NOTE ON GENERALIZED WITT ALGEBRAS

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Introduction. Throughout this note K will denote a field of characteristic $p > 0$. Let I be the set $\{1, 2, \ldots, m\}$, and $\mathfrak{G}$ a finite additive group of functions on I with values in K. We assume that $\mathfrak{G}$ is total in the sense that, for any $\lambda_1, \ldots, \lambda_m$ in K, $\sum_{i=1}^{m}\lambda_i\sigma(i) = 0$ for all σ in G implies all $\lambda_i = 0$. It is clear that $\mathfrak{G}$ is an elementary p-group. Let p^n be the order of $\mathfrak{G}$. A generalized Witt algebra $\mathfrak{L}$ is defined as an algebra over K with basis elements $\{e(\sigma, i) | \sigma \in \mathfrak{G}, i \in I\}$ and the multiplication table

$$e(\sigma, i)e(\tau, j) = \tau(i)e(\sigma + \tau, j) - \sigma(j)e(\sigma + \tau, i). \tag{0.0.1}$$

$\mathfrak{L}$ is a simple Lie algebra except when $p = 2$, $m = 1$.

In the first section of this note we shall prove that the outer derivation algebra of a generalized Witt algebra is abelian, assuming that K is infinite. We shall see that actually a result of Jacobson **(3)** is generalized.

It was shown in **(5)** that any generalized Witt algebra $\mathfrak{L}$ can be reformulated as follows: Let $\mathfrak{A}$ be a commutative associative algebra over K with a unity element, and $D_1, \ldots, D_m$ be derivations of $\mathfrak{A}$ such that:

(1) $[D_i, D_j] = D_iD_j - D_jD_i = 0$ for all i and j;

(2) If $f \in \mathfrak{A}$ and $\lambda_1, \ldots, \lambda_k$ in K are such that $D_if = \lambda_if$ for all i then $f = 0$ or f is a unit in $\mathfrak{A}$;

(3) $\sum_{i=1}^{m}f_iD_i = 0$, where $f_i \in \mathfrak{A}$, implies $f_i = 0$ for all i.

Now any generalized Witt algebra can be regarded as the subalgebra $\mathfrak{L}(\mathfrak{A}; D_1, \ldots, D_m)$ of the derivation algebra of $\mathfrak{A}$ consisting of all derivations of the form $f_1D_1 + \ldots + f_mD_m$. In the second section of this note we shall consider $\mathfrak{L}(\mathfrak{A}; D_1, \ldots, D_m)$ under the conditions (1) and (2) above only, and extend some results proved in **(5)**.

1. The derivation algebra of a generalized Witt algebra.

We prove the following

THEOREM 1.1. *Let $\mathfrak{L}$ be a generalized Witt algebra over an infinite field K of characteristic $p > 2$. Let $\{e(\sigma, i) | \sigma \in G, i \in I\}$ be a basis of $\mathfrak{L}$. Then any derivation of $\mathfrak{L}$ is the sum of an inner derivation and a derivation δ_1 given by*

$$\delta_1(e(\sigma, i)) = \phi(\sigma)e(\sigma, i) \tag{1.1.1}$$

where ϕ is a linear map of $\mathfrak{G}$ into K.

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Proof. First of all we show that we may assume (1.1.2): for any i, $1 \leqslant i \leqslant m$, $\sigma(i) = 0$ implies $\sigma = 0$. Suppose (1.1.2) is not satisfied. Since K is infinite and $\mathfrak{G}$ total, we may proceed as in the proof of Lemma 9.1 of **(5**, p. **533)** to obtain an $m \times m$ non-singular matrix (β_{ij}) such that if we define $\sigma[i]$ by

$$\sigma[i] = \sum_{j=1}^{m} \beta_{ij}\,\sigma(j), \qquad (i = 1, \ldots, m),$$

then, for any i, $\sigma[i] = 0$ implies $\sigma = 0$. Define a new basis $\{e[\sigma, i] | \sigma \in \mathfrak{G}, i \in I\}$ of $\mathfrak{L}$ by

