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THE BEST LEAST SQUARES APPROXIMATION PROBLEM FOR A 3-PARAMETRIC EXPONENTIAL REGRESSION MODEL

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Abstract

Given the data (p_i, t_i, f_i), $i = 1, \dots, m$, we consider the existence problem for the best least squares approximation of parameters for the 3-parametric exponential regression model. This problem does not always have a solution. In this paper it is shown that this problem has a solution provided that the data are strongly increasing at the ends.

1. Introduction

The mathematical model described by an exponential function or a linear combination of such functions is often used in applied research (see [5, 7–10, 14, 17, 19]). For example, in [14] it is used in the analysis of long-term selection experiments in biology. These models are also frequently used as a test function for testing numerical algorithms for function minimization (see [1, 3, 15, 17, 21]).

In this paper we consider the existence problem for the best least squares approximation of parameters in the 3-parametric exponential regression model

$$f(t; a, b, c) = a + be^{ct}. \tag{1.1}$$

Suppose we are given the experimental or empirical data (p_i, t_i, f_i), $i = 1, \dots, m$, $m \geq 3$, where $t_1 < t_2 < \cdots < t_m$ denote the values of the independent variable, f_i are the respective measured function values and $p_i > 0$ are the data weights. The least squares problem for the 3-parametric exponential function (1.1) then becomes

Whether there exists a point $(a^*, b^*, c^*) \in \mathbb{R}^3$, such that

$$F(a^*, b^*, c^*) = \inf_{(a,b,c)\in\mathbb{R}^3} F(a, b, c), \tag{1.2}$$

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where

$$F(a,b,c)=\sum_{i=1}^{m}p_i\left(a+be^{ct_i}-f_i\right)^2 \quad ? \tag{1.3}$$

The same problem can also be considered in l_p-norms, but usually one uses only l_1, l_2 and l_∞. Some remarks about this approximation problem in l_1- and l_∞-norms can be found in [1, 11] and [12], and some special cases for the l_2-norm are dealt with in [2]. The analysis of the solution existence problem in l_1- and l_∞-norms differs significantly from the l_2 case.

The problem (1.2)–(1.3) is a special case of the so-called nonlinear least squares problem, for which some special numerical methods and algorithms have been developed (see [1, 3, 13, 15]).

The problem (1.2)–(1.3) does not always have a solution (see Lemma 1). In this paper, we give sufficient conditions which guarantee the existence of a solution. The existence problem of the best approximation for some other model functions in the sense of ordinary least squares is taken up in [4] and [16] and in the sense of total least squares in [17] and [18].

2. The existence theorem

Suppose that we are given the data (p_i, t_i, f_i), $i = 1, \dots, m$, $m \geq 3$, such that $p_i > 0$, $i = 1, \dots, m$, and $t_1 < t_2 < \dots < t_m$.

The following lemma, which we are going to use in the proof of Theorem 1, shows that there exist data such that the problem (1.2)–(1.3) has no solution.

LEMMA 1. *If the points (t_i, f_i), $i = 1, \dots, m$, $m \geq 3$, all lie on some slanted line $y = kt + l$, $k \neq 0$, then the problem* (1.2)–(1.3) *has no solution.*

PROOF. Let (t_0, y_0) be an arbitrary point on the line $y = kt + l$. Then

$$y_0 + k(t_i - t_0) - f_i = 0, \quad i = 1, \dots, m.$$

Since

$$\lim_{c\to 0} F\left(y_0 - \frac{1}{c}, \frac{e^{-ckt_0}}{c}, ck\right) = \lim_{c\to 0}\sum_{i=1}^{m}p_i\left(y_0 + \frac{e^{ck(t_i-t_0)}-1}{c} - f_i\right)^2$$
$$= \sum_{i=1}^{m}p_i\left(y_0 + k(t_i - t_0) - f_i\right)^2 = 0,$$

this means that $\inf_{(a,b,c)\in\mathbb{R}^3} F(a,b,c) = 0$. Furthermore, since the graph of any function of the type (1.1) intersects the line $y = kt + l$, $k \neq 0$, in at most two points, and $m \geq 3$, it follows that $F(a, b, c) > 0$ for all $(a, b, c) \in \mathbb{R}^3$, and hence the problem (1.2)–(1.3) has no solution.

Therefore, in order to guarantee the existence of a solution of the problem (1.2)–(1.3), it is necessary to require that the data (p_i, t_i, f_i), $i = 1, \dots, m$, $m \geq 3$, satisfy some additional conditions.

