Algebraic cycles on the Jacobian of a curve with a linear system of given dimension

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Dedicated to Joseph Steenbrink on the occasion of his 60th birthday

Abstract

We present relations between cycles with rational coefficients modulo algebraic equivalence on the Jacobian of a curve. These relations depend on the linear systems that the curve admits. They are obtained in the tautological ring, the smallest subspace containing (an embedding of) the curve and closed under the basic operations of intersection, Pontryagin product and the pullback and pushdown induced by homotheties.

1. Introduction

In this paper we extend a theorem of Colombo and van Geemen for d-gonal curves to linear systems of higher dimension. To express our results in § 1.3 we recall in § 1.1 the Beauville decomposition theorem which enlightens the structure of the tautological ring described in § 1.2. The main points of the proof are summarised in § 1.4.

1.1 We use the notation $CH(X)$ for the Chow ring associated to a smooth algebraic variety X, and $A(X)$ for its quotient modulo algebraic equivalence. These rings will always be considered *tensored by* $\mathbb{Q}$. The brackets $[V]$ mean we consider the class of a subvariety V of X. Let X be an Abelian variety over $\mathbb{C}$ whose group law we denote m and dual variety we denote $\hat{X}$. It admits homotheties $x \mapsto kx$ which we will also denote k for $k \in \mathbb{Z}$. We have another product between algebraic cycles, the Pontryagin product, defined between two cycles α and β by $\alpha * \beta = m_*(p^*\alpha.q^*\beta)$ with p and q the two projections associated to the product $X \times X$. Beauville introduced in [Bea83] the Fourier transform $\mathcal{F}$, a $\mathbb{Q}$-linear automorphism between $A(X)$ and $A(\hat{X})$ which exchanges the two products. He used it to prove in [Bea86] the decomposition theorem which states that the operators k_* and k^* simultaneously diagonalize. For each codimension p the subspace $A^p(X)$ splits:

$$A^p(X) = \bigoplus_{i=p-g}^{g} A^p_{(i)}(X) \quad \text{where } \alpha \in A^p_{(i)}(X) \text{ if and only if for all } k \in \mathbb{Z} \quad k_*\alpha = k^{2g-2p+i}\alpha$$

$$\text{(or, equivalently, } k^*\alpha = k^{2p-i}\alpha). \quad (1)$$

It is conjectured that $A^p_{(i)}(X)$ vanishes for $i < 0$. This is proven if $p \in \{0, 1, g-2, g-1, g\}$.

1.2 Let C be a smooth projective complex curve of genus $g \geqslant 2$ and JC its Jacobian variety. As C embeds into JC, we can consider the associated cycle $[C]$ in $A^{g-1}(JC)$ and its decomposition[1]:

$$[C] = C_{(0)} + \cdots + C_{(g-1)} \quad \text{where } C_{(i)} \in A^{g-1}_{(i)}(JC). \tag{2}$$

Note that the cycle $[C]$ does not depend on the embedding of the curve, as we work modulo algebraic equivalence. The tautological ring R is the smallest subspace of $A(JC)$ which contains $[C]$ and is stable under the two products and the operators k_* and k^*. Actually, R is finite-dimensional: Beauville proved in [Bea04] that R is the subalgebra for the Pontryagin product generated by $C_{(0)}, \ldots, C_{(g-1)}$. So we can describe R as a quotient of the $\mathbb{Q}$-algebra $\mathbb{Q}[C_{(0)}, \ldots, C_{(g-1)}]$ by an ideal of relations we have to determine. However, to discuss the vanishing of algebraic cycles is always a difficult problem. Few results are known for R. The cycle $C_{(0)}$ is never zero. Neither are $C_{(1)}$ and $C_{(2)}$ for generic curves of respective genera $g \geqslant 3$ and $g \geqslant 11$ as Ceresa [Cer83] and Fakhruddin [Fak96] proved, respectively. More recently, Ikeda [Ike03] proved for each degree d the existence of a smooth plane curve C for which the component $C_{(i)}$ is nonzero in $CH^{g-1}_{(i)}(JC)$ when $i \leqslant d - 3$.

On the other hand, one could ask for vanishing results. Polishchuk has presented in [Pol05] an ideal I_g of relations which hold for all curves of genus g. For nongeneric curves, Colombo and van Geemen proved the main result.

THEOREM (Colombo–van Geemen [CV93]). *If C is a d-gonal curve then $C_{(i)} = 0$ in $A^{g-1}_{(i)}(JC)$ for $i \geqslant d - 1$.*

1.3 In this paper, we extend this theorem to curves which admit linear systems of higher dimension. More precisely, we state the following result, where we use the abbreviation g^r_d for linear systems of dimension r and degree d.

THEOREM 1. *Let C be a curve which admits a base point free g^r_d. For each integer $s \geqslant 0$ the following relation holds in $A^{g-r}_{(s)}(JC)$:*

$$\sum_{\substack{0 \leqslant a_1, \ldots, a_r \\ a_1 + \cdots + a_r = s}} \beta(d, a_1 + 1, \ldots, a_r + 1) C_{(a_1)} * \cdots * C_{(a_r)} = 0,$$

where

$$\beta(d, a_1, \ldots, a_r) = \sum_{i_1=1}^{d} \cdots \sum_{i_r=1}^{d} (-1)^{i_1 + \cdots + i_r} \binom{d}{i_1 + \cdots + i_r} i_1{}^{a_1} \ldots i_r{}^{a_r}.$$

In §7 we study applications of this theorem to plane and space curves. For almost all genera g there exists a curve with a g^2_d (respectively, a g^3_d) giving new relations. By new relations we mean that they could not be deduced from the $g^1_{d'}$ that the g^2_d (respectively, the g^3_d) induces and from knowledge of I_g. We list such relations for $g \leqslant 9$ in the Tables 1 and 2.

Do we obtain new vanishing statements $C_{(i)} = 0$ in $A^{g-1}_{(i)}(JC)$? No, because the monomial relations we obtain are $B(r, d, g)C_{(d-2r+1)} = 0$, where for each dimension r the integer $B(r, d, g)$ is the number of $(r - 2)$ planes which cut the curve (mapped to $\mathbb{P}^r$ by the g^r_d) in $2r - 2$ points. When such a situation arises, we can construct (by projection) a g^1_{d-2r+2}. In this case, the monomial relation above does not teach us more than Colombo and van Geemen's theorem. We explain this in §6.

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[1] Perhaps it is useful to recall that $C_{(i)}$ is denoted $\pi_{2g-2-i}C$ in [CV93] and that Polishchuk worked with the classes $p_i = \mathcal{F}(C_{(i-1)})$ in [Pol05].

1.4 The first sections are devoted to the proof of Theorem 1. Let C_d be the d-fold symmetric product of the curve C. Choosing an element of C_d provides a morphism $u_d : C_d \longrightarrow JC$. Throughout the whole paper, $\mathcal{G}$ will denote a g_d^r and G_n the truncated linear system of degree n; this is the set of divisors of C_n we can complete to a divisor of $\mathcal{G}$:

$$G_n = \{D \in C_n \mid \exists E \in C_{d-n}, D + E \in \mathcal{G}\} \quad \text{for } 1 \leqslant n \leqslant d.$$

These truncated linear systems may be considered as subvarieties[2] of the symmetric products C_n. The scheme structure is described in [ACGH85, paragraph 3 of ch. VIII]. They play an important role in Colombo and van Geemen's proof. We use the notation $\delta_{i_1,\dots,i_r}$ for the generalized diagonals $\delta_{i_1,\dots,i_r} = \{i_1 x_1 + \dots + i_r x_r \mid x_i \in C\}$ in the $(\sum i_u)$-fold symmetric product of C. The cornerstone of the present note is the following generalization of a formula[3] they obtained in the case $r = 1$, which expresses the algebraic classes $[G_n]$ (and, in particular, $[\mathcal{G}] = [G_d]$) as functions of the diagonals.

THEOREM 2. *If C admits a base point free g_d^r then the following equivalences hold in $A^{n-r}(C_n)$ for $r \leqslant n \leqslant d$:*

$$[G_n] = \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_r} \binom{d}{n - \sum_{u=1}^{r} i_u} \left(\prod_{u=1}^{r} \frac{(-1)^{i_u - 1}}{i_u}\right) \left[\delta_{i_1,\dots,i_r} + \left(n - \sum_{u=1}^{r} i_u\right) o\right]$$

where we choose a point o of C.

