Bull. Aust. Math. Soc. **80** (2009), 291–293
doi:10.1017/S0004972709000276

ON p-AUTOMORPHISMS THAT ARE INNER

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(Received 29 September 2008)

Abstract

Let G be a group and let $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))$ be the set of all automorphisms of G centralizing $G/\Phi(G)$ and $Z(\Phi(G))$. For each prime p and finite p-group G, we prove that $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) \leq \mathrm{Inn}(G)$ if and only if G is elementary abelian or $\Phi(G) = Z(G)$ and $Z(G)$ is cyclic.

2000 *Mathematics subject classification*: primary 20D45; secondary 20E36.
Keywords and phrases: automorphism, central automorphism, finite p-group.

1. Introduction and result

Throughout p denotes a prime number. Let G be a group. We denote by G', $\Phi(G)$, $Z(G)$, $\mathrm{Inn}(G)$, $\mathrm{Aut}(G)$ respectively the commutator subgroup, the Frattini subgroup, the centre, the inner automorphism group and the automorphism group of G. An automorphism α of G is called a central automorphism if $x^{-1}x^{\alpha} \in Z(G)$ for each $x \in G$. The central automorphisms of G form a normal subgroup $\mathrm{Aut_c}(G)$ of the full automorphism group $\mathrm{Aut}(G)$. Let $C_{\mathrm{Aut_c}(G)}(Z(G))$ be the group of all central automorphisms of G fixing $Z(G)$ elementwise. Curran and McCaughan [2] characterized finite p-groups G for which $\mathrm{Aut_c}(G) = \mathrm{Inn}(G)$. Curran [1] characterized finite p-groups G for which $C_{\mathrm{Aut_c}(G)}(Z(G)) \leq \mathrm{Inn}(G)$. In [6] we proved that if G is a finite p-group, then $C_{\mathrm{Aut}_c(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic. Let

$$\mathrm{Aut}^{\Phi}(G) = \{\phi \in \mathrm{Aut}(G) \mid x^{-1}x^{\phi} \in \Phi(G) \text{ for all } x \in G\}$$

and

$$C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = \{\phi \in \mathrm{Aut}^{\Phi}(G) \mid x^{\phi} = x \ \text{ for all } \ x \in Z(\Phi(G))\}.$$

By a well-known theorem of P. Hall the group $\mathrm{Aut}^{\Phi}(G)$ is a p-group. Clearly $\mathrm{Aut}^{\Phi}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$.

Müller [3], by using cohomological methods, proved that for a finite p-group G, $\mathrm{Aut}^{\Phi}(G)=\mathrm{Inn}(G)$ if and only if G is elementary abelian or extraspecial. Here we give a characterization for finite p-groups G for which $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))\leq \mathrm{Inn}(G)$.

THEOREM 1.1. *If G is a finite p-group, then $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))\leq \mathrm{Inn}(G)$ if and only if G is elementary abelian or $\Phi(G)=Z(G)$ and $Z(G)$ is cyclic.*

2. Proof of the theorem

If G is elementary abelian, then $\Phi(G)=1$ and so we have

$$C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))=\mathrm{Inn}(G)=1.$$

If $\Phi(G)=Z(G)$ and $Z(G)$ is cyclic, then by [6] we have $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))=C_{\mathrm{Aut}_c(G)}(Z(G))=\mathrm{Inn}(G)$.

Now let $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))\leq \mathrm{Inn}(G)$. Let G be an abelian p-group. We prove that $\exp(G)=p$. Suppose, on the contrary, that $\exp(G)=p^k$ for some $k>1$. We define the mapping $\theta: G\longrightarrow G$ by $\theta(x)=x^{1+p^{k-1}}$ for all $x\in G$. Then θ is a nontrivial automorphism of G, since $\exp(G)=p^k$. Also $\theta(x^p)=x^{p+p^k}=x^p$. Therefore θ is a nontrivial automorphism of G which fixes $G/\Phi(G)$ and $Z(\Phi(G))$. This contradicts the hypothesis. Assume that G is a nonabelian p-group. We first prove that $Z(G)\leq\Phi(G)$. Suppose, on the contrary, that there exists a maximal subgroup M of G such that $Z(G)\nleq M$. Take an element g in $Z(G)\backslash M$. Therefore $G=M\langle g\rangle$. Choose an element z of order p in $Z(G)\cap\Phi(G)$. Then it is easy to see that the map α defined by $(mg^k)^{\alpha}=mg^kz^k$ for every $m\in M$ and every $k\in\{0,1,\ldots,p-1\}$ is an automorphism which fixes $G/\Phi(G)$ and $Z(\Phi(G))$. By the hypothesis there exists an element $a\in G$ such that $\alpha=\theta_a$ where θ_a is the inner automorphism of G induced by a. Since $g\in Z(G)$, we have $gz=\alpha(g)=\theta_a(g)=a^{-1}ga=g$ whence $z=1$, which contradicts the hypothesis. Thus $Z(G)\leq\Phi(G)$.