DEFINITION 1. We say that the data (p_i, t_i, f_i), $i = 1, \dots, m$, $m \geq 3$, are *strongly monotonic at the ends*, provided

$$\left(f_m - \bar{f_m}\right)\left(f_{m-1} - \bar{f_m}\right) > 0 \quad \text{and} \quad \left(f_1 - \bar{f_1}\right)\left(f_2 - \bar{f_1}\right) > 0, \tag{2.1}$$

where $\bar{f_m} = \left(\sum_{i=1}^{m-1} p_i f_i\right) / \sum_{i=1}^{m-1} p_i$, $\bar{f_1} = \left(\sum_{i=2}^{m} p_i f_i\right) / \sum_{i=2}^{m} p_i$.

In this paper we show that the existence problem (1.2)–(1.3) always has a solution whenever the data are strongly monotonic at the ends.

REMARK 1. Before formulating and proving the existence theorem, let us notice the following simple but important facts:

- From the existence problem point of view, the origin of the coordinate system can be moved along the t-axis. Namely, after the transformation

$$\tau_i = T + t_i, \quad i = 1, \dots, m,$$

the functional F takes the form:

$$F(a, b, c) = \Phi(a, \beta, c) = \sum_{i=1}^{m} p_i \left(a + \beta e^{c\tau_i} - f_i\right)^2,$$

where $\beta = be^{-cT}$. The functional Φ is of the same type as F, and the map $(a, b, c) \mapsto (a, \beta, c)$ is a bijection of $\mathbb{R}^3$ onto $\mathbb{R}^3$.

- For every $\alpha \in \mathbb{R}$ we have

$$\sum_{i=1}^{m} p_i (\alpha - f_i)^2 \geq \sum_{i=1}^{m} p_i (\bar{f} - f_i)^2,$$

where $\bar{f} = \left(\sum_{i=1}^{m} p_i f_i\right) / \sum_{i=1}^{m} p_i$.

THEOREM 1. *Let the data (p_i, t_i, f_i), $i = 1, \dots, m$, $m \geq 3$, be strongly monotonic at the ends. Then the problem* (1.2)–(1.3) *has a solution if and only if the points (t_i, f_i), $i = 1, \dots, m$, are not collinear lying on a slanted line.*

PROOF. If the problem (1.2)–(1.3) has a solution, then by Lemma 1 the points (t_i, f_i), $i = 1, \dots, m$, do not lie on a slanted line.

Let us show the converse. Suppose that the points (t_i, f_i), $i = 1, \dots, m$, do not lie on a slanted line. We show that the problem (1.2)–(1.3) has a solution.

According to Remark 1 we can suppose that

$$t_1 < t_2 = 0 < t_3 < \cdots < t_m. \tag{2.2}$$

Since $F \geq 0$, there exists $F^\star := \inf_{(a,b,c)\in\mathbb{R}^3} F(a, b, c)$. Let (a_n, b_n, c_n) be a sequence in $\mathbb{R}^3$, such that

$$F^\star = \lim_{n\to\infty} F(a_n, b_n, c_n) = \lim_{n\to\infty} \sum_{i=1}^{m} p_i r_i^2(n) < +\infty, \tag{2.3}$$

where $r_i(n) = a_n + b_n\, e^{c_n t_i} - f_i$.

We show that the sequence (a_n, b_n, c_n) is bounded by showing that the sequences (a_n) and (b_n, c_n) are bounded. Without loss of generality, whenever we have a bounded sequence, we are going to assume it is convergent—otherwise by the Bolzano-Weierstrass theorem, we take a convergent subsequence.

I. We first show that the sequence (a_n) is bounded. Suppose it is not. Then, without loss of generality, by taking appropriate subsequences if necessary, only one of the following cases can occur: (a) $a_n \to +\infty$ or (b) $a_n \to -\infty$.

Since $r_2(n) = a_n + b_n - f_2$, in case (a) we have $b_n \to -\infty$, and in case (b) we have $b_n \to \infty$. Otherwise we would have $\lim_{n\to\infty} r_2^2(n) = \infty$, which by (2.3) would imply $F^\star = \infty$. Therefore we have to show that the functional F cannot attain its infimum in either of the following two ways:

1. $a_n \to +\infty, \quad b_n \to -\infty$; 2. $a_n \to -\infty, \quad b_n \to +\infty$.

