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A CHARACTERIZATION OF IDEALS OF C^*-ALGEBRAS

BY

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ABSTRACT. Let A be a C^*-algebra and let I be a C^*-subalgebra of A. Denote by $\overline{\varphi}$ an extension of a state φ of B to a state of A. It is shown that I is an ideal of A if and only if there exists a homomorphism Q from A^{**} onto I^{**} such that Q is the identity map on I^{**} and $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on I. Furthermore it is also shown that I is an essential ideal of A if and only if there exists an injective homomorphism from A into the multiplier algebra of I which is the identity map on I.

Dedicated to Professor Niro Yanagihara in Celebration of his Sixtieth Birthday

1. **Introduction.** Let A be a C^*-algebra and let B be a C^*-subalgebra of A. We denote by A^{**} the enveloping von Neumann algebra of A, which is identified with the second dual of A. Then the enveloping von Neumann algebra B^{**} of B is identified with the strong closure of B in A^{**} (e.g., [6, 3.7.9]). In [4], the author showed that B is a hereditary C^*-subalgebra of A if and only if there exists a projection of norm one Q from A^{**} onto B^{**} such that $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on B where $\overline{\varphi}$ denotes an extension of a state φ of B to a state of A. Since every closed ideal is a hereditary C^*-subalgebra, it is natural to investigate additional conditions which Q mentioned above should satisfy in order that a hereditary C^*-subalgebra should become an ideal. In this note, it is shown that a hereditary C^*-subalgebra become an ideal if and only if Q above is a homomorphism. Now recall that a closed ideal I of A is said to be essential if each non-zero closed ideal of A has a non-zero intersection with I. It is well known that if I is an essential ideal of A, then Q is injective on A (see [6, Proposition 3.12.8]). As a consequence of the above result, it is also shown that in order that a closed ideal I should be essential in A, it is necessary and sufficient that Q is injective on A. In the remainder of the paper, we discuss an application of this result to C^*-dynamical systems. In fact, let (A, G, α) be a C^*-dynamical system where G is amenable. Then it is shown that an α-invariant C^*-subalgebra I of A is an essential ideal of A if and only if the C^*-crossed product $I \times_\alpha G$ is an essential ideal of the C^*-crossed product $A \times_\alpha G$.

2. **Results.** Let A be a C^*-algebra and let I be a C^*-subalgebra of A. Troughout this paper, the identity of the von Neumann subalgebra I^{**} of A^{**} is always denoted by p, which is a projection of A^{**}. Let φ be a positive linear functional on I and denote by $\overline{\varphi}$ a norm-preserving extension of φ to A.

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THEOREM 2.1. *Let A be a C^*-algebra and let I be a C^*-subalgebra of A. Then the following conditions are equivalent:*

(1) I is an ideal of A.

(2) There exists a homomorphism Q from A^{**} onto I^{**} such that Q is the identity map on I^{**} and $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on I.

In addition, Q is uniquely determined as the form $Q(\cdot) = p \cdot p$ and Q maps A into the multiplier algebra $M(I)$ of I.

Furthermore, the homomorphism Q is injective on A if and only if I is essential.

PROOF. (1) $\Longrightarrow$ (2). Since I is an ideal, we have $I^{**} = A^{**}p$ for some open central projection p in A^{**}. Define a projection of norm one Q from A^{**} onto I^{**} by

$$Q(x) = xp$$

for $x \in A^{**}$. Then Q is a homomorphism and it follows from [4, Theorem 2.2] that $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on I. Take any element x from A. For any $y \in I$, we have

$$Q(x)y = xpy = xy \in I,$$

which implies that $Q(x) \in M(I)$.

(2) $\Longrightarrow$ (1). Since $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on I, I is a hereditary C^*-subalgebra of A (see [4, Theorem 2.2]), and hence Q is uniquely written as $Q(\cdot) = p \cdot p$ with the open projection $p \in A^{**}$ satisfying $I^{**} = pA^{**}p$. In order to prove condition (1), it suffices to show that p is a central projection in A^{**}. Take any element x from A^{**}. Denote by Ker$Q$ the kernel of Q. We now assert that

$$p(x - Q(x)) = (x - Q(x))p.$$

Assume that $p(x - Q(x)) \neq (x - Q(x))p$. Since $x - Q(x) \in \mathrm{Ker}Q$, there exists a positive element a in Ker$Q$ such that $pa \neq ap$. Since $Q(a) = pap$, we have $a^{\frac{1}{2}}p = 0$. Hence we obtain that $ap = 0$, which contradicts that $pa \neq ap$. We thus see that $p(x - Q(x)) = (x - Q(x))p$. Since $pQ(x) = Q(x)p$, we have

$$px = p(Q(x) + (x - Q(x))) = Q(x)p + (x - Q(x))p = xp.$$

Finally we assume that Q is injective on A. In order to prove that I is essential, we show that if a closed ideal J of A satisfies $I \cap J = \{0\}$, then $J = \{0\}$. Assume that $I \cap J = \{0\}$. Let x any element in J. We then have $xy = 0$ for all $y \in I$. Hence we have

$$Q(x)y = Q(xy) = 0.$$

Since $Q(x) \in M(I)$, the above equality shows that $Q(x) = 0$ i.e., $x = 0$. Thus we complete the proof. Q.E.D.

