

If R has 1, then the symbol n will denote both a positive integer and the near-ring element obtained by adding 1 the indicated number of times; in particular, for $x\in R$, xn is the n-th power of x in $(R,+)$. Even if R does not have 1, the symbol xn will have the same meaning.

3. An additive commutativity theorem for periodic near-rings

Theorem 1. *Let R be a periodic near-ring with multiplicative identity* 1, *and suppose that $N(R) \subseteq Z(R)$. Then $(R, +)$ is abelian.*

Proof. Note first that R is zero-symmetric—a fact we use without explicit mention. The proof of Lemma 1 of (**5**) shows that N is a normal subgroup of $(R, +)$, and that $RN \subseteq N$; we now wish to show that $(x+u)y - xy \in N$ for all $x, y \in R$ and all $u \in N$, so that N is an ideal. For such x, y and u, let $v = (x+u)y - xy$; and recall that $A_r(x)$ is an ideal for arbitrary $x \in Z(R)$. Now if $u^n = 0$, $u^{n-1}v = 0$; and since $u^{n-1} \in Z(R)$, we have $vu^{n-1} = 0$, hence $u \in A_r(vu^{n-2})$ and $vu^{n-2}v = 0 = v^2u^{n-2}$. Repeating the argument finitely many times ultimately yields $v^n = 0$, so our argument that N is an ideal is complete.

Since we wish to use the subdirect-sum structure theory, we need to know that homomorphic images of R inherit the hypothesis that nilpotent elements are multiplicatively central. To show this, note that $\bar{R} = R/N$ has no non-zero nilpotent elements and hence is zero-commutative; therefore, if $x \in \bar{R}$ and if we choose m, n such that $n > m$ and $x^n = x^m$, we get $0 = xx^{m-2}(x^{n-m+1} - x) = x^{n-m+1}x^{m-2}(x^{n-m+1} - x) = x^{m-2}(x^{n-m+1} - x)x = x^{m-2}(x^{n-m+1} - x)x^{n-m+1} = x^{m-2}(x^{n-m+1} - x)^2$. An obvious repetition shows that $\bar{R}$ has the $x^n = x$ property, hence for each x in R there exist arbitrarily large n for which $x - x^n \in N$. We can now carry out the proof of Lemma 1(d) in (**4**) to show that if $S = R/I$ is any homomorphic image of R, nilpotent elements of S are of form $u + I$ for u nilpotent in R, hence $N(S) \subseteq Z(S)$.

To prove our theorem, we now need consider only the case of subdirectly irreducible R. Moreover, since $1 + 1 \in Z(R)$ implies $(R, +)$ is abelian, we assume that $1 + 1 \notin N$. We begin by showing that 1 is the only non-zero idempotent of R. Note that since there exists $n > 1$ for which $x - x^n \in Z(R)$, R is zero-commutative (**3**, Lemma 3(A)); hence if e is a non-zero idempotent, $1 - e$ is an idempotent orthogonal to it. It is easy to show that $Re = A(1-e)$ and $R(1-e) = A(e)$, so that in particular Re and $R(1-e)$ are both ideals of R. Since their intersection is trivial, the subdirect irreducibility of R forces one of them to be trivial, hence $e = 1$.

Now every element of R has an idempotent power (**4**, Lemma 1(a)); thus, every non-nilpotent element of R is invertible and R/N is a near-field. Since $(R/N, +)$ is therefore abelian, additive commutators in R are nilpotent—a fact which permits a trivial modification of the proofs of Lemmas 4 and 5 of (**3**), yielding the result that distributive elements of R commute additively with each other.

Our next step is to show that if $b \in R$ and $b^2 = 1$, then $b = 1$ or $b = -1$. Since it is known that near-fields have this property (**8**, **9**), the fact that R/N is a near-field shows that $b - 1 \in N$ or $b + 1 \in N$; we may assume that not both of $b - 1$ and $b + 1$ are in N, for otherwise $1 + 1 = 1 + b - (-1 + b) \in N$. Suppose first that $b - 1 \in N$. Then $(b-1)(b+1) = (b-1)b + b - 1 = b(b-1) + b - 1 = b^2 - 1 = 0$; and since $b + 1$ is invertible, we get $b = 1$. Now consider the case $b + 1 \in N$. Note that $b + 1$ and 1, both being distributive, commute additively; therefore, b commutes additively with 1. It follows that $(b+1)(b-1) = (b+1)b - (b+1) = b(b+1) - (b+1) = b^2 + b - 1 - b = b^2 - 1 = 0$; and since $b - 1$ is invertible, we have $b = -1$.