I.1. Consider the case $(a_n \to \infty,\ b_n \to -\infty)$. Since $F^\star$ is a real number, and $r_2(n) = a_n + b_n - f_2$, the sequence $(a_n + b_n)$ has to be bounded. Let

$$l := \lim_{n\to\infty} (a_n + b_n).$$

If the sequence (c_n) is unbounded, by taking an appropriate subsequence, we may assume that the sequence (c_n) diverges toward $+\infty$ or $-\infty$. In both cases, by (2.2) we can choose $t_{i_0} \in \{t_1, \dots, t_m\}$, such that $c_n t_{i_0} > 0$. Since $b_n < 0$, we have

$$a_n + b_n e^{c_n t_{i_0}} \leq a_n + b_n(1 + c_n t_{i_0})$$

and taking the limits we obtain

$$\lim_{n\to\infty} (a_n + b_n e^{c_n t_{i_0}}) \leq \lim_{n\to\infty} (a_n + b_n) + \lim_{n\to\infty} b_n c_n t_{i_0} = l - \infty = -\infty. \tag{2.4}$$

This means that the sequence (c_n) cannot diverge, because otherwise, from (2.3) and (2.4), we would obtain $F^* = \infty$.

Suppose $c_n \to \tilde{c}$. If $\tilde{c} \neq 0$, then by the same argument as in the case of the unbounded sequence (c_n), we would obtain $F^* = \infty$. Therefore, we have to show that the functional F can not attain its infimum F^* when $a_n \to \infty$, $b_n \to -\infty$, $c_n \to 0$. Suppose on the contrary that $a_n \to \infty$, $b_n \to -\infty$ and $c_n \to 0$. Then by the Lagrange mean value theorem, for every $n \in \mathbb{N}$ there exist real numbers $\vartheta_i(n) \in (0, 1)$, $i = 1, \dots, m$, such that

$$a_n + b_n e^{c_n t_i} = a_n + b_n + b_n c_n t_i e^{\vartheta_i(n) c_n t_i}, \quad i = 1, \dots, m. \tag{2.5}$$

I.1.(a) If $\lim_{n\to\infty}(b_n c_n) = 0$, then by (2.5) and (2.3) we have

$$F^* = \sum_{i=1}^{m} p_i (l - f_i)^2.$$

By Remark 1,

$$F_1 := \sum_{i=1}^{m} p_i \left(\bar{f} - f_i\right)^2 \leq \sum_{i=1}^{m} p_i (l - f_i)^2 = F^*. \tag{2.6}$$

Let us show that there is a point in $\mathbb{R}^3$ at which the functional F attains a value which is smaller than F_1. Note that $\bar{f} \neq f_m$, because otherwise from

$$\left(\sum_{i=1}^{m} p_i\right) \bar{f} = \left(\sum_{i=1}^{m-1} p_i\right) \bar{f}_m + p_m f_m,$$

we would have $f_m = \bar{f}_m$, which contradicts the inequality (2.1). Therefore

$$\begin{aligned}\lim_{c\to\infty} F\left(\bar{f}, (f_m - \bar{f})e^{-ct_m}, c\right) &= \lim_{c\to\infty} \sum_{i=1}^{m} p_i \left[(f_m - \bar{f})e^{c(t_i - t_m)} - (f_i - \bar{f})\right]^2 \\ &= \sum_{i=1}^{m-1} p_i (\bar{f} - f_i)^2 < \sum_{i=1}^{m} p_i (\bar{f} - f_i)^2 = F_1.\end{aligned}$$

From this and (2.6) we conclude that there exists a real number $c' > 0$, such that $F\left(\bar{f}, (f_m - \bar{f})e^{-c' t_m}, c'\right) < F_1 \leq F^*$. This means that in this way the functional F can not attain its infimum F^*.