We state in §4 a relation between $[G_r], \dots, [G_n]$ for all $r \leqslant n \leqslant d$. It enables us to prove Theorem 2 by induction in §5. In §6, we note that $u_{d*}[G_d]$ is zero in $A(JC)$, because G_d is $\mathcal{G}$ and u_d contracts this projective space onto a point in the Jacobian.

Remark. Please note that in [VK07] van der Geer and Kouvidakis gave simpler relations than those of Theorem 1 using the Grothendieck–Riemann–Roch theorem. Don Zagier proved that the two sets of relations are equivalent.

2. Brief reminder about the Fourier transform and the tautological ring

The results recalled here are proven in [Bea83] and [Bea86]. We will identify JC and its dual to consider the Fourier transform $\mathcal{F} : A(JC) \to A(JC)$. It can be defined by the correspondence associated to the exponential of the class of the Poincaré line bundle in $A(JC \times JC)$. As $\mathcal{F} \circ \mathcal{F} = (-1)^g (-1)_*$, this is an isomorphism. It exchanges the two products, that is

$$\mathcal{F}(x.y) = (-1)^g \mathcal{F}(x) * \mathcal{F}(y) \quad \text{and} \quad \mathcal{F}(x * y) = \mathcal{F}x.\mathcal{F}y.$$

The decomposition (1) leads to a bigraduation, in the sense that

$$A_{(i)}^p(JC).A_{(j)}^q(JC) \subset A_{(i+j)}^{p+q}(JC).$$

The proof of the existence of the decomposition (1) gives

$$\mathcal{F}A_{(i)}^p(JC) = A_{(i)}^{g-p+i}(JC)$$

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[2] They have already been used by Kouvidakis in [Kou93] to construct curves in the symmetric product whose Neron Severi classes are known. Izadi has also used the truncated linear systems in [Iza05] to study how curves deform in Abelian varieties.

[3] Proposition 3.4 of [CV93] states that if $\mathcal{G}$ is g_d^1, then we have the equality in $A^{g-1}(JC)$:

$$u_{n*}[G_n] = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} \binom{d}{n-i} i_* C.$$

and so the Pontryagin product is also homogeneous of degree $-g$ with respect to the bigraduation

$$A_{(i)}^p(JC) * A_{(j)}^q(JC) \subset A_{(i+j)}^{p+q-g}(JC).$$

The subspace R is spanned by homogeneous elements for both graduations, so R is bigraded and if we note $R_{(i)}^p$ for the intersection $R_{(i)}^p = R \cap A_{(i)}^p(JC)$, we have

$$R = \bigoplus_{\substack{0 \leqslant p \leqslant g \\ 0 \leqslant i \leqslant p}} R_{(i)}^p.$$

3. Relations between truncated linear systems

Let us consider $\mathcal{G}$, a base point free g_d^r. It induces a nondegenerate morphism[4] $\Phi : C \to \mathbb{P}^r$ and truncated linear systems G_n. We will also consider for positive integers r, n the subvariety $\mathcal{H}_{\mathbb{P}^r}^n \subset (\mathbb{P}^r)^n$ of n-tuples whose components are contained in a hyperplane

$$\mathcal{H}_{\mathbb{P}^r}^n = \{(y_i) \in (\mathbb{P}^r)^n \mid \exists H \in \mathbb{P}^{r*}, \forall i \in \{1, \ldots, n\}, y_i \in H\}. \tag{3}$$

Now we introduce the morphisms $\Psi_\mathcal{P} : C^k \to C^n$ which are defined for integers $0 \leqslant k \leqslant n$ and for $\mathcal{P}$ an ordered k-partition of $\{1, \ldots, n\}$ (this is a partition of $\{1, \ldots, n\}$ into k ordered subsets $A_1, \ldots, A_k$). We set

$$\Psi_{(A_1, \ldots, A_k)} : C^k \longrightarrow C^n$$
$$x \longmapsto y$$

where $y_i = x_j$ if and only if j is the unique integer such that $i \in A_j$. We also denote by σ_k the quotient morphism $\sigma_k : C^k \to C_k$. We can now state the main result of this section.

PROPOSITION 1. *We have, for each integer $n \geqslant r$, the equality in $CH(C^n)$*

$$\Phi^{n*}[\mathcal{H}_{\mathbb{P}^r}^n] = \sum_{k=r}^n \frac{1}{k!} \sum_{\substack{\mathcal{P} \text{ ordered} \\ k\text{-partition of } \{1, \ldots, n\}}} \Psi_{\mathcal{P}*}(\sigma_k{}^*[G_k]).$$

We first state a set theoretic analogue of Proposition 1 in §3.2. We conclude in §3.3 where we prove that $(\Phi^n)^{-1}\mathcal{H}_{\mathbb{P}^r}^n$ is reduced. In §3.1 we compute the class of $\mathcal{H}_{\mathbb{P}^r}^n$ in $CH((\mathbb{P}^r)^n)$.

3.1 One has, by [Ful83, Proposition 8.4], that

$$CH(\mathbb{P}^r) = \frac{\mathbb{Q}[h]}{(h^{r+1})} \tag{4}$$

where h is the class of a hyperplane of $\mathbb{P}^r$. By [Ful83, Proposition 8.3.7] we have

$$CH((\mathbb{P}^r)^n) \simeq \frac{\mathbb{Q}[h_1, \ldots, h_n]}{(h_1^{r+1}, \ldots, h_n^{r+1})}$$

where h_i is the class of a hyperplane in $\mathbb{P}^r$ at the position i. Following this isomorphism, we can state the following.
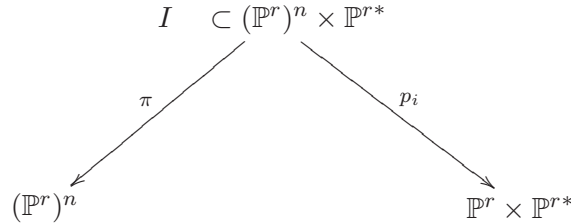
PROPOSITION 2. *For integers $0 \leqslant r \leqslant n$ we have, in $CH((\mathbb{P}^r)^n)$,*

$$[\mathcal{H}_{\mathbb{P}^r}^n] = \sum_{\substack{I \subset \{1, \ldots, n\} \\ \#I = n-r}} \left(\prod_{a \in I} h_a\right).$$

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[4]This amounts to saying that no hyperplane of $\mathbb{P}^r$ contains $\Phi(C)$.

Proof. Let us consider the incidence variety $\mathcal{I} = \{(x, H) \in \mathbb{P}^r \times \mathbb{P}^{r*} \mid x \in H\}$. Let $\mathcal{B}$ be a basis of $\mathbb{C}^{r+1}$, and $\mathcal{B}^*$ the dual basis. Such bases induce systems of coordinates on $\mathbb{P}^r$ and $(\mathbb{P}^r)^*$. If $([x_0 : \ldots : x_r], [a_0 : \ldots a_r])$ represents an element of $\mathbb{P}^r \times \mathbb{P}^{r*}$, the equation of $\mathcal{I}$ is $\sum a_i x_i = 0$. So $\mathcal{I}$ is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^r \times \mathbb{P}^{r*}$. Now let us consider the variety $I = \{(x_1, \ldots, x_n, H) \mid x_i \in H \ \forall i \in \{1, \ldots, n\}\}$.

$$
\begin{array}{ccc}
I & \subset (\mathbb{P}^r)^n \times \mathbb{P}^{r*} & \\
& \swarrow^{\pi} \qquad \searrow^{p_i} & \\
(\mathbb{P}^r)^n & & \mathbb{P}^r \times \mathbb{P}^{r*}
\end{array}
$$

The projection p_i maps $(x_1, \ldots, x_n, H)$ onto (x_i, H) while π maps it onto $(x_1, \ldots, x_n)$. We have $\mathcal{H}_{\mathbb{P}^r}^n = \pi(I)$, and I is the transverse intersection $\bigcap_{i=1}^n p_i^{-1}(\mathcal{I})$. Following the isomorphism

$$
CH((\mathbb{P}^r)^n \times \mathbb{P}^{r*}) \simeq \frac{\mathbb{Q}[h_1, \ldots, h_n, h]}{(h_1^{r+1}, \ldots, h_n^{r+1}, h^{r+1})}
$$

where h_i is the class of a hyperplane at position i and h the class of a hyperplane in $\mathbb{P}^{r*}$, we have in $CH^n((\mathbb{P}^r)^n \times \mathbb{P}^{r*})$ the equality

$$
[I] = \prod_{i=1}^n (h_i + h). \tag{5}
$$

Finally, taking the pushdown π_* is the same as considering the coefficient of h^r in the product (5). $\quad\square$