$$e[\sigma, i] = \sum_{j=1}^{m} \beta_{ij}\, e(\sigma, i).$$

Then by (0.0.1) we have

$$\begin{aligned} e[\sigma, i]\, e[\tau, j] &= \sum_{s,t} \beta_{is}\beta_{jt}\, e(\sigma, s)e(\tau, t) \\ &= \sum_{s,t} \beta_{is}\beta_{jt}\, (\tau(s)e(\sigma + \tau, t) - \sigma(t)e(\sigma + \tau, s)) \\ &= \tau[s]\, e[\sigma + \tau, t] - \sigma[t]\, e[\sigma + \tau, s]. \end{aligned}$$

Thus $\{e[\sigma, i]\}$ satisfies the same multiplication table as $\{e(\sigma, i)\}$ with $\sigma(i)$ replaced by $\sigma[i]$. But here $\sigma[i] = 0$ implies $\sigma = 0$. Suppose that the given derivation is the sum of an inner derivation and a derivation δ_1 given by $\delta_1(e[\sigma, i]) = \phi(\sigma)e[\sigma, i]$, where ϕ is an additive map of $\mathfrak{G}$ into K. Then clearly we have $\delta_1(e(\sigma, i)) = \phi(\sigma)e(\sigma, i)$ also. This shows that we can assume (1.1.2) from the beginning.

Now let δ be the given derivation, and let

$$\delta(e(\sigma, i)) = \sum_{\tau, j} \gamma(\sigma, i; \tau, j)e(\sigma + \tau, j)$$

with coefficients $\gamma(\sigma, i; \tau, j)$ in K. Then from

$$\delta(e(0, 1))e(\sigma, i) + e(0, 1)\delta(e(\sigma, i)) = \sigma(1)\delta(e(\sigma, i))$$

we obtain

$$\gamma(\sigma, i; \tau, j) = \gamma(0, 1; \tau, j)\tau(i)\tau(1)^{-1} \tag{1.1.3}$$

for $i \neq j$ and $\tau \neq 0$, and

$$\sum_{j} \gamma(0, 1; \tau, j)\sigma(j) + \gamma(\sigma, i; \tau, i)\tau(1) = \gamma(0, 1; \tau, i)\tau(i). \tag{1.1.4}$$

By (1.1.3) and (1.1.4) we see easily that

$$\begin{aligned} \delta(e(\sigma, i)) &= \sum_{j} \gamma(\sigma, i; 0, j)e(\sigma, j) \\ &\quad + e(\sigma, i) \sum_{\tau \neq 0} \sum_{j} \gamma(0, 1; \tau, j)\tau(1)^{-1}e(\tau, j). \end{aligned}$$

Hence δ is the sum of an inner derivation and a derivation δ_1 of the form

$$\delta_1(e(\sigma, i)) = \sum_{j} \gamma(\sigma, i, j)e(\sigma, j) \tag{1.1.5}$$

with coefficients $\gamma(\sigma, i, j)$ in K.

We shall show that $\gamma(\sigma, i, j) = 0$ if $i \neq j$, that $\gamma(\sigma, 1, 1) = \ldots = \gamma(\sigma, m, m)$, and that $\gamma(\sigma, 1, 1)$ is additive with respect to σ. If $m = 1$, then the additivity of $\gamma(\sigma, 1, 1)$ follows immediately from

$$\delta_1(e(\sigma, 1))e(\tau, 1) + e(\sigma, 1)\delta_1(e(\tau, 1)) = \delta_1(e(\sigma, 1)e(\tau, 1)).$$