We complete the proof by borrowing a computational trick from the end of (**9**).

Specifically, if h is any invertible element of R, then $b=(-h)h^{-1}\neq 1$ and $b^2=1$; hence $b=-1$ and hence h commutes multiplicatively with -1. Since nilpotent elements also commute with -1, we have $-1\in Z$ and hence $(R,+)$ is abelian.

4. The common centre

The *common centre* $Z_c(R)$ is defined to be $Z(R)\cap \S(R)$. It is a natural set to consider, but seemingly not so useful as the centre to be introduced in Section 5.

Theorem 2. (I) *Let R be a near-ring such that for each $x\in R$, there is an integer $n(x)>1$ for which $x-x^{n(x)}\in Z_c(R)$. Then the set N is an ideal of R.*

(II) *Suppose, moreover, that each homomorphic image of R without zero-divisors has a non-trivial distributive element. Then $(R,+)$ is nilpotent of class at most* 2.

Proof. (I) It is clear that $0^n=0$ for all $n>1$; and since $0-0^n\in Z_c(R)$ for some such n, we have $0\in Z_c(R)$ and hence R is zero-symmetric. The proof of Lemma 3(A) in (3) may therefore be carried over to show that R is zero-commutative. Referring again to (3), we obtain from Lemma 1 and the proof of Lemma 3(B) the result that N is an ideal.

(II) The near-ring $\bar{R}=R/N$ has no non-zero nilpotent elements, hence is a subdirect sum of homomorphic images $\bar{R}_\alpha$ with no non-zero divisors of zero (see (2), Lemma 3). Then, using the fact that distributive idempotents are multiplicatively central, we can adapt the procedure of (1), Section 3 to embed each $\bar{R}_\alpha$ in a near-field. Thus $(R/N,+)$ is abelian, so $(x,y)\in N$ for all $x,y\in R$. From the definition of $Z_c(R)$, we get a sequence $\langle n_1,n_2,\ldots\rangle$ of integers greater than 1, for which $x-x^{n_1}, x^{n_1}-x^{n_1n_2},\ldots$ are all in $\S(R)$; thus $N\subseteq \S(R)$, and hence all (x,y) belong to $\S(R)$.

The following theorem extends Theorem 1, and also the theorem of (5). Its proof—though not its statement—is contained in (5).

Theorem 3. *Let R be a periodic d-g near-ring with $N\subseteq Z_c(R)$. Then R is a commutative ring.*

5. The strong common centre

The *strong common centre*, which we shall denote by $Z_0(R)$, is defined to be

$$\{x\in Z(R)\mid \{x\}\cup xR\subseteq \S(R)\}.$$

One of its advantages is indicated by the following theorem.

Theorem 4. *If R is any d-g near-ring, $Z_0(R)$ is a commutative subring of R.*

Proof. Let $a,b\in Z_0(R)$; note that if t is distributive or anti-distributive, then $(a-b)t=at-bt$. Represent the arbitrary element $r\in R$ as $t_1+t_2+\cdots+t_k$, where each t_i is either distributive or anti-distributive.

Clearly $a-b\in\S(R)$; moreover $(a-b)r=(a-b)\Sigma t_i=\Sigma(a-b)t_i=\Sigma at_i-bt_i\in \S(R)$. Since each at_i and bt_i is in $\S(R)$, and since $a,b\in Z(R)$, this last sum can be re-written as $t_1a+t_2a+\cdots+t_ka+(-t_kb-t_{k-1}b-\cdots-t_1b)=(t_1+\cdots+t_k)a-(t_1+\cdots+t_k)b=r(a-b)$; and it has now been shown that $a-b\in Z_0(R)$. Since it is immediate from the definition that $ab\in Z_0(R)$, and since multiplicatively commutative near-rings are distributive, our proof is complete.