3.2 We establish here the set theoretical equality between cycles leading to Proposition 1.

PROPOSITION 3. *For integers $0 \leqslant r \leqslant n$ we have*

$$
(\Phi^n)^{-1} \mathcal{H}_{\mathbb{P}^r}^n = \bigcup_{k=r}^n \left(\bigcup_{\substack{\mathcal{P} \ k\text{-partition} \\ of \ \{1, \ldots, n\}}} \Psi_{\mathcal{P}}(\sigma_k^{-1} G_k) \right).
$$

Proof. Let X be an irreducible component of $(\Phi^n)^{-1} \mathcal{H}_{\mathbb{P}^r}^n$. The generic point x of X is an ordered n-tuple with exactly k distinct elements $x_1, \ldots, x_k$ repeated according to a certain partition $\mathcal{P}$ of $\{1, \ldots, n\}$ into k sets. Then x belongs to $(\Phi^n)^{-1} \mathcal{H}_{\mathbb{P}^r}^n$ if and only if the points $\Phi(x_i)$ belong to a same hyperplane, that is $(x_1, \ldots, x_k) \in \sigma_k^{-1} G_k$, and then $x \in \Psi_{\mathcal{P}}(\sigma_k^{-1} G_k)$. As the other inclusion is straightforward, we have

$$
(\Phi^n)^{-1} \mathcal{H}_{\mathbb{P}^r}^n = \bigcup_{k=1}^n \bigcup_{\substack{\mathcal{P} \ k\text{-partition} \\ of \ \{1, \ldots, n\}}} \Psi_{\mathcal{P}}(\sigma_k^{-1} G_k).
$$

Suppose that $k \leqslant r$. We know that k points of $\mathbb{P}^r$ always lie in a hyperplane, so $G_k = C_k$ and $\sigma_k^{-1} G_k = C^k$. So

$$
\bigcup_{\substack{\mathcal{P} \ k\text{-partition} \\ of \ \{1, \ldots, n\}}} \Psi_{\mathcal{P}}(C^k) \subset \bigcup_{\substack{\mathcal{P} \ r\text{-partition} \\ of \ \{1, \ldots, n\}}} \Psi_{\mathcal{P}}(C^r)
$$

and the proposition follows. $\quad\square$

3.3 The following proposition is a sufficient statement for the cycle theoretic application of Proposition 1.

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PROPOSITION 4. *The inverse image $(\Phi^n)^{-1}\mathcal{H}^n_{\mathbb{P}^r}$ is of dimension r, and generically reduced along each of its irreducible components of dimension r.*

We learned the following proof from the referee.

Proof. One knows by Proposition 3 the decomposition of $(\Phi^n)^{-1}\mathcal{H}^n_{\mathbb{P}^r}$ into irreducible components of dimension r. Thus, it suffices to show that at a generic point of each of these components the tangent space to $(\Phi^n)^{-1}\mathcal{H}^n_{\mathbb{P}^r}$ is r-dimensional. Let us denote by $\mathcal{L}$ the line bundle associated to the linear system $\mathcal{G}$ and $W \subset H^0(\mathcal{L})$ the space of sections of $\mathcal{L}$. Thus, $(\Phi^n)^{-1}\mathcal{H}^n_{\mathbb{P}^r}$ is the degeneracy locus of the bundle map

$$ev : W \otimes \mathcal{O}_{C^n} \longrightarrow \bigoplus_{i=1}^{n} pr_i^*\mathcal{L}$$

given by evaluation

$$s \mapsto (s(x_1), \dots, s(x_n))$$

above a point $x = (x_1, \dots, x_n)$ of C^n. At a point $x \in C^n$ where ev has only a one-dimensional kernel generated by σ, the tangent space to the degeneracy locus is exactly $\ker d\sigma$ where

$$d\sigma : T_{C^n,x} = \bigoplus_{i=1}^{n} T_{C,x_i} \longrightarrow \frac{(\bigoplus_{i=1}^{n} pr_i^*\mathcal{L})_x}{Im_{ev_x}}$$

is the projection modulo Im_{ev_x} composed with the direct sum of the $d\sigma_i : T_{C,x_i} \to \mathcal{L}_{x_i}$. Note that all of the differentials $d\sigma_i$ are surjective if and only if the section σ vanishes transversally at each x_i. In this case, the tangent space is r-dimensional.

To conclude, consider the description of the irreducible components of $(\Phi^n)^{-1}\mathcal{H}^n_{\mathbb{P}^r}$ given in Proposition 3. As Φ is nondegenerate, and by Bertini, at the generic point x of each component the kernel $\ker ev_x$ is generated by one section which vanishes transversally at each x_i. $\qquad\square$

4. Classes of the truncated linear systems

Let $I, I_1, \dots, I_r, J_1, \dots, J_s$ be subsets of $\{1, \dots, n\}$ and let $o_1, \dots, o_s$ be points of C. We use the notation:

$$\Delta^I = \{(x_1, \dots, x_n) \in C^n \mid \forall (i,j) \in I^2, x_i = x_j\},$$
$$O^I_{o_j} = \{(x_1, \dots, x_n) \in C^n \mid \forall i \in I, x_i = o_j\},$$

and

$$\Delta^{I_1} \dots \Delta^{I_r} O^{J_1}_{o_1} \dots O^{J_s}_{o_s} \quad \text{for the intersection.}$$

THEOREM 3. *Assume that the curve C admits a base point free g^r_d and let D be a reduced divisor of this linear system whose points of the support are pairwise distinct. We have the following equalities in $CH^{n-r}(C^n)$ and $CH^{n-r}(C_n)$, respectively:*

(i) *we have*

$$\sigma_n^*[G_n] = \sum_{\substack{I_1, \dots, I_r \subset \{1, \dots, n\} \\ o_1, \dots, o_s \text{ distinct}}} \left(\prod_{u=1}^{r} (-1)^{i_u-1}(i_u-1)! \right) [\Delta^{I_1} \dots \Delta^{I_r} O^{\{a_1\}}_{o_1} \dots O^{\{a_s\}}_{o_s}];$$

(ii) *we have*

$$[G_n] = \sum_{\substack{1 \leqslant i_1 \leqslant \dots \leqslant i_r \\ o_1, \dots, o_s \text{ distinct}}} \left(\prod_{u=1}^{r} \frac{(-1)^{i_u-1}}{i_u} \right) [\delta_{i_1, \dots, i_r} + o_1 + \dots + o_s],$$

where we note that $s = n - \sum_{i=1}^{r} i_u$.

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The sum of the first statement is taken over the choice of r subsets $I_1, \ldots, I_r$ of $\{1, \ldots, n\}$ which are pairwise disjoint. We do not order these sets, except by their growing cardinals $i_1 \leqslant \cdots \leqslant i_r$. For each choice of $I_1, \ldots, I_r$, we denote by $a_1, \ldots, a_{n-\sum i_u}$ the elements of $\{1, \ldots, n\} \setminus \bigcup_{k=1}^r I_k$ we order from the smallest to the largest.

In both statements, the points $o_1, \ldots, o_{n-\sum i_u}$ are chosen pairwise distinct in the support of the divisor D. These points are considered ordered in the sum (i) and unordered in the sum (ii).