I.1.(b). If $\lim_{n\to\infty}(b_n c_n) = k \neq 0$, then from (2.5) we obtain

$$\lim_{n\to\infty} \left(a_n + b_n e^{c_n t_i}\right) = k t_i + l, \quad i = 1, \dots, m.$$

Therefore

$$F^* = \lim_{n\to\infty} F(a_n, b_n, c_n) = \sum_{i=1}^{m} p_i (k t_i + l - f_i)^2.$$

In order to show that in this way also the functional F cannot attain its infimum, we define m functions $\phi_r : (-\infty, \infty) \to \mathbb{R}$, $r = 1, \ldots, m$, by the formula:

$$\phi_r(c) = \begin{cases} \sum_{i=1}^{m} p_i \left[kt_r + \frac{e^{kc(t_i - t_r)} - 1}{c} - (f_i - l) \right]^2, & c \neq 0, \\ \sum_{i=1}^{m} p_i (kt_i + l - f_i)^2, & c = 0. \end{cases} \tag{2.7}$$

Note that $\phi_r(c) = F(l + kt_r - 1/c, e^{-kct_r}/c, kc)$ for every $c \neq 0$. Furthermore, since

$$\lim_{c \to 0} \frac{e^{kc(t_i - t_r)} - 1}{c} = k(t_i - t_r), \tag{2.8}$$

we have

$$\lim_{c \to 0} \phi_r(c) = \sum_{i=1}^{m} p_i (kt_i + l - f_i)^2 = \phi_r(0) = F^*.$$

This means that all functions ϕ_r, $r = 1, \ldots, m$, are continuous on $\mathbb{R}$.

The derivative of the function ϕ_r ($\forall r = 1, \ldots, m$) is

$$\frac{d\phi_r(c)}{dc} = \begin{cases} 2 \sum_{i=1}^{m} p_i \left[kt_r + \frac{e^{kc(t_i - t_r)} - 1}{c} - (f_i - l) \right] \frac{e^{kc(t_i - t_r)}[kc(t_i - t_r) - 1] + 1}{c^2}, & c \neq 0, \\ \sum_{i=1}^{m} p_i (kt_i + l - f_i) k^2 (t_i - t_r)^2, & c = 0. \end{cases}$$

Since

$$\lim_{c \to 0} \frac{e^{kc(t_i - t_r)}[kc(t_i - t_r) - 1] + 1}{c^2} = \frac{k^2 (t_i - t_r)^2}{2},$$

by (2.8) we obtain

$$\lim_{c \to 0} \frac{d\phi_r(c)}{dc} = \sum_{i=1}^{m} p_i (kt_i + l - f_i) k^2 (t_i - t_r)^2 = \frac{d\phi_r(0)}{dc}, \quad r = 1, \ldots, m, \tag{2.9}$$

which means that all derivatives $d\phi_r(c)/dc$, $r = 1, \ldots, m$, are also continuous on $\mathbb{R}$.

Let us show that there exists an $r \in \{1, \ldots, m\}$, such that $d\phi_r(0)/dc \neq 0$. Namely, if $d\phi_r(0)/dc = 0$ for all $r = 1, \ldots, m$, then, since $k \neq 0$, by (2.9) the vector

$$\mathbf{e} := (kt_1 + l - f_1, kt_2 + l - f_2, \ldots, kt_m + l - f_m)^T$$

would be orthogonal to each of the m vectors

$$\mathbf{e}_1 := \begin{bmatrix} 0 \\ (t_2 - t_1)^2 \\ (t_3 - t_1)^2 \\ \vdots \\ (t_m - t_1)^2 \end{bmatrix}, \quad \mathbf{e}_2 := \begin{bmatrix} (t_1 - t_2)^2 \\ 0 \\ (t_3 - t_2)^2 \\ \vdots \\ (t_m - t_2)^2 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_m := \begin{bmatrix} (t_1 - t_m)^2 \\ \vdots \\ (t_{m-1} - t_m)^2 \\ 0 \end{bmatrix}.$$

Since the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ are linearly independent in the space $\mathbb{R}^m$, this would imply that $\mathbf{e}$ is a null-vector, that is, that the points (t_i, f_i), $i = 1, \ldots, m$, lie on the line $y = kt + l$, $k \neq 0$, which contradicts our assumption. Therefore there exists an $r_0 \in \{1, \ldots, m\}$, such that $d\phi_{r_0(0)}/dc \neq 0$.

Now, since $d\phi_{r_0}(0)/dc \neq 0$, the function ϕ_{r_0} is either strictly increasing or strictly decreasing on some ε-neighbourhood of the point $c = 0$. Therefore, there is a point $c' \in (-\varepsilon, \varepsilon)$, such that $\phi_{r_0}(c') < \phi_{r_0}(0)$. From (2.7) we now obtain

$$\phi_{r_0}(c') = F\left(l + kt_{r_0} - \frac{1}{c'}, \frac{e^{-kc't_{r_0}}}{c'}, kc'\right) < \phi_{r_0}(0) = F^\star.$$

Therefore, also in this way, the functional F cannot attain its infimum.