Hence we shall assume that $m > 1$. Then from

$$\delta_1(e(\sigma, 1))e(\tau, j) + e(\sigma, i)\delta_1(e(\tau, j)) = \delta_1(e(\sigma, i)e(\tau, j))$$

we have, for $i \neq j$,

$$\gamma(\sigma, i, j)\sigma(i) - \gamma(\tau, i, j)\tau(i) = \gamma(\sigma + \tau, i, j)(\sigma(i) - \tau(i)); \tag{1.1.6}$$

$$\sum_k \gamma(\sigma, i, k)\tau(k) = \gamma(\sigma, i, j)\sigma(j) - \gamma(\tau, j, j)\tau(i) + \gamma(\sigma + \tau, j, j)\tau(i) - \gamma(\sigma + \tau, i, j)\sigma(j). \tag{1.1.7}$$

Setting $\sigma = 0$ in (1.1.7) and using the fact that G is total, we have

$$\gamma(0, i, k) = 0 \tag{1.1.8}$$

for all i and k. Set $\tau = -\sigma$, in (1.1.6) and use (1.1.8). Then we have, for any σ and $i \neq j$,

$$\gamma(\sigma, i, j) + \gamma(-\sigma, i, j) = 0. \tag{1.1.9}$$

Replace τ in (1.1.6) by $-\tau$, and use (1.1.9). Then we have

$$\gamma(\sigma, i, j)\sigma(i) - \gamma(\tau, i, j)\tau(i) = \gamma(\sigma - \tau, i, j)(\sigma(i) + \tau(i)).$$

Combining this with (1.1.6) yields

$$\gamma(\sigma - \tau, i, j)(\sigma(i) + \tau(i)) = \gamma(\sigma + \tau, i, j)(\sigma(i) - \tau(i)). \tag{1.1.10}$$

Since $\mathfrak{G}$ is an elementary p-group and $p \neq 2$, $\sigma - \tau$ and $\sigma + \tau$ may be regarded as two arbitrary elements in $\mathfrak{G}$. Hence by (1.1.10) it follows that, for $i \neq j$;

$$\gamma(\sigma, i, j) = \alpha_{ij}\sigma(i), \tag{1.1.11}$$

where α_{ij} are in K and independent of σ. Substituting this in (1.1.7) we obtain

$$\gamma(\sigma, i, i)\tau(i) + \sum_{k \neq i} \alpha_{ik}\sigma(i)\tau(k) = \gamma(\sigma + \tau, j, j)\tau(i) - \gamma(\tau, j, j)\tau(i) - \alpha_{ij}\tau(i)\sigma(j), \tag{1.1.12}$$

which shows that $(\gamma(\sigma + \tau, j, j) - \gamma(\tau, j, j))\tau(i)$ is additive with respect to τ. Hence

$$\gamma(\sigma + \tau, j, j) - \gamma(\tau, j, j) = \gamma(\sigma - \tau, j, j) - \gamma(-\tau, j, j) \tag{1.1.13}$$

for all σ and τ. Let $\sigma = \tau$ in the above and use (1.1.8). Then

$$\gamma(2\tau, j, j) - \gamma(\tau, j, j) = -\gamma(-\tau, j, j). \tag{1.1.14}$$

By (1.1.13) and (1.1.4) we have

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$$\gamma(\sigma + \tau, j, j) = \gamma(\sigma - \tau, j, j) + \gamma(2\tau, j, j)$$

which shows that $\gamma(\sigma, j, j)$ is additive with regard to σ, since, as before, $\sigma + \tau$ and $\sigma - \tau$ can be regarded as two arbitrary elements in $\mathfrak{G}$. Now from (1.1.12) we obtain

$$\gamma(\sigma, i, i)\tau(i) + \sum_{k \neq i} \alpha_{ik}\sigma(i)\tau(k) = \gamma(\sigma, j, j)\tau(i) - \alpha_{ij}\tau(i)\sigma(j)$$

for all σ and τ. Using the fact that G is total, we see from the above that $\alpha_{ik} = 0$ for $k \neq i$ and that $\gamma(\sigma, i, i) = \gamma(\sigma, j, j)$ for any i and j. Set $\gamma(\sigma, i, i) = \phi(\sigma)$. Then ϕ is additive, and we have (1.1.1) as desired. Thus Theorem 1.1 is proved.