Proof. Statement (ii) is a consequence of statement (i) and of the pullback–pushdown formula applied to the degree $n!$ morphism σ_n:

$$\sigma_{n*} \circ \sigma_n{}^* = n! \, Id_{CH(C_n)}.$$

To conclude, one counts the diagonals $\Delta^{I_1} \ldots \Delta^{I_r} O_{o_1}^{\{a_1\}} \ldots O_{o_s}^{\{a_s\}}$ that σ_n maps to $\delta_{i_1, \ldots, i_r} + o_1 + \cdots + o_s$ and the number of antecedents of a generic point $x \in \delta_{i_1, \ldots, i_r} + o_1 + \cdots + o_s$ by σ_n. We know that $G_r = C_r$ and $\sigma_r^{-1} G_r = C^r$. We deduce that $\sigma_r^*[G_r] = [C^r]$. This is Theorem 3 for $n = r$. Suppose that Theorem 3 is proven for $n \leqslant m - 1$. By Proposition 1, we have

$$\sigma_m^*[G_m] = \Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m] - \sum_{k=r}^{m-1} \frac{1}{k!} \sum_{\substack{\mathcal{P} \text{ ordered } k\text{-partition} \\ \text{of } \{1, \ldots, m\}}} \Psi_{\mathcal{P}*}(\sigma_k{}^*[G_k]) \tag{6}$$

and Proposition 2 gives the class of $\mathcal{H}_{\mathbb{P}^r}^m$ in $CH((\mathbb{P}^r)^m)$

$$[\mathcal{H}_{\mathbb{P}^r}^m] = \sum_{\substack{I \subset \{1, \ldots, m\} \\ \#I = m-r}} \left(\prod_{a \in I} h_a \right).$$

By the definition of D we have

$$\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m] = \sum_{\substack{o_1, \ldots, o_{m-r} \in \mathrm{Supp}(D) \\ 1 \leqslant a_1 < \cdots < a_{m-r} \leqslant m}} O_{o_1}^{\{a_1\}} \ldots O_{o_{m-r}}^{\{a_{m-r}\}}, \tag{7}$$

where the points o_i are chosen in the support of D and are not necessarily distinct. By the induction hypothesis the classes $\sigma_k^*[G_k]$ for $k \leqslant m - 1$ are sums of classes of varieties $\Delta^{J_1} \ldots \Delta^{J_r} O_{o_1}^{\{b_1\}} \ldots O_{o_s}^{\{b_s\}}$ where $(J_1, \ldots, J_r, \{b_1\}, \ldots, \{b_s\})$ is a partition of $\{1, \ldots, k\}$. By definition of the morphisms $\Psi_{\mathcal{P}}$, the classes $\Psi_{\mathcal{P}*}[\Delta^{J_1} \ldots \Delta^{J_r} O_{o_1}^{\{b_1\}} \ldots O_{o_s}^{\{b_s\}}]$ are again classes of the form $[\Delta^{I_1} \ldots \Delta^{I_r} O_{o_1}^{A_1} \ldots O_{o_s}^{A_s}]$ with $(I_1, \ldots, I_r, A_1, \ldots, A_s)$ a partition of $\{1, \ldots, m\}$. So we can express the cycle $\sigma_m^*[G_m]$ as a linear combination of such classes.

We will fix a partition $(I_1, \ldots, I_r, A_1, \ldots, A_s)$ of $\{1, \ldots, m\}$ and look for which integers k of $\{r, \ldots, m-1\}$ and which partitions $(J_1, \ldots, J_r, \{b_1\}, \ldots, \{b_s\})$ of $\{1, \ldots, k\}$ there exists a k-ordered partition $\mathcal{P}$ of $\{1, \ldots, m\}$ such that

$$\Psi_{\mathcal{P}*}[\Delta^{J_1} \ldots \Delta^{J_r} O_{o_1}^{\{b_1\}} \ldots O_{o_s}^{\{b_s\}}] = \Delta^{I_1} \ldots \Delta^{I_r} O_{o_1}^{A_1} \ldots O_{o_s}^{A_s}.$$

We also have to determine the coefficients associated to these classes in the sum (6).

We will consider the partitions such that for all $u \in \{1, \ldots, r\}$, $\Psi_{\mathcal{P}}(\Delta^{J_u}) = \Delta^{I_u}$. If we denote by i_u and j_u the cardinals of the sets I_u and J_u, we have by definition of the varieties Δ^{I_u} and the morphisms $\Psi_{\mathcal{P}}$ the inequalities $1 \leqslant j_u \leqslant i_u$ for all $u \in \{1, \ldots, r\}$. The classes $\Delta^{J_1} \ldots \Delta^{J_r} O_{o_1}^{\{b_1\}} \ldots O_{o_s}^{\{b_s\}}$ come from the cycles $\sigma_k^*[G_k]$ with $k = \sum_{u=1}^r j_u + s$. All of these contributions are associated to the coefficient

$$-\frac{1}{k!} \prod_{u=1}^r (j_u - 1)! (-1)^{j_u - 1}$$

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which appears in the sum (6). To choose $J_1, \ldots, J_r, \{b_1\}, \ldots, \{b_s\}$ amounts to choosing r disjoint subsets of cardinals $j_1, j_2, \ldots, j_r$. We count $k!/j_1! \ldots j_r!$ such ways.

To finish we have to choose an ordered partition $\mathcal{P}$ of $\{1, \ldots, m\}$ such that for all $u \in \{1, \ldots, r\}$ we have $\Psi_{\mathcal{P}}(\Delta^{J_u}) = \Delta^{I_u}$, and that for each $v \in \{1, \ldots, s\}$ we have $\Psi_{\mathcal{P}}(O_{o_v}^{\{b_v\}}) = O_{o_v}^{A_v}$. It amounts to choosing ordered partitions of the sets I_u in j_u subsets. The symbol $\{^a_b\}$ stands for the Stirling number of the second kind. This is the number of ways to partition a set of a objects into b nonempty sets.[5] So the number of ways to partition such a set into b ordered sets is $b!\{^a_b\}$. In this case, we count $\prod_{u=1}^r j_u!\{^{i_u}_{j_u}\}$ admissible ways to choose $\mathcal{P}$.

We will distinguish four cases depending on whether the sets I_u are singletons (in this case $\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m]$ contributes) and whether the sets A_v are singletons.

First case: both I_u and A_v are singletons. Such classes must come from $\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m]$, with coefficient 1 by (7). So it is equal to the following product where for all $u \in \{1, \ldots, r\}$ we have $i_u = 1$:

$$\prod_{u=1}^r (-1)^{i_u-1}(i_u-1)!.$$

Second case: the sets I_u are singletons, but one of the sets A_v is not. There is, as above, one contribution with coefficient 1 which comes from $\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m]$. The cardinals $i_1, \ldots, i_r$ which are equal to one make the other contributions come from the cycles $\sigma_{r+s}^*[G_{r+s}]$. Precisely, they come from classes $\Delta^{J_1} \ldots \Delta^{J_r} O_{o_1}^{\{b_1\}} \ldots O_{o_s}^{\{b_s\}}$ with $j_1 = \cdots = j_r = 1$. The coefficient associated is -1 which cancels the contribution of $\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m]$.

Third case: one of the sets I_u is not a singleton, but the sets A_v are singletons. In this case, the pullback $\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m]$ does not contribute. On the other hand, we have to enumerate the contributions from the cycles $\sigma_{\sum j_u+s}^*[G_{\sum j_u+s}]$ for the integers j_u which satisfy $1 \leqslant j_u \leqslant i_u$ and $\sum j_u + s < m$. The second inequality is equivalent to $\sum j_u < \sum i_u$. So the sum is taken over the integers j_u which verify $1 \leqslant j_u \leqslant i_u$ and such that there exists $u \in \{1, \ldots, r\}$ making j_u different from i_u. It is equivalent to consider the sum for all j_u such that $1 \leqslant j_u \leqslant i_u$ and to subtract the contribution corresponding to the term for which $i_u = j_u$ for each $u \in \{1, \ldots, m\}$. The total contribution is then

$$-\left(\sum_{j_1=1}^{i_1} \cdots \sum_{j_r=1}^{i_r} \prod_{u=1}^r (-1)^{j_u-1}(j_u-1)!\{^{i_u}_{j_u}\} - \prod_{u=1}^r (-1)^{i_u-1}(i_u-1)! \right).$$

Note that the first sum can be factorized by

$$\sum_{j_1=1}^{i_1} (-1)^{j_1-1}(j_1-1)!\{^{i_1}_{j_1}\}$$

which vanishes by Lemma 1. So the class of $\Delta^{I_1} \ldots \Delta^{I_r} O_{o_1}^{A_1} \ldots O_{o_s}^{A_s}$ again appears with the coefficient

$$\prod_{u=1}^r (-1)^{i_u-1}(i_u-1)!.$$

Fourth case: one of the sets I_u is not a singleton and one of the sets A_v is not a singleton either. In a similar manner, the cycle $\Phi^{m*}[\mathcal{H}_{\mathbb{P}^r}^m]$ does not contribute whereas the

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[5]One could consult [GKP89, ch. 6] to learn more about the Stirling numbers of the second kind.