I.2. Consider the case $a_n \to -\infty$, $b_n \to +\infty$. Reasoning similarly as in Case I.1, one can show that in this way the functional F cannot attain its infimum.

Therefore we have proved that the sequence (a_n) is bounded, so we may assume it is also convergent. Let $a_n \to a^\star$.

II. We will now show that the sequence (b_n, c_n) is bounded. Suppose it is not. Without loss of generality, by taking appropriate subsequences if necessary, only one of the following cases can occur:

1.	$b_n \to \pm\infty$,	$c_n \to +\infty$;	2.	$b_n \to \pm\infty$,	$c_n \to -\infty$;
3.	$b_n \to \pm\infty$,	$c_n \to c^\star \in \mathbb{R}$;			
4.	$b_n \to b^\star \neq 0$,	$c_n \to +\infty$;	5.	$b_n \to b^\star \neq 0$,	$c_n \to -\infty$;
6.	$b_n \to 0$,	$c_n \to +\infty$;	7.	$b_n \to 0$,	$c_n \to -\infty$.

Because of the assumption (2.2), in Cases 1–5 it is easy to see that the sequence $F(a_n, b_n, c_n)$ diverges to $+\infty$. Namely, in the first and fourth case we have $r_i^2(n) \to \infty$ $(i \geq 3)$, in the second and fifth case $r_1^2(n) \to \infty$, while in the third case $r_i^2(n) \to \infty$ for all $i = 1, \ldots, m$. This means that the infimum $F^\star$ cannot be attained in any of these five cases. Hence, it remains to consider Cases 6 and 7.

Case 6. Suppose that $b_n \to 0$ and $c_n \to +\infty$. Note that

$$L := \lim_{n\to\infty} b_n e^{c_n t_m} < \infty,$$

because otherwise $r_m^2(n) \to \infty$, which contradicts (2.3). Because of the assumption $t_1 < t_2 < \cdots < t_m$, we have

$$\lim_{n\to\infty} b_n e^{c_n t_i} = \lim_{n\to\infty} b_n e^{c_n t_m} \cdot \lim_{n\to\infty} e^{c_n(t_i - t_m)} = L \cdot 0 = 0, \quad i = 1, \ldots, m-1.$$

Therefore

$$F^\star = \lim_{n\to\infty} F(a_n, b_n, c_n) \geq \sum_{i=1}^{m-1} p_i (a^\star - f_i)^2.$$

By Remark 1,

$$\sum_{i=1}^{m-1} p_i(a^\star - f_i)^2 \geq \sum_{i=1}^{m-1} p_i(\bar{f}_m - f_i)^2,$$

and therefore we obtain

$$F^\star \geq \sum_{i=1}^{m-1} p_i(\bar{f}_m - f_i)^2.$$

We are going to find a point $(b', c') \in \mathbb{R}^2$, such that $F(\bar{f}_m, b', c') < F^\star$. For that purpose we examine the behaviour of the functional F on the curve $\Gamma_1 \subset \mathbb{R}^3$ defined by

$$\Gamma_1 = \left\{(\bar{f}_m, b, c) \in \mathbb{R}^3 : b = (f_m - \bar{f}_m)e^{-ct_m}, c \geq 0\right\}.$$

The restriction of the functional F to the curve Γ_1 (see Figure 1(a)) can be considered as a function $\varphi : [0, \infty) \to \mathbb{R}$ given by

$$\varphi(c) = F\left(\bar{f}_m, (f_m - \bar{f}_m)e^{-ct_m}, c\right) = \sum_{i=1}^{m-1} p_i\left[(f_m - \bar{f}_m)e^{c(t_i - t_m)} - (f_i - \bar{f}_m)\right]^2.$$

Note that

$$\lim_{c\to\infty} \varphi(c) = \sum_{i=1}^{m-1} p_i(f_i - \bar{f}_m)^2 \leq F^\star. \tag{2.10}$$

The derivative of the function φ equals

$$\begin{aligned}\frac{1}{2}\frac{d\varphi(c)}{dc} &= \sum_{i=1}^{m-1} p_i\left[(f_m - \bar{f}_m)e^{c(t_i - t_m)} - (f_i - \bar{f}_m)\right](f_m - \bar{f}_m)(t_i - t_m)e^{c(t_i - t_m)} \\ &= e^{c(t_{m-1} - t_m)}\sum_{i=1}^{m-1} p_i\left[(f_m - \bar{f}_m)e^{c(t_i - t_m)} - (f_i - \bar{f}_m)\right] \\ &\quad \times (f_m - \bar{f}_m)(t_i - t_m)e^{c(t_i - t_{m-1})}.\end{aligned} \tag{2.11}$$