When is the derivation δ defined by $\delta(e(\sigma, i)) = \phi(\sigma)e(\sigma, i)$, where ϕ is an additive function on G, inner? Let

$$\delta(e(\sigma, i)) = e(\sigma, i) \sum_{\tau, j} \alpha_{\tau, j} e(\tau, j)$$

with $\alpha_{\tau,j} \in K$. Then

$$0 = e(0, i) = \sum_{\tau, j} \alpha_{\tau, j}\tau(i)e(\tau, j).$$

Hence $\tau(i) = 0$, $\tau = 0$, whenever $\alpha_{\tau,j} \neq 0$. From this it follows that δ is inner if, and only if, $\phi(\sigma) = \sum_j \alpha_j \sigma(j)$ with $\alpha_j \in K$. Such additive functions ϕ form clearly an m-dimensional vector space over K. On the other hand, if $\mathfrak{G}$ is an elementary group of order p^n, then all the additive functions on $\mathfrak{G}$ with values in K form an n-dimensional vector space over K. Hence we have

COROLLARY 1.2. *Let $\mathfrak{L}$ be a generalized Witt algebra with basis $\{e(\sigma, i) | \sigma \in \mathfrak{G}, i \in I\}$, where $\mathfrak{G}$ is an elementary p-group of order p^n, and $I = \{1, 2, \ldots, m\}$. Let $\mathfrak{D}$ and $\mathfrak{J}$ be the derivation algebra and the algebra of inner derivations of $\mathfrak{L}$, respectively. Then $\mathfrak{D}/\mathfrak{J}$ is an abelian algebra of dimension $n - m$, provided that the characteristic of K is greater than 2.*

From the above corollary it follows immediately that the number m is uniquely determined by $\mathfrak{L}$. This is, however, proved in (**5**, p. 546). Also, if $m = n$, then every derivation of $\mathfrak{L}$ is inner. This is a result of Jacobson (**3**).

2. Generalized orthogonal systems. Let $\mathfrak{A}$ be a finite-dimensional commutative associative algebra over the algebraically closed ground field K. We assume that $\mathfrak{A}$ has a unity element.

An ordered set $(D_1, \ldots, D_m)$ of derivations of $\mathfrak{A}$ will be called a *generalized orthogonal* (*g.o.*) *system* if the following conditions (2.1.1.)-(2.1.2) are satisfied:

(2.1.1.) $[D_i, D_j] = D_iD_j - D_jD_i = 0$ *for all* i *and* j;

(2.1.2) *If* $f \in \mathfrak{A}$ *and* $\lambda_1, \ldots, \lambda_m \in K$ *are such that* $D_if = \lambda_if$ *for all* i, *then* $f = 0$ *or* f *is a unit of* $\mathfrak{A}$.

A g.o. system $(D_1, \ldots, D_m)$ will be called an *o. system* if it satisfies the following condition:

(2.1.3.) $\sum_{i=1}^m f_i D_i = 0$, *where* $f_i \in \mathfrak{A}$, *implies* $f_i = 0$ *for all* i.

LEMMA 2.1. *The conditions* (2.1.1.)–(2.1.2) *imply the following:*

(2.1.4) $D_i f = 0$ *for all* $i = 1, \ldots, m$ *implies* $f \in K$.

Proof. The set $\mathfrak{B}$ of all $f \in \mathfrak{A}$ such that $D_i f = 0$ for all i is clearly a subalgebra of $\mathfrak{A}$, and, moreover, if $0 \neq f \in \mathfrak{B}$ then by (2.1.2) f^{-1} exists and belongs to $\mathfrak{B}$, since $D_i f^{-1} = -f^{-2} D_i f = 0$. Therefore, $\mathfrak{B}$ is a finite extension field of K. Since K is algebraically closed, we have $\mathfrak{B} = K$.