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cycles $\Psi_{\mathcal{P}*}[\Delta^{J_1}\ldots\Delta^{J_r}O_{o_1}^{\{b_1\}}\ldots O_{o_s}^{\{b_s\}}]$ from the pushdown $\sigma^*_{\sum j_u+s}[G_{\sum j_u+s}]$ contribute as soon as the inequalities $1\leqslant j_u\leqslant i_u$ and $\sum j_u+s<m$ hold. However, we have $m=\sum i_u+\sum \#A_v$, and at least one of the cardinals $\#A_v$ is strictly greater than one. So the first inequality implies the second, and the class of $\Delta^{I_1}\ldots\Delta^{I_r}O_{o_1}^{A_1}\ldots O_{o_s}^{A_s}$ appears with the coefficient

$$\sum_{j_1=1}^{i_1}\cdots\sum_{j_r=1}^{i_r}\prod_{u=1}^{r}(-1)^{j_u-1}(j_u-1)!\{{}^{i_u}_{j_u}\}.$$

This concludes the proof of Theorem 3 admitting Lemma 1. □

LEMMA 1. *For an integer $i\geqslant 2$ the following sum vanishes*

$$G(i)=\sum_{j=1}^{i}(-1)^{j-1}(j-1)!\{{}^{i}_{j}\}.$$

Proof. Use the well-known identity $\{{}^{i}_{j}\}=j\{{}^{i-1}_{j}\}+\{{}^{i-1}_{j-1}\}$ (one can consult [GKP89, Table 250]) and thus replace $\{{}^{i}_{j}\}$ in the definition of $G(i)$. □

Now we can simplify Theorem 3 to obtain Theorem 2. Note that for integers $i_1,\ldots,i_r$ such that $\sum_{u=1}^{r}i_u\leqslant n$ and points of the curve $o,o_1,\ldots,o_{n-\sum i_u}$, we have

$$[\delta_{i_1,\ldots,i_r}+o_1+\cdots+o_{n-\sum i_u}]=\left[\delta_{i_1,\ldots,i_r}+\left(n-\sum i_u\right)o\right]\quad\text{in }A^{n-r}(C_n).$$

The number of ways to choose $n-\sum i_u$ distinct points in the support of a divisor $D\in\mathcal{G}$, that is, among d elements, is $\binom{d}{n-\sum i_u}$. Theorem 2 follows.

5. Relations in the tautological ring modulo algebraic equivalence

We explain in §5.1 how to deduce from Theorem 2 the relations of Theorem 1 between the components $C_{(i)}$. In §5.2 we discuss the nullity of these relations. We use in §5.2 two lemmas which are proved in §5.3.

5.1 By definition, all of the divisors of a linear system are linearly equivalent. The morphism u_d contracts the r-dimensional variety $\mathcal{G}$ into a point of JC so, by definition of the pushdown, we have

$$u_{d*}[\mathcal{G}]=0\quad\text{in }CH^{g-r}(JC).\tag{8}$$

We have already calculated the class of $\mathcal{G}$ (because $\mathcal{G}=G_d$) in terms of translates of the generalized diagonals $\delta_{i_1,\ldots,i_r}+o_1+\cdots+o_s=\{i_1x_1+\cdots+i_rx_r+o_1+\cdots+o_s\mid x_i\in C\}$. The morphism u_d maps these diagonals onto translates of the variety $i_1C+\cdots+i_rC$ and is one to one. In order to simplify the problem, we will consider relations modulo algebraic equivalence, but one could also express relations modulo the translations in the Chow ring of the Jacobian. From the definition of the Pontryagin product we have $i_{1*}C*\cdots*i_{r*}C=n[i_1C+\cdots+i_rC]$ where n is the degree of the addition morphism.

$$i_1C\times\cdots\times i_rC$$
$$\downarrow{\scriptstyle n:1}$$
$$i_1C+\cdots+i_rC$$

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To each sequence $i_1, \ldots, i_r$ of integers we associate integers $d_1, \ldots, d_u$ such that

$$i_1 = \cdots = i_{d_1}, \quad i_{d_1} \neq i_{d_1+1},$$
$$i_{d_1+1} = \cdots = i_{d_1+d_2}, \quad i_{d_1+d_2} \neq i_{d_1+d_2+1},$$
$$\vdots$$
$$i_{d_1} + \cdots + i_{d_{u-1}+1} = \cdots = i_{d_1} + \cdots + i_{d_u} \quad \text{and} \quad d_1 + \cdots + d_u = r$$

and we write γ for the product

$$\gamma(d, i_1, \ldots, i_r) = d_1! \ldots d_u!$$

The number of antecedents of a generic point of $i_1 C + \cdots + i_r C$ is $d_1! \ldots d_u!$, so we have

$$i_{1*}C * \cdots * i_{r*}C = d_1! \ldots d_u! \, [i_1 C + \cdots + i_r C]. \tag{9}$$

By (8), Theorem 2, and (9), we obtain

$$\sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant n} \frac{(-1)^{i_1+\cdots+i_r}}{i_1 \ldots i_r} \binom{d}{i_1+\cdots+i_r} \frac{1}{d_1! \ldots d_u!} i_{1*}C * \cdots * i_{r*}C = 0.$$

For all integers $i_1, \ldots, i_r$ such that $1 \leqslant i_1 \leqslant \cdots \leqslant i_r$ we count $\binom{r}{d_1, \ldots, d_u}$ different r-tuples $(j_1, \ldots, j_r)$ we obtain by permutations of the i_k. So we can write

$$\sum_{i_1=1}^{d} \cdots \sum_{i_r=1}^{d} \frac{(-1)^{i_1+\cdots+i_r}}{i_1 \ldots i_r} \binom{d}{i_1+\cdots+i_r} i_{1*}C * \cdots * i_{r*}C = 0. \tag{10}$$

Now we use the decomposition (2), the action of the operator i_* onto the components (that is $i_* C_{(a)} = i^{a+2} C_{(a)}$), and multilinearity to write

$$i_{1*}C * \cdots * i_{r*}C = \sum_{a_1=1}^{g} \cdots \sum_{a_r=1}^{g} i_1^{a_1+2} \ldots i_r^{a_r+2} C_{(a_1)} * \cdots * C_{(a_r)}.$$

Then we use the inclusions $A_{(s)}^p(JC) * A_{(t)}^q(JC) \subset A_{(s+t)}^{p+q-g}(JC)$ and we project (10) onto $A_{(s)}^{g-r}(JC)$ for all $s \geqslant 0$ to prove Theorem 1.

5.2 The relations we quote are obtained in $A_{(s)}^{g-r}(JC)$. For small values of s, these relations are trivial. For example, for $s = 0$, the relation is monomial with $C_{(0)}^{*r}$. It should be trivial, otherwise we would obtain the vanishing of a small power of the theta divisor. The first nontrivial relation is obtained for $s = d - r + 1$ as stated in the following proposition.

PROPOSITION 5. *We have*

$$\beta(d, a_1, \ldots, a_r) = \begin{cases} 0 & \text{if } a_1 + \cdots + a_r < d - r + 1 \\ (-1)^d a_1! \ldots a_r! & \text{if } a_1 + \cdots + a_r = d - r + 1. \end{cases}$$

Proof. We denote $t = i_1 + \cdots + i_r$ and expand $i_r^{a_r} = (t - i_1 - \cdots - i_{r-1})^{a_r}$ using Newton's multinomial formula to get

$$\beta(d, a_1, \ldots, a_r) = \sum_{t \geqslant 0} (-1)^t \binom{d}{t} \sum_{\substack{b, b_1, \ldots, b_{r-1} \\ b+b_1+\cdots+b_{r-1}=a_r}} (-1)^{b_1+\cdots+b_{r-1}} \binom{a_r}{b, b_1, \ldots, b_{r-1}}$$

$$\times t^b \sum_{i_1=0}^{t} \sum_{i_2=0}^{t-i_1} \cdots \sum_{i_{r-1}=0}^{t-i_1-\cdots-i_{r-2}} i_1^{a_1+b_1} \ldots i_{r-1}^{a_{r-1}+b_{r-1}}.$$

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By Lemma 2 below, the lower part of the sum above is a polynomial of degree

$$\sum_{i=1}^{r-1} a_i + \sum_{i=1}^{r-1} b_i + b + r - 1 = \sum_{i=1}^{r} a_i + r - 1$$

in t whose leading coefficient is

$$\frac{\prod_{i=1}^{r-1}(a_i + b_i)!}{(\sum_{i=1}^{r-1} a_i + \sum_{i=1}^{r-1} b_i + r - 1)!}.$$