Now we prove that $Z(G)\lneqq Z(M)$ for every maximal subgroup M of G. Suppose, for a contradiction, that M is a maximal subgroup of G such that $Z(G)=Z(M)$. We have $C_G(M)=Z(M)$, since $Z(G)\leq\Phi(G)$ and M is maximal subgroup. Let $g\in G\backslash M$ and z be an element of order p in $Z(G)\leq\Phi(G)$. Then it is easy to see that the map β on G defined by $(mg^k)^{\beta}=mg^kz^k$ for every $m\in M$ and every $k\in\{0,1,\ldots,p-1\}$ is an automorphism which fixes $G/\Phi(G)$ and $Z(\Phi(G))$ elementwise. By assumption we have $\alpha=\theta_a$ for some $a\in G$ whence $a\in C_G(M)=Z(M)=Z(G)$, which contradicts the hypothesis. Thus $Z(G)\neq Z(M)$.

Hence, by [5], G has one of the following forms:

(i) $G=E_1E_2\cdots E_s$, where $[E_i,E_j]=1$ for all $i\neq j$, $|E_i|=p^2|Z(G)|$ and $Z(G)=Z(E_i)$ for all $1\leq i\leq s$; or

(ii) $G=EF$ is the central product of the Frattinian subgroups E and F where $C_F(Z(\Phi(F)))=\Phi(F)$ and where $E=C_G(F)$ satisfies $\Phi(E)\leq Z(G)$.

Moreover in case (ii), either $E=Z(G)$ (and therefore $G=F$), or E is a central product as in case (i).

https://doi.org/10.1017/S0004972709000276 Published online by Cambridge University Press

If the group G is as in case (i), then $Z(G) = \Phi(G)$. Therefore $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = C_{\mathrm{Aut}_c(G)}(Z(G)) \le \mathrm{Inn}(G)$. On the other hand $\mathrm{Inn}(G) \le C_{\mathrm{Aut}_c(G)}(Z(G))$ since G is nilpotent of class 2. Therefore $C_{\mathrm{Aut}_c(G)}(Z(G)) = \mathrm{Inn}(G)$. Hence, by [6], $Z(G)$ is cyclic. We now complete the proof by showing that G can not have a form as case (ii). Suppose, for a contradiction, that G satisfies case (ii). If $G = F$ then $C_G(Z(\Phi(G))) = \Phi(G)$, which is impossible by [5, Proposition 3]. Let $G \ne F$. Thus E is a central product as in case (i) and so $\Phi(E) = Z(E)$. Since $G = EF$, we have $\Phi(G) = G'G^p = E'F'E^pF^p = E'E^pF'F^p = \Phi(E)\Phi(F) = Z(E)\Phi(F)$ and hence $Z(\Phi(G)) \le Z(E)Z(\Phi(F))$. Since $\Phi(F) = C_F(Z(\Phi(F)))$, by [5, Proposition 3] there exists $\alpha \in C_{\mathrm{Aut}^{\Phi}(F)}(Z(\Phi(F))) \setminus \mathrm{Inn}(F)$.

Since $E = C_G(F)$ and $C_F(Z(\Phi(F))) = \Phi(F)$, $E \cap F \le Z(\Phi(F))$ and hence the map φ on G defined by $(xy)^{\varphi} = xy^{\alpha}$ for every $x \in E$ and for every $y \in F$ is well-defined. Since $Z(\Phi(G)) \le Z(E)Z(\Phi(F))$, it is easy to check that $\varphi \in C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))$ and so it is an inner automorphism of G. It follows that α is an inner automorphism of F, which is impossible. □

COROLLARY 2.1. *If G is a finite p-group, then $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = \mathrm{Inn}(G)$ if and only if G is elementary abelian or $\Phi(G) = Z(G)$ and $Z(G)$ is cyclic.*

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https://doi.org/10.1017/S0004972709000276 Published online by Cambridge University Press