THEOREM 2.2. *For any g.o. system* $(D_1, \ldots, D_m)$ *there exists a non-void subset* $S = \{i_1, \ldots, i_r\}$ *of indices* $1, \ldots, m$ *such that* (2.2.1)–(2.2.2), *below, hold:*

(2.2.1) $(D_{i_1}, \ldots, D_{i_r})$ *is an o. system;*

(2.2.2) *There exists* $\alpha_{is} \in K$ *such that*

$$D_i = \sum_{s \in S} \alpha_{is} D_s, \qquad (i = 1, \ldots, m).$$

Proof. Let S be a minimal subset of the indices $1, \ldots, m$ with respect to the property: there exist $a_{is} \in \mathfrak{A}$ such that

$$D_i = \sum_{s \in S} a_{is} D_s \qquad (i = 1, \ldots, m). \tag{2.2.3}$$

We may assume without loss of generality that $S = \{1, \ldots, r\}$. Let V be the set of all r-tuples $(f_1, \ldots, f_r)$ of elements $f_s \in \mathfrak{A}$ such that $\sum_s f_s D_s = 0$. Define addition in V componentwise, scalar multiplication by $\alpha(f_1, \ldots, f_r) = (\alpha f_1, \ldots, \alpha f_r)$, $\alpha \in K$. Then V is a finite-dimensional vector space over K. We shall prove (2.1.3) for $(D_1, \ldots, D_r)$ by showing that $V = 0$. Suppose $V \neq 0$. Since $\sum f_s D_s = 0$ implies $\sum_s (D_i f_s) D_s = 0$, the mapping $(f_1, \ldots, f_r) \to (D_i f_1, \ldots, D_i f_r)$ is a linear transformation of V. Since $D_i(D_j f) = D_j(D_i f)$ for all f $\in \mathfrak{A}$, i, and j, and since K is algebraically closed, there exists a non-zero $(f_1, \ldots, f_r) \in V$ and $\lambda_1, \ldots, \lambda_m \in K$ such that

$$(D_i f_1, \ldots, D_i f_r) = \lambda_i (f_1, \ldots, f_r)$$

for $i = 1, \ldots, m$. Then $D_i f_s = \lambda_i f_s$ for all i and s. Then from (2.1.2) it follows that f_s is either 0 or a unit in $\mathfrak{A}$. Since not all f_s are zero, we may assume $f_1 \neq 0$; f_1 is a unit. Then $D_1 = -f_1^{-1} f_2 D_2 - \ldots - f_1^{-1} f_r D_r$. Then every D_i can be written as a linear combination of $D_2, \ldots, D_r$ with coefficients in $\mathfrak{A}$. This contradicts the minimality of S. Thus $V = 0$, and hence (2.1.3) is proved for $(D_1, \ldots, D_r)$.

Now, from (2.1.1) and (2.2.3) it follows that $\sum_s (D_k a_{is}) D_s = 0$ for all $i, k = 1, \ldots, m$. Therefore by (2.2.1), we have $D_k a_{is} = 0$ and hence, by Lemma 2.1, $a_{is} = \alpha_{is} \in K$ for all i and s. This proves (2.2.2).

In order to show that $(D_1, \ldots, D_r)$ is an *o.* system, it remains to be shown that $D_s f = \lambda_s f$, $\lambda_s \in K$, for $s = 1, \ldots, r$ implies that $f = 0$ or f is a unit. This, however, follows easily from (2.2.2) and (2.1.2). Thus the proof of Theorem 2.2 is complete.

COROLLARY 2.3. *A g.o. system $(D_1, \ldots, D_m)$ is an o. system if, and only if, $D_1, \ldots, D_m$ are linearly independent over K.*

COROLLARY 2.4. *If there exists a g.o. system of derivations of $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to the group algebra over K of an abelian p-group of type $(p, p, \ldots, p)$.*

Proof. By Theorem 2.2, there exists an *o.* system of derivations of $\mathfrak{A}$. Then Corollary 2.3 follows from Lemma 2.1 above and Theorem 6.10 of **(5)**.

