Canad. Math. Bull. Vol. 67 (3), 2024, pp. 796–804
http://dx.doi.org/10.4153/S0008439524000171

Moments of the central L-values of the Asai lifts

Wenzhi Luo

Abstract. We study some analytic properties of the Asai lifts associated with cuspidal Hilbert modular forms, and prove sharp bounds for the second moment of their central L-values.

1 Introduction

Let $\mathbf{F}$ be a fixed real quadratic field over $\mathbf{Q}$, with ring of integers $O = O_{\mathbf{F}}$ and the real imbeddings $\sigma_1 = 1,\ \sigma_2$. For simplicity, we assume the narrow class number of $\mathbf{F}$ is 1, so the totally positive units are squares of units and every ideal has a totally positive generator. Let $SL(2, O)$ be the Hilbert modular group. For any ideal $\mathfrak{C} \subset O$, the Hecke congruence subgroups $\Gamma_0(\mathfrak{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O), \quad c \equiv 0 \pmod{\mathfrak{C}}\right\}$, act discontinuously on the upper half-space $\mathbf{H}^2$ in the usual way with finite co-volumes, i.e., for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{C}) \text{ and } z = (z_1,\ z_2) \in \mathbf{H}^2,$$

we have

$$\gamma(z) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)},\ \frac{\sigma_2(a)z_1 + \sigma_2(b)}{\sigma_2(c)z_1 + \sigma_2(d)} \right).$$

Denote by $M_k(\Gamma_0(\mathfrak{C}))(k \in 2\mathbf{Z} \text{ and } \geq 2)$, the space of Hilbert modular forms of parallel even weight (k, k), level $\mathfrak{C}$ with trivial character, i.e., the space of holomorphic functions $f(z)$ on $\mathbf{H}^2$ such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{C}), f(\gamma(z)) = N(cz + d)^k f(z)$, where for $z = (z_1,\ z_2) \in \mathbf{H}^2$,

$$N(cz + d)^k = (\sigma_1(c)z_1 + \sigma_1(d))^k \cdot (\sigma_2(c)z_2 + \sigma_2(d))^k.$$

Any $f(z)$ in $M_k(\Gamma_0(\mathfrak{C}))$ has the following Fourier expansion (we assume that the different of $\mathbf{F}$ is generated by $\delta = \delta_{\mathbf{F}} > 0$, where and henceforth $\xi > 0$ for $\xi \in \mathbf{F}$ means that ξ is a totally positive element in $\mathbf{F}$, and denote $\nu^{(i)} = \sigma_i(\nu)$, the ith conjugate of ν

Received by the editors October 17, 2023; accepted February 25, 2024.
Published online on Cambridge Core March 4, 2024.
This research is partially supported by a Simons Foundation Collaboration Grant.
AMS subject classification: **11F41, 11F30, 11f66.**
Keywords: Hilbert modular form, Asai lift, central L-values, Petersson formula.

for $i = 1, 2$):

$$f(z) = \sum_{\nu \in O,\ \nu \geq 0} a(\nu) \exp(2\pi i \mathrm{Tr}(\nu z)), \tag{1}$$

where

$$\mathrm{Tr}(\nu z) = \sum_{i=1}^{2} \nu^{(i)} z_i \delta^{(i)-1}.$$

Since any $f(z)$ in $M_k(\Gamma_0(\mathcal{C}))$ is invariant under $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is an unit in O, we have $a(\varepsilon^2 \nu) = a(\nu)$.

$f(z) \in M_k(\Gamma_0(\mathcal{C}))$ is called a Hilbert modular cusp form if the Fourier expansion of $f(g(z))N(cz+d)^{-k}$ (see [Lu, p. 130]) has no constant term for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{F})$. Space of all such cusp forms is denoted by $S_k(\Gamma_0(\mathcal{C}))$.

It is well-known (see [Ga]) that $\dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C}))$ is finite, and (see [Sh]) $J =: \dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C})) \sim \frac{\mathrm{vol}(\Gamma_0(\mathcal{C})\backslash \mathbf{H}^2)}{(4\pi)^2}(k-1)^2$ as $k \to \infty$. Moreover,

$$\begin{aligned} \mathrm{vol}(\Gamma_0(\mathcal{C})\backslash \mathbf{H}^2)) &= [SL(2, O) : \Gamma_0(\mathcal{C})] \ \mathrm{vol}(SL(2, O)\backslash \mathbf{H}^2) \\ &= 2N(\mathcal{C}) \prod_{\mathcal{P}|\mathcal{C}} (1 + N(\mathcal{P})^{-1}) \times \pi^{-2} \zeta_{\mathbf{F}}(2) D^{3/2}, \end{aligned}$$

where $\zeta_{\mathbf{F}}(s)$ is the Dedekind zeta-function of $\mathbf{F}$ and $D = D_{\mathbf{F}}$ is the discriminant. The Petersson inner product on $S_k(\Gamma)$ is defined by

$$< g_1,\ g_2 >= \int_{\Gamma\backslash \mathbf{H}^2} g_1(z)\overline{g_2(z)} \prod_{i=1}^{2} y_i^{k-2} dx_i dy_i,$$

where $z = (z_1,\ z_2)$ with $z_i = x_i + y_i\sqrt{-1},\ i = 1, 2$.