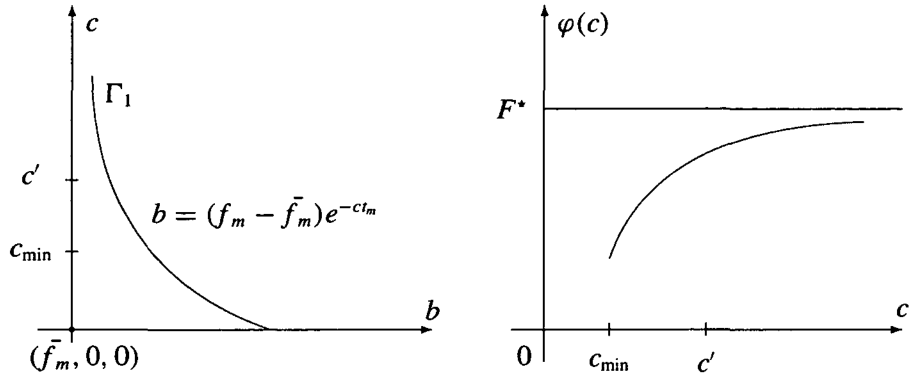
Let us show that for c large enough, the sign of the derivative $d\phi/dc$ equals the sign of the $(m-1)$-st summand in (2.11).

Because $t_i - t_m < 0$, $\forall i = 1, \dots, m-2$, we have

$$\lim_{c\to\infty} \sum_{i=1}^{m-2} p_i\left[(f_m - \bar{f}_m)e^{c(t_i - t_m)} - (f_i - \bar{f}_m)\right](f_m - \bar{f}_m)(t_i - t_m)e^{c(t_i - t_{m-1})} = 0.$$

Furthermore, because $t_{m-1} - t_m < 0$ and, by our assumption $(f_m - \bar{f}_m)(f_{m-1} - \bar{f}_m) > 0$, we have

$$\lim_{c\to\infty} p_{m-1}\left[(f_m - \bar{f}_m)e^{c(t_{m-1} - t_m)} - (f_{m-1} - \bar{f}_m)\right](f_m - \bar{f}_m)(t_{m-1} - t_m) > 0.$$



(a) The curve Γ_1. (b) The function φ.

FIGURE 1.

This means that the derivative of the function φ is strictly positive whenever c is large enough. Therefore there is a real number $c_{\min}$ such that the function φ is strictly increasing on the interval $(c_{\min}, \infty)$. Hence for every $c' \in (c_{\min}, \infty)$ we have (see Figure 1(b))

$$\varphi(c') = F\left(\bar{f}_m, (f_m - \bar{f}_m)e^{-c't_m}, c'\right) < \lim_{c\to\infty} \varphi(c). \tag{2.12}$$

For the chosen $c' \in (c_{\min}, \infty)$, let $b' = (f_m - \bar{f}_m)e^{-c't_m}$. Then $(\bar{f}_m, b', c') \in \mathbb{R}^3$ and from (2.12) and (2.10) we get

$$F\left(\bar{f}_m, b', c'\right) < \lim_{c\to\infty} \varphi(c) \le F^\star.$$

This means that in this way also one cannot obtain the infimum of the functional F, because we found a point $(\bar{f}_m, b', c') \in \mathbb{R}^3$, where the functional F attains a value smaller than $F^\star$.

Case 7. Suppose that $b_n \to 0$ and $c_n \to -\infty$.

Arguing similarly as in the sixth case, one can show that in this case also the functional F cannot attain its infimum in this way.