The equality $\sum_{t=0}^{d}(-1)^t \binom{d}{t} t^a = d!\{^a_d\}(-1)^d$ is the result (6.19) in [GKP89]. However, the Stirling number $\{^a_d\}$ vanishes for $a < d$ and equals one when $a = d$. So when $\sum_{i=1}^{r} a_i < d - r + 1$, the sum $\beta(d, a_1, \ldots, a_r)$ vanishes. In the case of equality $\sum_{i=1}^{r} a_i = d - r + 1$, it remains to compute the sum

$$(-1)^d d! \sum_{\substack{b,b_1,\ldots,b_{r-1} \\ b+b_1+\cdots+b_{r-1}=a_r}} \binom{a_r}{b, b_1, \ldots, b_{r-1}} (-1)^{b_1+\cdots+b_{r-1}} \frac{\prod_{i=1}^{r-1}(a_i + b_i)!}{(\sum_{i=1}^{r-1} a_i + \sum_{i=1}^{r-1} b_i + r - 1)!}$$

which is $(-1)^d \prod_{i=1}^{r} a_i!$ by Lemma 3 below. $\qquad\square$

5.3 We now provide the lemmas used in §5.2.

LEMMA 2. *For all positive integers $a_1, a_2, \ldots, a_n$ the sum*

$$\sum_{\substack{0 \leqslant i_1,\ldots,i_n \\ i_1+\cdots+i_n \leqslant t}} i_1^{a_1} \ldots i_n^{a_n}$$

is a polynomial of degree $a_1 + \cdots + a_n + n$ in t. Its leading term is

$$\frac{\prod_{i=1}^{n} a_i!}{(n + \sum_{i=1}^{n} a_i)!}.$$

Proof. Suppose that Lemma 2 is proven up to rank n, and write the sum above with $n+1$ terms in this way:

$$\sum_{i_1=0}^{t} i_1^{a_1} \left(\sum_{\substack{0 \leqslant i_2,\ldots,i_{n+1} \\ i_2+\cdots+i_{n+1} \leqslant t-i_1}} i_2^{a_2} \ldots i_{n+1}^{a_{n+1}} \right). \tag{11}$$

By the induction hypothesis, the term between parentheses is a degree $(a_2+\cdots+a_{n+1}+n)$ polynomial in $t-i_1$. Its leading coefficient is $a_2!\ldots a_{n+1}!/(a_2+\cdots+a_{n+1}+n)!$. We denote it by P. There exists a polynomial Q whose degree is less than $a_2+\cdots+a_{n+1}+n$ and such that

$$P(t) = \frac{a_2!\ldots a_{n+1}!}{(a_2+\cdots+a_{n+1}+n)!} t^{a_2+\cdots+a_{n+1}+n} + Q(t).$$

So (11) is equal to

$$\frac{a_2!\ldots a_{n+1}!}{(a_2+\cdots+a_{n+1}+n)!} \sum_{i_1=0}^{t} i_1^{a_1}(t-i_1)^{a_2+\cdots+a_{n+1}+n} + \sum_{i_1=0}^{t} Q(t-i_1) i_1^{a_1}. \tag{12}$$

We note $N = a_2+\cdots+a_{n+1}+n$ for brevity and develop $(t-i_1)^N$ using Newton's binomial formula. The first sum of expression (12) becomes

$$\sum_{k=0}^{N}(-1)^k t^{N-k} \binom{N}{k} \left(\sum_{i_1=0}^{t} i_1^{a_1+k} \right). \tag{13}$$

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Now we use the induction hypothesis when $n = 1$ to prove that the sum between parentheses in (13) is a degree $a_1 + k + 1$ polynomial in t whose leading term is $1/(a_1 + k + 1)$. So (13) is a polynomial of degree $a_1 + N + 1$ in t whose leading term is

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} \frac{1}{a_1 + k + 1}.$$

The simplification of this last sum given by formula (5.51) in [GKP89] enables us to conclude. □

In § 5.2 we have used the following equality.

LEMMA 3. *For r positive integers $a_1, a_2, \ldots, a_r$, we have*

$$\sum_{\substack{0 \leqslant b, b_1, \ldots, b_{r-1} \\ b+b_1+\cdots+b_{r-1}=a_r}} \binom{a_r}{b, b_1, \ldots, b_{r-1}} (-1)^{b_1+\cdots+b_{r-1}} \frac{\prod_{i=1}^{r-1}(a_i + b_i)!}{(\sum_{i=1}^{r-1} a_i + \sum_{i=1}^{r-1} b_i + r - 1)!} = \frac{\prod_{i=1}^{r} a_i!}{(\sum_{i=1}^{r} a_i + r - 1)!}.$$

Proof. Write the sum in the following way

$$\sum_{\substack{b, b_1, \ldots, b_{r-1} \\ b+b_1+\cdots+b_{r-1}=a_r}} \prod_{i=1}^{r-1} \left((-1)^{b_i} \frac{(a_i + b_i)!}{b_i!} \right) \left(\frac{1}{b! (\sum_{i=1}^{r} a_i + r - 1 - b)!} \right).$$

One could consult [GKP89, p. 321] for the following equalities which hold for all $i \in \{1, \ldots, r-1\}$:

$$\sum_{b_i \geqslant 0} \frac{(-1)^{b_i}(a_i + b_i)!}{b_i!} X^{b_i} = a_i! \sum_{b_i \geqslant 0} \binom{a_i + b_i}{b_i} (-X)^{b_i} = \frac{a_i!}{(1 + X)^{a_i+1}}.$$

We also note that

$$\sum_{b \geqslant 0} \frac{X^b}{b! (\sum_{i=1}^{r} a_i + (r-1) - b)!} = \frac{\sum_{b \geqslant 0} \binom{\sum_{i=1}^{r} a_i + r - 1}{b} X^b}{(\sum_{i=1}^{r} a_i + r - 1)!}$$

$$= \frac{(1 + X)^{\sum_{i=1}^{r} a_i + r - 1}}{(\sum_{i=1}^{r} a_i + r - 1)!},$$

so the sum we study is the product of $\prod_{i=1}^{r} a_i! / (\sum_{i=1}^{r} a_i + r - 1)!$ by the coefficient of X^{a_r} in the formal series

$$\frac{1}{(1 + X)^{a_1+1}} \cdots \frac{1}{(1 + X)^{a_{r-1}+1}} (1 + X)^{a_1+\cdots+a_r+r-1}.$$

This is the coefficient of X^{a_r} in $(1 + X)^{a_r}$, so it is 1. □

6. Linear systems and $(2r - 2)$-secant $r - 2$ planes

The goal of this section is to analyze the relations $C_{(i)} = 0$ we can deduce from Theorem 1. First of all, for $r = 1$, one recovers Colombo and van Geemen's theorem. The study of relations when $r \geqslant 2$ makes the following coefficients appear:

$$A(r, d, g) = \sum_{i=0}^{r-1} \frac{(-1)^i}{d - 2r + 2} \binom{i + g + r - d - 2}{i} \binom{d - 2r}{r - 1 - i} \binom{d - r + 1 - i}{r - i}.$$

Using the relations of Theorem 1 we will prove the following.

THEOREM 4. *If C admits a base point free g_d^r and if $A(r, d, g) \neq 0$, then we have*

$$C_{(i)} = 0 \quad \text{in } A_{(i)}^{g-1} \text{ for } i \geqslant d - 2r + 1.$$

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However, we will explain how Theorem 4 can also be deduced from the theorem of Colombo and van Geemen because $A(r, d, g)$ is the number of projective spaces of dimension $r - 2$ which cut the curve in $2r - 2$ points. So Theorem 4 does not give us new relations, but it is interesting to recover a number of geometrical significance such as the Castelnuovo number from the relations of Theorem 1. It could also suggest the existence of a partial converse to the theorem of Colombo and van Geemen.

6.1 The case of a g_d^1

If C admits a base point free g_d^1 then Theorem 1 gives, for $0 \leqslant a \leqslant g - 1$,

$$\beta(d, a + 1)C_{(a)} = 0 \quad \text{in } A_{(a)}^{g-1}.$$

The coefficients $\beta(d, a)$ can be expressed in terms of the Stirling numbers of the second kind. One can consult formula (6.19) of [GKP89]:

$$\beta(d, a) = \sum_{i=1}^{d} (-1)^i \binom{d}{i} i^a = d! \{_d^a\} (-1)^d.$$

Lastly, $\{_d^{a+1}\}$ is nonzero if and only if $0 \leqslant d \leqslant a + 1$, so we have $C_{(a)} = 0$ for $a \geqslant d - 1$. This is Colombo and van Geemen's result.