Theorem 5. *Let R be a d-g near-ring and $n>1$ a fixed positive integer. If $x-x^n\in Z_0(R)$ for all $x\in R$, then R is a commutative ring.*

Proof. One consequence of Theorem 4 is the existence of infinitely many positive integers n for which $x-x^n\in Z_0(R)$; thus $N\subseteq Z_0(R)$. By the proof of Theorem 2, additive commutators are in N; therefore each element commutes additively with its conjugates, and we easily obtain

$$x-y-x+y=-x+y+x-y \text{ for all } x,y\in R.$$

Multiplying this equation on the left by an arbitrary $d\in D(R)$, and then making the substitution $x=r$, $y=-s$ for elements $r,s\in D(R)$, we get $(d2)(r,s)=0$ for all $d,r,s\in D(R)$; and utilising the fact that $(r,s)\in Z(R)$ now gives

$$d((r,s)2)=0 \text{ for all } d,r,s\in D(R).$$

Since R is d-g, this translates as

$$(r,s)2\in A(R) \text{ for all } r,s\in D(R). \tag{1}$$

Let d be an arbitrary element of $D(R)$. Since $d-d^n$ and $d2-(d2)^n$ are both in $Z_0(R)$, Theorem 4 implies that $d^n(2^n-2)=d^n(2j)\in Z_0(R)$, hence is multiplicatively central. Thus,

$$(x+y)(d^n(2j))=x(d^n(2j))+y(d^n(2j)) \text{ for all } x,y\in R.$$

Using distributivity of d^n, we get

$$((x+y)(2j)-y(2j)-x(2j))d^n=0. \tag{2}$$

Let R_1 be the factor near-ring $R/A(d^n)$; then R_1 is a near-ring inheriting all the original hypotheses on R. Let $\bar{D}(R_1)$ be the set of distributive elements of R_1 which are images of elements of $D(R)$ under the canonical homomorphism. In view of (1) and (2), R_1 has the properties

$$(x+y)(2j)=x(2j)+y(2j)=0 \text{ for all } x,y\in R_1 \tag{3}$$

and

$$(r,s)2=0 \text{ for all } r,s\in\bar{D}(R_1). \tag{4}$$

It follows from (4) and the fact that additive commutators are in §(R_1) that

$$(r,s2)=0 \text{ for all } r,s\in\bar{D}(R_1)., \tag{5}$$

and that

$$(r+s)2=(s+r)2 \text{ for all } r,s\in\bar{D}(R_1). \tag{6}$$

From (3) we have

$$(r+s)(2j)=r(2j)+s(2j) \text{ for all } r,s\in\bar{D}(R_1); \tag{7}$$

https://doi.org/10.1017/S0013091500003618 Published online by Cambridge University Press

applying (5) and (6) and some additive cancellation yields

$$(s+r)(2j-1)=r(2j-1)+s(2j-1),$$
$$(s+r)(2j-3)+(r+s)2=r(2j-2)+s(2j-2)+r+s,$$
$$(s+r)(2j-4)+s+r+r+s=r(2j-2)+s(2j-2),$$
$$(r+s)(2j-4)=r(2j-4)+s(2j-4) \text{ for all } r,s\in\bar{D}(R_1). \tag{8}$$

Now (8) has the same form as (7), hence by repeating the argument and noting that $j=(2^n-2)/2$ was odd, we ultimately get $(r+s)2=r2+s2$. Thus, elements of $\bar{D}(R_1)$ commute and $(R_1,+)$ is abelian.

Returning to the original near-ring R, we now have $(x,y)d^n=0$ for all $x,y\in R$ and all $d\in D(R)$; and since $d-d^n\in Z_0(R)$, the definition of $Z_0(R)$ shows directly that $(x,y)(d-d^n)=0$ as well. Thus $(x,y)d=0$ for all $x,y\in R$ and $d\in D(R)$; and since R is distributively-generated, we have

$$(x,y)\in A(R) \text{ for all } x,y\in R. \tag{9}$$

Otherwise expressed, (9) states that $(xR,+)$ is abelian for each $x\in R$; and we shall use this result to show that

$$(x-x^n)y=xy-x^ny \text{ for all } x,y\in R. \tag{10}$$

Specifically, let $y=s_1+s_2+\cdots+s_k$, where each s_i is either distributive or anti-distributive; then, since $(-x^n)s_i=-x^ns_i=x(-x^{n-1}s_i)\in xR$ for each $i=1,\ldots,k$, we get $(x-x^n)y=(x-x^n)\Sigma s_i=\Sigma(x-x^n)s_i=\Sigma(xs_i+(-x^n)s_i)=x\Sigma s_i+x^n(-s_n-s_{n-1}-\cdots-s_1)$ $=xy+x^n(-y)=xy-x^ny$, and (10) is proved.