In this case

$$L := \lim_{n\to\infty} b_n e^{c_n t_1} < \infty,$$

because otherwise $r_1^2(n) \to \infty$, which contradicts (2.3). Furthermore, because of the assumption $t_1 < t_2 < \cdots < t_m$, we have

$$\lim_{n\to\infty} b_n e^{c_n t_i} = \lim_{n\to\infty} b_n e^{c_n t_1} \cdot \lim_{n\to\infty} e^{c_n(t_i - t_1)} = L \cdot 0 = 0, \quad i = 1, \dots, m-1.$$

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Therefore

$$F^\star = \lim_{n\to\infty} F(a_n, b_n, c_n) \geq \sum_{i=2}^{m} p_i(a^\star - f_i)^2.$$

By Remark 1,

$$\sum_{i=2}^{m} p_i(a^\star - f_i)^2 \geq \sum_{i=2}^{m} p_i(\bar{f}_1 - f_i)^2,$$

and thus we obtain

$$F^\star \geq \sum_{i=2}^{m} p_i(\bar{f}_1 - f_i)^2.$$

In this case, by using the function $\varphi : (-\infty, 0] \to \mathbb{R}$ given by

$$\varphi(c) = F\left(\bar{f}_1, (f_1 - \bar{f}_1)e^{-ct_1}, c\right) = \sum_{i=2}^{m} p_i\left[(f_1 - \bar{f}_1)e^{c(t_i - t_1)} - (f_i - \bar{f}_1)\right]^2,$$

we are going to show that there exists a point $(b', c') \in \mathbb{R}^2$, such that $F(\bar{f}_m, b', c') < F^\star$.

Note that

$$\lim_{c\to-\infty} \varphi(c) = \sum_{i=2}^{m} p_i(f_i - \bar{f}_1)^2 \leq F^\star. \tag{2.13}$$

The derivative of the function φ is given by

$$\begin{aligned}\frac{1}{2}\frac{d\varphi}{dc} &= \sum_{i=2}^{m} p_i\left[(f_1 - \bar{f}_1)e^{c(t_i - t_1)} - (f_i - \bar{f}_1)\right](f_1 - \bar{f}_1)(t_i - t_1)e^{c(t_i - t_1)} \\ &= e^{c(t_2 - t_1)} \sum_{i=2}^{m} p_i\left[(f_1 - \bar{f}_1)e^{c(t_i - t_1)} - (f_i - \bar{f}_1)\right](f_1 - \bar{f}_1)(t_i - t_1)e^{c(t_i - t_2)}.\end{aligned}$$

Because $t_i - t_1 > 0, \forall i = 3, \dots, m$, we have

$$\lim_{c\to\infty} \sum_{i=3}^{m} p_i\left[(f_1 - \bar{f}_1)e^{c(t_i - t_1)} - (f_i - \bar{f}_1)\right](f_1 - \bar{f}_1)(t_i - t_1)e^{c(t_i - t_2)} = 0.$$

Furthermore, because $t_2 - t_1 > 0$, and by our assumptions $(f_1 - \bar{f}_1)(f_2 - \bar{f}_1) > 0$, we have

$$\lim_{c\to-\infty} p_2\left[(f_1 - \bar{f}_1)e^{c(t_2 - t_1)} - (f_2 - \bar{f}_1)\right](f_1 - \bar{f}_1)(t_2 - t_1) < 0.$$

This means that the derivative of the function φ is strictly negative for sufficiently large negative real c. Therefore there exists a real number $c_{\max}$ such that the function φ

is strictly decreasing on the interval $(-\infty, c_{\max})$. Hence for every $c' \in (-\infty, c_{\max})$ we have

$$\varphi(c') = F\left(\bar{f_1}, (f_1 - \bar{f_1})e^{-c't_1}, c'\right) < \lim_{c\to\infty} \varphi(c). \tag{2.14}$$

For the chosen $c' \in (-\infty, c_{\max})$, let $b' = (f_1 - \bar{f_1})e^{-c't_1}$. Then $(\bar{f_1}, b', c') \in \mathbb{R}^3$, and from (2.14) and (2.13), we get

$$F\left(\bar{f_1}, b', c'\right) < \lim_{c\to\infty} \varphi(c) \le F^\star.$$

This means that in this way also one cannot obtain the infimum of the functional F, because we found a point $(\bar{f_1}, b', c') \in \mathbb{R}^3$, where the functional F attains a value smaller than $F^\star$.

Hence the sequence (a_n, b_n, c_n) is bounded. By the Bolzano-Weierstrass theorem, we may assume that the sequence (a_n, b_n, c_n) is convergent (otherwise we take a convergent subsequence). Let $(a_n, b_n, c_n) \to (a^\star, b^\star, c^\star)$. By the continuity of the functional F, we have $\inf_{(a,b,c)\in\mathbb{R}^3} F(a, b, c) = \lim_{n\to\infty} F(a_n, b_n, c_n) = F(a^\star, b^\star, c^\star)$. This completes the proof of Theorem 1.