COROLLARY 2.5. *The conditions* (2.1.1)–(2.1.2) *imply the following: If $f, a_1, \ldots, a_m \in \mathfrak{A}$ are such that $D_i f = a_i f$ for all i, then $f = 0$ or f is a unit in $\mathfrak{A}$.*

Proof. Corollary 2.5 follows immediately from Theorem 2.2 above, and Lemma 6.3 of **(5)**.

The following theorem, which also follows immediately from Theorem 2.2, above, and Theorem 6.10 of **(5)**, is a partial generalization of Theorem 6.10 of **(5)**.

THEOREM 2.6. *If $(D_1, \ldots, D_m)$ is a g.o. system, then the subalgebra of the derivation algebra of $\mathfrak{A}$, consisting of all derivations of the form $\sum f_i D_i$, where $f_i \in \mathfrak{A}$, is isomorphic to a generalized Witt algebra.*

Now let $(D_0, \ldots, D_m)$ be a set of derivations of $\mathfrak{A}$, satisfying (2.1.1), and let $a_0, \ldots, a_m \in \mathfrak{A}$ be such that $D_i a_j = D_j a_i$ for all i and j. Then the set $\mathfrak{L} = \mathfrak{L}(D_0, \ldots, D_m; a_0, \ldots, a_m)$ of all derivations of the form $\sum f_i D_i$, where $f_i \in \mathfrak{A}$ satisfy $\sum_i (D_i f_i - a_i f_i) = 0$, forms a subalgebra of the derivation algebra of $\mathfrak{A}$. A special case of such algebras was considered for the first time by Frank **(2)**, and another by Albert and Frank **(1)**. The general case where $(D_0, \ldots, D_m)$ is an *o.* system was considered by Jennings and Ree **(4)**. Here we consider the case where $(D_0, \ldots, D_m)$ is an arbitrary g.o. system.

THEOREM 2.7. *If $(D_0, \ldots, D_m)$ is a g.o. system, then the algebra $L(D_0, \ldots, D_m; a_0, \ldots, a_m)$ is isomorphic either to a generalized Witt algebra or to an algebra of the form $L(D_0', \ldots, D_r'; a_0', \ldots, a_r')$, where $(D_0', \ldots, D_r')$ is an o. system.*

Proof. If $m = 0$, then $(D_0, \ldots, D_m)$ is an o. system, and so our theorem is clear. We shall proceed by induction on m. Assume that Theorem 2.7 is true for $m - 1$. If $(D_0, \ldots, D_m)$ is an *o.* system then our theorem is clear. If $(D_0, \ldots, D_m)$ is not an *o.* system, then, by Theorem 2.2, we may assume without loss of generality that $D_m = \alpha_0 D_0 + \ldots + \alpha_{m-1} D_{m-1}$ with $\alpha_i \in K$. We have

$$D_k\left(a_m - \sum_{i=0}^{m-1} \alpha_i a_i\right) = D_m a_k - \sum_{i=0}^{m-1} \alpha_i D_i a_k = 0$$

for $k = 0, 1, \ldots, m$. Hence

$$a_m - \sum_{i=0}^{m-1} \alpha_i a_i = \alpha$$

belongs to K by Lemma 2.1.