6.2 Proof of Theorem 4

Proof. By Proposition 5, the first nontrivial relation is obtained in $A_{(d-2r+1)}^{g-d+2r-1}(JC)$:

$$\sum_{\substack{1 \leqslant a_1 \leqslant \cdots \leqslant a_r \\ a_1 + \cdots + a_r = d - 2r + 1}} \frac{(a_1 + 1)! \ldots (a_r + 1)!}{\gamma(a_1, \ldots, a_r)} C_{(a_1)} * \cdots * C_{(a_r)}. \tag{14}$$

Now multiply (14) by $C_{(0)}^{*(g+r-d-2)}$ and order the terms with respect to the powers of $C_{(0)}$. We introduce t which is the number of integers a_i which are equal to zero. So we have $a_1 = \cdots = a_t = 0$ and $a_{t+1} > 0$. By definition of γ we have $\gamma(a_1, \ldots, a_r) = t! \, \gamma(a_{t+1}, \ldots, a_r)$, and

$$\sum_{t=0}^{r-1} \sum_{\substack{1 \leqslant a_{t+1} \leqslant \cdots \leqslant a_r \\ a_{t+1} + \cdots + a_r = d - 2r + 1}} \frac{(a_{t+1} + 1)! \ldots (a_r + 1)!}{t! \gamma(a_{t+1}, \ldots, a_r)} C_{(0)}^{*(t+g+r-2-d)} * C_{(a_{t+1})} * \cdots * C_{(a_r)} = 0. \tag{15}$$

By Fourier transform, $R_{(d-2r+1)}^{g-1}$ is isomorphic to $R_{(d-2r+1)}^{d-2r+2}$. This subspace is spanned by $C_{(d-2r+1)}$, so its dimension is bounded by 1. The relations Polishchuk obtained enable us to express $C_{(0)}^{*(t+g+r-2-d)} * C_{(a_{t+1})} * \cdots * C_{(a_r)}$ in terms of $C_{(0)}^{*(g+2r-d-3)} * C_{(d-2r+1)}$. The corollary of Lemma 0.3 in [Pol05] gives

$$C_{(0)}^{*(g-1-\sum_{i=1}^{k} n_i)} * C_{(n_1)} * \cdots * C_{(n_k)}$$
$$= \left((-1)^{k-1} \frac{(g - 1 - \sum_{i=1}^{k} n_i)!}{(g + k - 2 - \sum_{i=1}^{k} n_i)!} \frac{(\sum_{i=1}^{k} n_i)!}{n_1! \ldots n_k!} \right) C_{(0)}^{*(g+k-2-\sum_{i=1}^{k} n_i)} * C_{(1-k+\sum_{i=1}^{k} n_i)}.$$

Thus (15) becomes a monomial relation whose coefficient is

$$\sum_{t=0}^{r-1} (-1)^{r+t} \frac{(t + g + r - d - 2)!(d - r + 1 - t)!}{t!(r - t)!} \left(\sum_{\substack{1 \leqslant a_{t+1} \leqslant \cdots \leqslant a_r \\ a_{t+1} + \cdots + a_r = d - 2r + 1}} \frac{(r - t)!}{\gamma(a_{t+1}, \ldots, a_r)} \right). \tag{16}$$

We note that $\gamma(a_{t+1}, \ldots, a_r) = \gamma(a_{t+1} - 1, \ldots, a_r - 1)$ and reindex the sum to obtain

$$\sum_{\substack{1 \leqslant a_{t+1} \leqslant \cdots \leqslant a_r \\ a_{t+1} + \cdots + a_r = d-2r+1}} \frac{(r-t)!}{\gamma(a_{t+1}, \ldots, a_r)} = \sum_{\substack{0 \leqslant b_1 \leqslant \cdots \leqslant b_{r-t} \\ b_1 + \cdots + b_{r-t} = d-3r+1+t}} \frac{(r-t)!}{\gamma(b_1, \ldots, b_{r-t})}.$$

As $(r-t)!/\gamma(b_1, \ldots, b_{r-t})$ is the number of $(r-t)$-tuples we can associate to the integers $\{b_1\}$, $\{b_2\}, \ldots, \{b_{r-t}\}$, the sum corresponds to the number of positive integers whose sum is $d-3r+1+t$. If we consider the formal series $1/(1-X) = 1 + X + X^2 + \cdots$, the integer we are looking for appears as the coefficient of $X^{d-3r+1+t}$ in $(1/(1-X))^{r-t}$, that is $\binom{d-2r}{d-3r+1+t}$. Now factor (16) by $(g+r-d-2)!(d-2r+2)!$ to make the coefficient $A(r,d,g)$ above appear. Note that $C_{(d-2r+1)}$ is (up to a coefficient) the Fourier transform of $C_{(0)}^{*(g+2r-d-3)} * C_{(d-2r+1)}$, so when $A(r,d,g) \neq 0$, we have $C_{(d-2r+1)} = 0$ modulo algebraic equivalence. This concludes the proof of Theorem 4 admitting Lemma 4 below. $\square$

The following lemma is a consequence of the relations obtained by Polishchuk. Marini has also noted and proven it in [Mar05].

LEMMA 4. *Let i be a positive integer. If $C_{(i)} = 0$ in $A_{(i)}^{g-1}$, then $C_{(i+1)} = 0$ in $A_{(i+1)}^{g-1}$.*

Proof. If $g \in \{2, 3\}$, the only components which arise are $C_{(0)}$ and $C_{(1)}$, and the cycle $C_{(0)}$ never vanishes because $-\mathcal{F}(C_{(0)})$ is the class of the theta divisor (see [Bea04]). Now suppose that $g \geqslant 4$. As the gonality of the curve is less than g, apply the theorem of Colombo and van Geemen to obtain $C_{(g-2)} = 0$ and the implication $C_{(g-3)} = 0 \Rightarrow C_{(g-2)} = 0$. Now suppose that $i \leqslant g - 4$. The point is that $R_{(i+1)}^{g-1}$, which is spanned by $C_{(i+1)}$, is isomorphic to $R_{(i)}^{i+1}$, which is spanned by $C_{(0)}^{*(g-i-2)} * C_{(i+1)}$ by [Pol05, Corollary 0.3]. Now consider the cycle $C_{(0)} * C_{(1)} * C_{(i)}$ which is proportional to $C_{(0)}^{*(g-i-2)} * C_{(i+1)}$ by the same corollary. $\square$

6.3 The Castelnuovo number

The Grassmannian $\mathbb{G}(r-1, r+1)$ which parametrises the $(r-2)$-dimensional linear subspaces of $\mathbb{P}^r$ is a $(2r-2)$-fold. The subspaces containing a point of the curve correspond to a codimension one subvariety of $\mathbb{G}(r-1, r+1)$. So one can expect to find a finite number of $\mathbb{P}^{r-2}$ which contains $2r-2$ points of the curve. Actually, the formula of Castelnuovo computes this number when it is finite:

$$B(r,d,g) = \sum_{i=0}^{r-1} \frac{(-1)^i}{r-i} \binom{d-r-i+1}{r-1-i} \binom{d-r-i}{r-1-i} \binom{g}{i}.$$

One could consult [ELMS89] for a proof. When $B(r,d,g)$ is nonzero, consider $2r-2$ such points and then the hyperplanes which contain these points. The linear system we obtain in this way is a g_{d-2r+2}^1 and we can apply the theorem of Colombo and van Geemen. So when $B(r,d,g) \neq 0$, we have $C_{(i)} = 0$ in $A_{(i)}^{g-1}$ for $i \geqslant d - 2r + 1$. We conclude this second proof of Theorem 4 with the following lemma.

LEMMA 5. *For each integer $r \geqslant 1$, we have the equality in $\mathbb{Q}[d,g]$:*

$$A(r,d,g) = B(r,d,g).$$

We use the algorithm of Zeilberger to prove that $A(r,d,g) = B(r,d,g)$. We apply it thanks to the software Mathematica and the package ZW that Paule and Schorne have developed [PS95]. It enables us to compute a linear recurrence relation

$$Q_2(r,d,g)X(r+2,d,g) + Q_1(r,d,g)X(r+1,d,g) + Q_0(r,d,g)X(r,d,g) = 0$$

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TABLE 1. Relations deduced from Theorem 1 for plane curves of genus $g \leqslant 9$.