Before proceeding, we recall Fröhlich's classical theorem (6) that a distributively-generated near-ring R is distributive if and only if $(R^2,+)$ is abelian. Since $(R/A(R),+)$ is abelian by (9), $R/A(R)$ is therefore a ring, which is multiplicatively commutative by a well-known theorem of Herstein (7); thus

$$w[x,y]=[x,y]w=0 \text{ for all } x,y,w\in R. \tag{11}$$

It follows, in particular, that

$$x^2y=xyx \text{ for all } x,y\in R. \tag{12}$$

We now write $x=\Sigma s_i$ for appropriate distributive and anti-distributive elements s_i, and write $yx=[y,x]+xy$. Note that $[y,x]\in N$ by (11), and recall that $N\subseteq Z_0(R)$. Thus, using (11) we obtain $yx^2=([y,x]+xy)x=\Sigma([y,x]+xy)s_i=\Sigma xys_i$—that is,

$$yx^2=xyx \text{ for all } x,y\in R. \tag{13}$$

It follows from (12) and (13) that

$$x^ny=yx^n \text{ for all } x,y\in R. \tag{14}$$

From $x-x^n\in Z_0(R)$, we get $(x-x^n)y=y(x-x^n)$; and (10) can be invoked to yield $xy-x^ny=yx-yx^n$. Applying (14) now yields multiplicative commutativity of R, hence distributivity as well; and Fröhlich's theorem shows that $(R^2,+)$ is abelian. Thus, for each $x\in R$ both $x-x^n$ and x^n commute additively with R^2, hence so does x.

Therefore $R^2 \subseteq \S(R)$; and since $x - x^n \in \S(R)$ for each $x \in R$, we see that $\S(R) = R$. This completes the proof.

A natural conjecture is that the restriction to fixed n in the hypotheses of Theorem 5 can be dropped. The following theorem is a step in that direction.

Theorem 6. *Let R be a d-g near-ring in which $(R, +)$ is a torsion group; and suppose that for each $x \in R$, there exists an integer $n(x) > 1$ for which $x - x^{n(x)} \in Z_0(R)$. Then R is a commutative ring.*

Proof. Let $d \in D(R)$ and let $dk = 0$, where $k = 2^q j$ and j is odd; we assume without loss that $q \geq 1$. Then $d(2j) \in N \subseteq Z_0(R)$; and beginning just before equation (2), we may simply repeat the remainder of the proof of Theorem 5, with obvious trivial modifications.

Experience to date would suggest the following conjecture: if R is an arbitrary near-ring with 1, and if for each $x \in R$ there is an integer $n(x) > 1$ for which $x - x^{n(x)} \in Z_0(R)$, then $(R, +)$ is abelian. The following theorem—the final one in this paper—is the best we have been able to achieve in this direction.

Theorem 7. *Let n be a positive even integer and R a near-ring with 1 such that $x - x^n \in Z_0$ for each $x \in R$. Then $(R, +)$ is abelian.*

Proof. Taking $x = 2$ and -2 in turn shows that

$$-2^n - 2 \in Z_0 \text{ and } 2 - 2^n \in Z_0; \tag{15}$$

using the fact that each of these is multiplicatively central gives

$$x(2^n + 2) + y(2^n + 2) = (y + x)(2^n + 2) \text{ for all } x, y \in R \tag{16}$$

and

$$x(2^n - 2) + y(2^n - 2) = (y + x)(2^n - 2) \text{ for all } x, y \in R. \tag{17}$$

Now (15) shows that $x(2^n - 2)$ and $x(2^n + 2) \in \S(R)$ for each $x \in R$, hence

$$x(4) \in \S(R) \text{ for all } x \in R. \tag{18}$$

Combining (16), (17) and (18) yields

$$x(4) + y(4) = (y + x)(4) \text{ for all } x, y \in R. \tag{19}$$

By repeating the above argument for 3 and -3 we get $x(6) \in \S(R)$; hence, in view of (18) we have

$$x(2) \in \S(R) \text{ for all } x \in R. \tag{20}$$

It now follows from (16), (19) and (20) that $x(2) + y(2) = (x + y)(2)$ for all $x, y \in R$—that is, $(R, +)$ is abelian.

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MATHEMATICS DEPARTMENT
BROCK UNIVERSITY
ST. CATHARINES, ONTARIO, CANADA