If $\alpha = 0$ then $\mathfrak{L} = \mathfrak{L}(D_0, \ldots, D_m; a_0, \ldots, a_m)$ and $\mathfrak{L}_1 = \mathfrak{L}(D_0, \ldots, D_{m-1}; a_0, \ldots, a_{m-1})$ coincide. This is seen as follows: Let $\sum_0^m f_i D_i \in L$. Then by definition, $\sum_0^m (D_i f_i - a_i f_i) = 0$, and hence

$$\sum_{i=0}^{m-1} (D_i(f_i + \alpha_i f_m) - a_i(f_i + \alpha_i f_m)) = 0.$$

On the other hand,

$$\sum_{i=0}^{m} f_i D_i = \sum_{i=0}^{m-1} (f_i + \alpha_i f_m) D_i.$$

Therefore, $\sum_0^m f_i D_i \in \mathfrak{L}_1$ and hence $\mathfrak{L} \leqslant \mathfrak{L}_1$ is proved. Since $\mathfrak{L}_1 \leqslant \mathfrak{L}$ is clear, we have $\mathfrak{L} = \mathfrak{L}_1$.

If $\alpha \neq 0$ then $\mathfrak{L} = \mathfrak{L}(D_0, \ldots, D_m; a_0, \ldots, a_m)$ coincides with the set $\mathfrak{L}_2$ of all derivations of the form $\sum_0^{m-1} g_i D_i$, where g_i runs over $\mathfrak{A}$. This is seen as follows: Clearly we have $L \leqslant L_2$. Now, for an arbitrary element $\sum_0^{m-1} g_i D_i$ in $\mathfrak{L}_2$, define $f_0, f_1, \ldots, f_m$ by the formulae:

$$f_m = \alpha^{-1} \sum_{i=0}^{m-1} (D_i g_i - a_i g_i);$$

$$f_i = g_i - \alpha_i f_m, \qquad (0 \leqslant i < m).$$

Then it is easily seen that $\sum_0^{m-1} g_i D_i = \sum_0^m f_i D_i$, and that

$$\sum_{i=0}^{m} (D_i f_i - a_i f_i) = 0.$$

Therefore $\sum_0^{m-1} g_i D_i \in \mathfrak{L}$, and hence $\mathfrak{L}_2 \leqslant \mathfrak{L}$ is proved. Thus we have $\mathfrak{L} = \mathfrak{L}_2$. Since $\mathfrak{L}_2$ is a generalized Witt algebra, this completes the proof of Theorem 2.7.

Consider now a set of derivations $(D_1, \ldots, D_m)$ of $\mathfrak{A}$ satisfying only the condition (2.1.1) and denote by $\mathfrak{L}$ the subalgebra of the derivation algebra of $\mathfrak{A}$ consisting of all derivations of the form $f_i D_i$, where $f_i \in \mathfrak{A}$. Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$, and let $\mathfrak{D}$ be the set of all $f \in \mathfrak{N}$ such that $D_k(D_j(\ldots(D_i f)\ldots)) \in \mathfrak{N}$ for any $i, j, \ldots, k$ (the number of indices $i, j, \ldots, k$ is arbitrary). It is easily seen that $\mathfrak{D}$ is an ideal of $\mathfrak{A}$ and that $f \in \mathfrak{D}$ implies $D_i f \in \mathfrak{D}$ for all i. Therefore every D_i induces a derivation $\bar{D}_i$ of the algebra $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{D}$. Since $[\bar{D}_i, \bar{D}_j] = 0$ follows from $[D_i, D_j] = 0$, we can consider the subalgebra $\mathfrak{L}$ of the derivation algebra of $\bar{\mathfrak{A}}$ consisting of all derivations of the form $\sum \bar{f}_i \bar{D}_i$, where $\bar{f}_i \in \bar{\mathfrak{A}}$. Denote by $\bar{f}$ the image of $f \in \mathfrak{A}$ under the natural homomorphism: $\mathfrak{A} \to \bar{\mathfrak{A}}$. Since $\sum f_i D_i = 0$ implies $\sum \bar{f}_i \bar{D}_i = 0$, a mapping ϕ is uniquely defined by $\phi(\sum f_i D_i) = \sum \bar{f}_i \bar{D}i$. It is easily seen that ϕ is a homomorphism of $\mathfrak{L}$ onto $\bar{\mathfrak{L}}$. The kernel $\mathfrak{J}$ of ϕ consists of elements $\sum f_i D_i$ such that $\sum \bar{f}_i \bar{D}_i = 0$. Note that $\sum \bar{f}_i \bar{D}_i = 0$ if and only if $\sum f_i(D_i g) \in \mathfrak{D}$ for all $g \in \mathfrak{A}$. From this it follows immediately that the ideal $[\mathfrak{J}, \mathfrak{J}]$ of $\mathfrak{L}$ is contained in the algebra $\mathfrak{L}_1$ consisting of all derivations of the form $\sum f_i D_i$, where $f_i \in \mathfrak{D}$. For a positive integer k, denote by $\mathfrak{L}_k$ the algebra of all derivations of the form $\sum f_i D_i$, where $f_i \in \mathfrak{D}^k$. It is easily seen that $[\mathfrak{L}_k, \mathfrak{L}_1] \leqslant \mathfrak{L}_{k+1}$ for any k. Since $\mathfrak{D} \leqslant \mathfrak{N}$, it follows that $\mathfrak{D}$ is nilpotent, say, $\mathfrak{D}^t = 0$. Then $\mathfrak{L}_t = 0$, and hence $\mathfrak{L}_1$ is nilpotent, and $\mathfrak{J}$ is solvable.