Genus	g_d^2	$g_{d'}^1$ deduced from the g_d^2	Relations deduced from the g_d^2 with Theorem 1	Consequences
$g = 5$	g_5^2	g_3^1	$3C_{(0)} * C_{(2)} + C_{(1)}^{*2} = 0$	
$g = 6$	g_5^2	g_4^1	$\mathbf{3C_{(0)}C_{(2)} + C_{(1)}^2 = 0}$	
$g = 7$	g_6^2	g_4^1	$2C_{(0)} * C_{(3)} + C_{(1)} * C_{(2)} = 0$	
$g = 8$	g_7^2	g_5^1	$\mathbf{8C_{(1)} * C_{(3)} + 3C_{(2)}^{*2} = 0}$	$C_{(1)} * C_{(3)} = C_{(2)}^{*2} = 0$
	g_6^2	g_4^1	$\mathbf{C_{(1)} * C_{(2)} = 0}$	$C_{(0)} * C_{(1)} * C_{(2)} = 0$
				$C_{(1)}^{*3} = 0$
$g = 9$	g_7^2	g_5^1	$\mathbf{8C_{(1)} * C_{(3)} + 3C_{(2)}^{*2} = 0}$	$C_{(1)} * C_{(3)} = C_{(2)}^{*2} = 0$
				$C_{(0)} * C_{(1)} * C_{(3)} = C_{(0)} * C_{(2)}^{*2} = 0$
				$C_{(1)}^{*2} * C_{(2)} = 0$
	g_6^2	g_4^1	$\mathbf{C_{(1)} * C_{(2)} = 0}$	$C_{(0)} * C_{(1)} * C_{(2)} = 0$
				$C_{(0)}^{*2} * C_{(1)} * C_{(2)} = 0$
				$C_{(0)} * C_{(1)}^{*3} = 0$

with coefficients $Q_i \in \mathbb{Q}[r, d, g]$ that both A and B satisfy. We conclude because the first terms are the same: $A(1, d, g) = B(1, d, g)$ and $A(2, d, g) = B(2, d, g)$. To learn more about Zeilberger's algorithm, one can consult [PWZ96]. One can also find an explanation of the ZW package in [PS95].

7. New relations in the tautological ring for nets and webs

We have seen in the last section that the relations $C_{(i)} = 0$ we can deduce from Theorem 1 could also be deduced from the theorem of Colombo and van Geemen. Do we obtain new relations modulo algebraic equivalence? Yes, we give the new relations for plane and space curves of low genus in Tables 1 and 2.

The integer $A(2, d, g) = (d-1)(d-2)/2 - g$ is the well-known number of nodes for a plane curve of degree d and genus g. So for every plane curve the integer $A(2, d, g)$ is positive. Conversely, when the integers d and g satisfy $g \geqslant 0$ and $(\sqrt{8g+1} + 3)/2 \leqslant d$, there exists a plane curve with such invariants. As explained in the last section, when the curve is singular, one can consider the line passing through a singular point to obtain a g_{d-2}^1. Let us denote $d' = d - 2$ in this case. When the curve is smooth, it only admits a g_{d-1}^1. We denote $d' = d - 1$ in this case. We indicate in Table 1 the relations[6] deduced from Theorem 1. The new relations (those which cannot be deduced from I_g and from Colombo and van Geemen's theorem applied to the $g_{d'}^1$ the curve admits)[7] are written in bold type.

Remark. For g sufficiently large, one can always find d such that a genus g curve exists with a g_d^2 giving us new relations. Let us sketch the proof. The first non-trivial relation given by Theorem 1

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[6]Keep in mind that Polishchuk used the cycles $p_{i+1} = \mathcal{F}(C_{(i)})$ to describe R. So to translate the relations, one has to change $C_{(i)}$ into p_{i+1} and the Pontryagin products into intersection products.

[7]More precisely, to define relations one should consider the kernel $\mathcal{R}$ of the map $\pi : \mathbb{C}[C_{(0)}, \ldots, C_{(g-1)}] \longrightarrow R$ which maps the indeterminate $C_{(i)}$ to the cycle $C_{(i)}$. One could also define (with Polishchuk's formula) a Fourier transform on $\mathbb{C}[C_{(0)}, \ldots, C_{(g-1)}]$ compatible with $\mathcal{F}$. In these terms, a new relation is an element of $\mathcal{R}$ which is not in the smallest ideal containing I_g, $C_{(d'-1)}$ and stable under Fourier transform.

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Table 2. Relations deduced from Theorem 1 for space curves of genus $g \leqslant 9$.

Genus	g_d^3	$g_{d'}^1$ deduced from g_d^3	Relations deduced from g_d^3 with Theorem 1	Consequences
$g = 6$	g_7^3	g_3^1	$C_{(0)} * C_{(1)}^{*2} = 0$	
$g = 7$	g_8^3	g_4^1	$9C_{(0)} * C_{(1)} * C_{(2)} + C_{(1)}^{*3} = 0$	
$g = 8$	g_8^3	g_4^1	$\mathbf{9C_{(0)} * C_{(1)} * C_{(2)} + C_{(1)}^{*3} = 0}$	$\mathbf{C_{(1)} * C_{(2)} = 0}$ $\mathbf{C_{(1)}^{*3} = C_{(0)} * C_{(1)} * C_{(2)} = 0}$
$g = 9$	g_9^3	g_5^1	$\mathbf{8C_{(0)} * C_{(1)} * C_{(3)}}$ $\mathbf{+ 3C_{(0)} * C_{(2)}^{*2}}$ $\mathbf{+ 2C_{(1)}^{*2} * C_{(2)} = 0}$	$\mathbf{C_{(1)} * C_{(3)} = C_{(2)}^{*2} = 0}$ $\mathbf{C_{(1)}^{*2} * C_{(2)} = 0}$ $\mathbf{C_{(0)} * C_{(2)}^{*2} = 0}$ $\mathbf{C_{(0)} * C_{(1)} * C_{(3)} = 0}$
	g_8^3	g_4^1	$\mathbf{9C_{(0)} * C_{(1)} * C_{(2)} + C_{(1)}^{*3} = 0}$	$\mathbf{C_{(0)}^{*m} * C_{(1)} * C_{(2)} = 0}$ $\mathbf{\text{for } m \in \{0\ldots 2\}}$ $\mathbf{C_{(0)}^{*n} * C_{(1)}^{*3} = 0}$ $\mathbf{\text{for } n \in \{0, 1\}}$

is

$$\sum_{\substack{1 \leqslant a,b \\ a+b=d-3}} (a+1)!(b+1)!C_{(a)} * C_{(b)} = 0 \quad \text{in } R_{(d-3)}^{g-2}. \tag{17}$$

The relations of I_g correspond to subspaces $R_{(i)}^{d+i}$ for $0 \leqslant d \leqslant g/2 - 1$ and $d + 1 \leqslant i \leqslant g - d - 1$ (one has to see how the relations of the Theorem 0.1 of [Pol05] are parametrized). So when the inequality $d < g/2 + 2$ holds, there is no relation of I_g which belongs to $R_{(d-3)}^{g-2}$. Finally, if $d' = d - 1$, there is no relation deduced from the existence of the $g_{d'}^1$ in $R_{(d-3)}^{g-2}$. If $d' = d - 2$, the only relation we can deduce from the $g_{d'}^1$ in $R_{(d-3)}^{g-2}$ is $C_{(0)} * C_{(d-3)} = 0$ which is different from (17).

The integer $A(3, d, g)$ is the number of quadrisecants to the space curve $\Phi(C)$. Let us recall Cayley's formula

$$A(3, d, g) = \frac{(d-2)(d-3)^2(d-4)}{12} - \frac{g(d^2 - 7d + 13 - g)}{2}.$$

When $A(3, d, g)$ is non-zero, the curve admits a g_{d-4}^1 and so $C_{(d-5)}$ is zero modulo algebraic equivalence. We note $d' = d - 4$ in this case. Otherwise, we deduce from the existence of a g_{d-3}^1 the vanishing of $C_{(d-4)}$. We note $d' = d - 3$. We give in Table 2 the results we obtain by Theorem 1 for space curves of low genus. As above, a new relation is written in bold type. It means the relation cannot be deduced from the theorem of Colombo and van Geemen applied to the $g_{d'}^1$ and from the relations of I_g. With the same argument as in the case of a g_d^2, one can show that for g sufficiently large there exists a genus g space curve for which Theorem 1 gives us new relations.

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