In the above theorem, the condition that Q be injective is very strong. If, in condition(2), we assume such a condition instead of the condition that $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on I, we can show that I is an ideal. In fact, we have the following.

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PROPOSITION 2.2. *Let A be a C^*-algebra and let I be a C^*-subalgebra of A. Then the following conditions are equivalent:*

(1) I is an essential ideal of A.

(2) There exists an injective homomorphism Q from A into the multiplier algebra M(I) of I which is the identity map on I.

PROOF. We have only to prove the implication (2) $\Longrightarrow$ (1). For any $x \in A$ and $y \in I$, we have

$$Q(xy) = Q(x)y \in M(I)I \subset I,$$

and hence $xy - Q(xy) \in A$. Since we have $Q(xy - Q(xy)) = 0$ and since Q is injective on A, we obtain that $xy - Q(xy) = 0$. We thus see that $xy = Q(xy) \in I$, which implies that I is an ideal of A. It then follows from the last paragraph of the proof of the above theorem that I is essential in A. Q.E.D.

By a C^*-dynamical system, we mean a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and a group homomorphism α from G into the automorphism group of A such that $G \ni t \longrightarrow \alpha_t(x)$ is continuous for each x in A. We denote by $A \times_\alpha G$ the C^*-crossed product of A by G, which is the enveloping C^*-algebra of the Banach *-algebra $L^1(A, G)$ of all Bochner integrable A-valued functions on G (see [6, 7.6]).

Let π be a representation of A on a Hilbert space H. Then the covariant representation $(\overline{\pi}, \lambda, L^2(H, G))$ is given by

$$(\overline{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t),$$
$$(\lambda_s\xi)(t) = \xi(s^{-1}t)$$

for $a \in A$, $s \in G$ and $\xi \in L^2(H, G)$ where $L^2(H, G)$ denotes the Hilbert space of all square integrable H-valued functions on G. The *regular representation* of $A \times_\alpha G$ induced by (π, H) is the representation $(\overline{\pi} \times \lambda, L^2(H, G))$ defined by

$$((\overline{\pi} \times \lambda)(x)\xi)(t) = \int_G (\overline{\pi}(x(s))\lambda_s\xi)(t)\, ds$$

for $x \in L^1(A, G)$ and $\xi \in L^2(H, G)$ (see [6, 7.7]). Let π be faithful. Then $(\overline{\pi} \times \lambda)(A \times_\alpha G)$ is called the reduced C^*-crossed product of A by G, which is denoted by $A \times_{\alpha r} G$ ([6, 7.7]).

It is well known that I is an α-invariant ideal of A, then $I \times_\alpha G$ is an ideal of $A \times_\alpha G$ ([2, Proposition 12], or [3, Lemma 4]).

COROLLARY 2.3. *Let (A, G, α) be a C^*-dynamical system. Let I be an α-invariant C^*-subalgebra of A. Then the following conditions are equivalent:*

(1) I is an ideal of A.

(2) There exists a homomorphism Q from A^{**} onto I^{**} such that Q is the identity map on I^{**} and $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on I.

(3) $I \times_\alpha G$ is an ideal of $A \times_\alpha G$.

(4) $I \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$ and there exists a homomorphism $\hat{Q}$ from $(A \times_\alpha G)^{**}$ onto $(I \times_\alpha G)^{**}$ such that $\hat{Q}$ is the identity map on $(I \times_\alpha G)^{**}$ and $\overline{\psi} \circ \hat{Q} = \overline{\psi}$ for every state ψ on $I \times_\alpha G$.

PROOF. The equivalence of (1) and (2) and that of (3) and (4) follow from Theorem 2.1. We have only to show the implication (3) $\Longrightarrow$ (1). Condition (3) shows that $I \times_{\alpha r} G$

is an ideal of $A \times_{\alpha r} G$. Consider the dual coaction δ of G on $A \times_{\alpha r} G$. Clearly $I \times_{\alpha r} G$ is invariant under δ (see [5, 4.1]). Denote by $(A \times_{\alpha r} G) \times_\delta G$ (resp. $(I \times_{\alpha r} G) \times_\delta G$) the crossed product of $A \times_{\alpha r} G$ (resp. $I \times_{\alpha r} G$) by δ. It then follows from [5, Proposition 4.5] that $(I \times_{\alpha r} G) \times_\delta G$ is an ideal of $(A \times_{\alpha r} G) \times_\delta G$. By duality for the dual coaction (e.g., [5, Theorem 6.3]), $I \otimes C(L^2(G))$ is an ideal of $A \otimes C(L^2(G))$, where $C(L^2(G))$ denotes the C^*-algebra of all compact operators on $L^2(G)$. This means that I is an ideal of A. Q.E.D.