3. Some remarks on data strongly monotonic at the ends

As proved in Theorem 1, if the data (p_i, t_i, f_i), $i = 1, \ldots, m$, $m \ge 3$, are strongly monotonic at the ends, then the problem (1.2)–(1.3) has a solution if and only if the points (t_i, f_i), $i = 1, \ldots, m$, do not lie on a slanted line. The class of data which are strongly monotonic at the ends is a rather wide one. For example, it is not difficult to show that the data (p_i, t_i, f_i), $i = 1, \ldots, m$, $m \ge 3$, are strongly monotonic at the ends, provided they are either strictly increasing or strictly decreasing.

We are going to show that the wide class of data which are strongly monotonic in the mean (see [2, 6]) are also strongly monotonic at the ends.

DEFINITION 2. Let (p_i, t_i, f_i), $i = 1, \ldots, m$, $m \ge 3$, be the data. In the case when

$$\frac{\sum_{k=1}^{i} p_k f_k}{\sum_{k=1}^{i} p_k} < f_{i+1} < \frac{\sum_{k=i+2}^{m} p_k f_k}{\sum_{k=i+2}^{m} p_k}, \quad i = 1, \ldots, m-2, \tag{3.1}$$

we say that the data are *strongly increasing in the mean*, and in the case when

$$\frac{\sum_{k=1}^{i} p_k f_k}{\sum_{k=1}^{i} p_k} > f_{i+1} > \frac{\sum_{k=i+2}^{m} p_k f_k}{\sum_{k=i+2}^{m} p_k}, \quad i = 1, \ldots, m-2, \tag{3.2}$$

that they are *strongly decreasing in the mean.*

We say that the data are *strongly monotonic in the mean*, provided they are either strongly increasing in the mean, or they are strongly decreasing in the mean.

In other words, the data (p_i, t_i, f_i), $i = 1, \dots, m$, are strongly increasing in the mean [strongly decreasing in the mean] if and only if the ordinate of each datum (except the first and the last one), is strictly bigger [strictly smaller] than the mean value of the ordinate of the data preceding it, and also strictly smaller [strictly bigger] than the mean value of the ordinate of the data following it.

PROPOSITION 1. *If the data are strongly monotonic in the mean, then they are also strongly monotonic at the ends.*

PROOF. Suppose the data (p_i, t_i, f_i), $i = 1, \dots, m$, are strongly increasing in the mean. From the inequality (3.1) we obtain

$$\bar{f}_m \cdot \sum_{i=1}^{m-1} p_i = \sum_{i=1}^{m-2} p_i f_i + p_{m-1} f_{m-1} = \left(\sum_{i=1}^{m-2} p_i\right)\left(\frac{\sum_{k=1}^{m-2} p_k f_k}{\sum_{k=1}^{m-2} p_k}\right) + p_{m-1} f_{m-1}$$
$$< \left(\sum_{i=1}^{m-2} p_i\right) f_{m-1} + p_{m-1} f_{m-1} = f_{m-1} \cdot \sum_{i=1}^{m-1} p_i,$$

which implies $\bar{f}_m < f_{m-1}$. Since by (3.1) $f_{m-1} < f_m$, it follows that $f_m > f_{m-1} > \bar{f}_m$, which implies the inequality

$$(f_m - \bar{f}_m)(f_{m-1} - \bar{f}_m) > 0.$$

It remains to prove $(f_1 - \bar{f}_1)(f_2 - \bar{f}_1) > 0$.

From the inequality (3.1) we have

$$\bar{f}_1 \cdot \sum_{i=2}^{m} p_i = p_2 f_2 + \sum_{i=3}^{m} p_i f_i = p_2 f_2 + \left(\sum_{i=3}^{m} p_i\right)\left(\frac{\sum_{k=3}^{m} p_k f_k}{\sum_{k=3}^{m} p_k}\right)$$
$$> p_2 f_2 + \left(\sum_{i=3}^{m} p_i\right) f_2 = f_2 \cdot \sum_{i=2}^{m} p_i,$$

and therefore $\bar{f}_1 > f_2$. Since by (3.1) $f_2 > f_1$, it follows that $\bar{f}_1 > f_2 > f_1$, which implies the required inequality

$$(f_1 - \bar{f}_1)(f_2 - \bar{f}_1) > 0.$$

Therefore, if the data are strongly increasing in the mean, then they are also strongly increasing at the ends.

For data which are strongly decreasing in the mean, the proof is similar.

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