Consider now the algebra $\bar{\mathfrak{L}}$, assuming that every non-unit element in $\mathfrak{A}$ is contained in the radical $\mathfrak{N}$. We shall prove that $(\bar{D}_1, \ldots, \bar{D}_m)$ is a g.o. system of $\bar{\mathfrak{A}}$. Suppose that $\bar{D}_i \bar{f} = \lambda_i \bar{f}$ for all i, and that $\bar{f}$ is a non-unit in $\bar{\mathfrak{A}}$. Then $D_i f = \lambda_i f + g_i$, where $g_i \in \mathfrak{D}$. Since $\bar{f}$ is not a unit f is also not a unit, and hence by our assumption $f \in \mathfrak{N}$. Then from $D_i f - \lambda_i f \in \mathfrak{D}$ it follows easily that $f \in \mathfrak{D}$. Therefore $\bar{f} = 0$, and hence $(\bar{D}_1, \ldots, \bar{D}_m)$ is proved to be a g.o. system. Then, by Theorem 1.6, $\bar{\mathfrak{L}}$ is isomorphic to a generalized Witt algebra.

An associative algebra $\mathfrak{A}$ is called *completely primary* if the set of non-unit elements coincide with the radical of $\mathfrak{A}$. Summarizing the above, we have

THEOREM 2.8. *Suppose that the commutative associative algebra $\mathfrak{A}$ is completely primary. Then for any set of derivatives $(D_1, \ldots, D_m)$ of $\mathfrak{A}$, which satisfies the condition* (2.1.1.), *the algebra $\mathfrak{L}$ consisting of all derivations of the form $\sum f_i D_i$, where $f_i \in \mathfrak{A}$, has a solvable ideal $\mathfrak{J}$ such that $\mathfrak{L}/\mathfrak{J}$ is isomorphic to a generalized Witt algebra.*

Similarly we may obtain the following

THEOREM 2.9. *Suppose that the commutative associative algebra $\mathfrak{A}$ is completely primary. Then for any set of derivations $(D_1, \ldots, D_m)$ of $\mathfrak{A}$, which satisfies the condition* (2.1.1), *an algebra $\mathfrak{L}$ of the form $\mathfrak{L}(D_0, \ldots, D_m; a_0, \ldots, a_m)$ has a solvable ideal $\mathfrak{J}$ such that $\mathfrak{L}/\mathfrak{J}$ is isomorphic either to a generalized Witt algebra or to an algebra of the form $\mathfrak{L}(E_0, \ldots, E_r; b_0, \ldots, b_r)$, where $(E_0, \ldots, E_r)$ is an o. system of derivations of the group algebra over K of an abelian group of type $(p, \ldots, p)$.*

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