In the above corollary, since p is G-invariant, Q is a G-invariant homomorphism. This fact will be used in the next proposition.

PROPOSITION 2.4. *Let (A, G, α) be a C^*-dynamical system. Let I be an α-invariant C^*-subalgebra of A. If $I \times_\alpha G$ is an essential ideal of $A \times_\alpha G$, then I is an essential ideal of A. If G is amenable, the converse also holds.*

PROOF. Suppose that $I \times_\alpha G$ is an essential ideal of $A \times_\alpha G$. Let Q and $\hat{Q}$ be as in Corollary 2.3. By Theorem 2.1, $\hat{Q}$ is injective on $A \times_\alpha G$, hence so on the multiplier algebra $M(A \times_\alpha G)$. Let (π, u, H) be the covariant representation of A corresponding to the universal representation $(\pi \times u, H)$ of $A \times_\alpha G$ (cf. [6, 7.6.4]). As described in [4, Remark 3.3], $\hat{Q}$ has the form $\hat{Q} = (\pi^{**} \circ Q) \times u$ where π^{**} denotes the normal extension of π to A^{**}. Since $\hat{Q}\,|_A$ is nothing but $\pi^{**} \circ Q$ and $A \subset M(A \times_\alpha G)$, Q is injective on A. By Theorem 2.1, I is an essential ideal of A.

Suppose that G is amenable and that I is an essential ideal of A. Since $I \times_\alpha G$ is an ideal of $A \times_\alpha G$, we have only to prove that $I \times_\alpha G$ is essential. Let π be a faithful representation of I on a Hilbert space H, which we can regard as a faithful representation of $M(I)$. In addition, α is extended to an automorphism group, also denoted by α, of $M(I)$. Denote by $(\overline{\pi}, \lambda, L^2(H, G))$ the covariant representation of $M(I)$ induced by (π, H). Consider the covariant representation $(\overline{\pi \circ Q}, \lambda, L^2(H, G))$ of A induced by $(\pi \circ Q, H)$. Since Q is a G-invariant homomorphism, i.e., $\alpha \circ Q = Q \circ \alpha$, it is easy to verify that $\overline{\pi \circ Q} = \overline{\pi} \circ Q$. Now we consider the (faithful) regular representation $(\overline{\pi} \times \lambda, L^2(H, G))$ of $I \times_\alpha G$ ([6, Corollary 7.7.8]). For any $x \in L^1(A, G)$, $(\overline{\pi \circ Q} \times \lambda)(x)$ belongs to the C^*-algebra generated by $\overline{\pi}(M(I)) \cup \lambda_G$. Hence, we see that $(\overline{\pi \circ Q} \times \lambda)(x) \in (\overline{\pi} \times \lambda)(I \times_\alpha G)''$. Since $(\overline{\pi \circ Q} \times \lambda)(x)(\overline{\pi} \times \lambda)(I \times_\alpha G) \subset (\overline{\pi} \times \lambda)(I \times_\alpha G)$, $(\overline{\pi \circ Q} \times \lambda)(x)$ is a multiplier of $(\overline{\pi} \times \lambda)(I \times_\alpha G)$. Thus $\overline{\pi \circ Q} \times \lambda$ maps $A \times_\alpha G$ into the multiplier algebra $M((\overline{\pi} \times \lambda)(I \times_\alpha G))$ of $(\overline{\pi} \times \lambda)(I \times_\alpha G)$, and $\overline{\pi \circ Q} \times \lambda = \overline{\pi} \times \lambda$ on $I \times_\alpha G$. Since $\pi \circ Q$ is injective, $\overline{\pi \circ Q} \times \lambda$ is faithful on $A \times_\alpha G$. Identifying $(\overline{\pi} \times \lambda)(I \times_\alpha G)$ with $I \times_\alpha G$, $\overline{\pi \circ Q} \times \lambda$ is regarded as an injective homomorphism from $A \times_\alpha G$ into $M(I \times_\alpha G)$ which is the identity map on $I \times_\alpha G$. By Proposition 2.2, $I \times_\alpha G$ is an essential ideal of $A \times_\alpha G$. This completes the proof. Q.E.D.

REFERENCES

1. C. J. K. Batty, M. Kusuda, *Weak expectations in C^*-dynamical systems*, J. Math. Soc. Japan **40** No. 4 (1988) 662–669.

2. P. Green, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978) 191-250.

3. M. Kusuda, *Hereditary C^*-subalgebras of C^*-crossed products*, Proc. Amer. Math. Soc. **102** No. 1 (1988) 90–94.

4. M. Kusuda, *Unique state extension and hereditary C^*-subalgebras*, Math. Ann. (to appear).

5. M. B. Landstad, J. Phillips, I. Raeburn, C. E. Sutherland, *Representations of crossed products by coactions and principal bundles*, Trans. Amer. Math. Soc. **299** No. 2 (1987) 747-784.

6. G. K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London and New York, 1979.

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