Now, let f be a cuspidal Hilbert modular form of parallel weight (k, k) for even $k \geq 2$ and with respect to $GL^+(2, O) \supset SL(2, O)$. We assume f is a normalized Hecke eigenform with Fourier coefficients $a_f(\nu) = a_f(1)\lambda_f(\nu)N(\nu)^{(k-1)/2}$, $\nu \in O$, where $\lambda_f(\mu)$ is the eigenvalue of $f(z)$ for the Hecke operator $T_{(\mu)}$ (see, e.g., [Ga]). We have

$$\lambda_f(\mu)\lambda_f(\nu) = \sum_{(d),\ d|(\mu,\nu),\ d>0} \lambda_f\left(\frac{\mu\nu}{d^2}\right).$$

The standard L-function associated with f is defined, for $\Re(s) > 1$, by

$$L(s,\ f) = \sum_{(\mu),\ \mu>0} \lambda_f(\mu)N(\mu)^{-s},$$

which has Euler product

$$\prod_{(\pi),\ \pi>0} (1 - \lambda_f(\pi)N(\pi)^{-s} + N(\pi)^{-2s})^{-1},$$

where π stands for prime element of O. It is well-known that $L(s, f)$ has analytic continuation to the whole complex plane as an entire function. Let

$$\Lambda(s,f) = (2\pi)^{-2s}\Gamma^2(s+(k-1)/2)L(s,f).$$

We then have the functional equation

$$\Lambda(s,f) = \varepsilon_f D^{1-2s}\Lambda(1-s,f),$$

where ε_f is the root number of absolute value 1.

Asai [As] defined a new Dirichlet series by restricting the coefficients on rational integers,

$$L(s,\mathrm{As}(f)) = \zeta(2s)\sum_{m=1}^{\infty}\lambda_f(m)m^{-s},\ \Re(s) > 1.$$

He showed that the function

$$\Lambda(s,\mathrm{As}(f)) = D^{s/2}(2\pi)^{-2s}\Gamma(s+k-1)\Gamma(s)L(s,\mathrm{As}(f))$$

admits analytic continuation to the whole s-plane with possible simple poles at $s = 0, 1$, and satisfies the functional equation

$$\Lambda(s,\mathrm{As}(f)) = \Lambda(1-s,\mathrm{As}(f)).$$

Moreover, if

$$\begin{aligned} L(s,f) &= \prod_{(\pi),\ \pi>0}(1-\lambda_f(\pi)N(\pi)^{-s}+N(\pi)^{-2s})^{-1} \\ &= \prod_{(\pi),\ \pi>0}[(1-\alpha_f(\pi)N\pi^{-s})(1-\beta_f(\pi)N\pi^{-s})]^{-1}, \end{aligned}$$

then we have

$$L(s,\mathrm{As}(f)) = \prod_p L_p(s),$$

where

$$L_p^{-1}(s) = \begin{cases} (1-\alpha_f(\pi_1)\alpha_f(\pi_2)p^{-s})(1-\alpha_f(\pi_1)\beta_f(\pi_2)p^{-s}) & \\ (1-\beta_f(\pi_1)\alpha_f(\pi_2)p^{-s})(1-\beta_f(\pi_1)\beta_f(\pi_2)p^{-s}), & \text{if}\quad p=\pi_1\pi_2, \pi_1\neq\pi_2; \\ (1-\alpha_f(\pi)p^{-s})(1-\beta_f(\pi)p^{-s})(1-p^{-2s}), & \text{if}\quad p=\pi; \\ (1-\alpha_f^2(\pi)p^{-s})(1-\beta_f^2(\pi)p^{-s})(1-p^{-s}), & \text{if}\quad p=\pi^2. \end{cases}$$

Ramakrishnan [Ra] and Krishnamurthy [Kr] proved that $\Lambda(s,\mathrm{As}(f))$ is in fact the L-function associated with an automorphic form on $GL(4, A_{\mathbf{Q}})$, the Asai lift $\mathrm{As}(f)$ of f. Then, in view of the Splitting Formula in [As] and assuming $D = D_{\mathbf{F}}$ is odd, we have

$$L(s, f\otimes f^t) = L(s,\mathrm{As}(f))\ L(s,\mathrm{As}(f)\otimes\chi_D),$$

where

$$\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$$

is the Kronecker symbol, and

$$f^t(z_1, z_2) = f(z_2, z_1).$$

If f is a base change from an Hecke eigenform $h \in S_k(SL_2(\mathbf{Z}))$, then f is symmetric, i.e., $f = f^t$, and

$$L(s, \mathrm{As}(f)) = L(s, \mathrm{sym}^2(h))\, L(s, \chi_D),$$

while if f is a base change from an Hecke eigenform $h \in S_k(\Gamma_0(D), \chi_D)$, then also $f = f^t$, and

$$L(s, \mathrm{As}(f)) = L(s, \mathrm{sym}^2(h))\, \zeta(s)$$

(see [As, Section 5]).

Moreover, Prasad and Ramakrishnan [PR] established the following (special case of) cuspidal criterion for $\mathrm{As}(f)$.

Theorem 1.1 (Prasad and Ramakrishnan) *With the same notation as above. If f is non-dihedral, then $\mathrm{As}(f)$ is non-cuspidal iff f and f^t are twist-equivalent; if f is dihedral, then $\mathrm{As}(f)$ is non-cuspidal iff f is induced from a quadratic extension K of F which is biquadratic over* $\mathbf{Q}$.

Choosing an orthonormal basis $\{f_j(z)\}_{j=1}^J$ of $S_k(\Gamma_0(\mathfrak{C}))$ and denote the Fourier coefficients of $f_j(z)$ by $a_j(\cdot)$. We normalize the Fourier coefficients $a_j(\mu)$ by

$$\psi_j(\mu) = \left(\frac{N(\mathfrak{C})((k-1)!)^2 D^{k+1}}{((4\pi)^2 N(\mu))^{k-1}}\right)^{1/2} a_j(\mu).$$

We then have the Petersson formula for Hilbert modular forms as proved in [Lu],

$$\sum_{j=1}^{J} \bar{\psi}_j(\nu)\psi_j(\mu) = \chi_\nu(\mu) D^{3/2} N(\mathfrak{C})(k-1)^2 + N(\mathfrak{C})(k-1)^2 D(2\pi)^2 \sum_{\varepsilon \in U} \sum_{c \in \mathfrak{C}^\times/U}$$

$$\frac{1}{|N(c)|} S(\nu, \mu\varepsilon^2; c)\; NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|), \tag{2}$$

where χ_ν is the characteristic function of the set $\{\nu\varepsilon^2,\ \varepsilon \in U\}$, U is the unit group of $\mathbf{F}$,

$$S(\nu, \mu; c) = \sum_{h \pmod{c}}{}^{*}\, e\left(\frac{\nu h + \mu \bar{h}}{c}\right)$$

is the generalized Kloosterman sum, and $e(x) = \exp(2\pi i \mathrm{Tr}(x))$ for $x \in \mathbf{F}$. We will assume that in the above formula, the c's are chosen among their associates the representatives satisfying $|N(c)|^{1/2} \ll |c^{(i)}| \ll |N(c)|^{1/2},\ i = 1, 2$.

If the L^2-normalized basis element $f_j = \tilde{f}_j/\|\tilde{f}_j\|$ is a newform, where $\tilde{f}_j$ is the corresponding *arithmetically* normalized newform with the first Fourier coefficient 1, then $\psi_j(\mu) = \psi_j(1)\, \lambda_j(\mu)$, where $\lambda_j(\cdot)$ denotes the (normalized) Hecke eigenvalues of f_j as noted above. For $\mathfrak{C} = (1)$, from the integral representation for $L(s,\ \tilde{f}_j \otimes \overline{\tilde{f}_j})$, and the factorization $L(s,\ \tilde{f}_j \otimes \overline{\tilde{f}_j}) = \zeta_{\mathbf{F}}(s)\ L(s,\ \mathrm{ad}(\tilde{f}_j))$, we have

$$|a_j(1)|^{-2} = \|\tilde{f}_j\|^2 = 16D^{1+k}(4\pi)^{-2k-2}\Gamma^2(k)\ L(1,\ \mathrm{ad}(\tilde{f}_j))/L(1,\ \chi_D).$$

Thus for $\mathcal{C} = (1)$,

$$\bar{\psi}_j(\nu)\ \psi_j(\mu) = \frac{(4\pi)^4 L(1,\ \mathrm{ad}(\tilde{f}_j))}{16L(1,\ \chi_D)}\lambda_j(\nu)\ \lambda_j(\mu).$$

For each $j,\ 1 \le j \le J$ and any $\varepsilon > 0$, we have (see [Ta])

$$\lambda_j(\mu) \ll N(\mu)^{\varepsilon},$$

and by a straightforward extension of results of [Iw] and [HL] that

$$k^{-\varepsilon} \ll L(1,\ \mathrm{ad}(\tilde{f}_j)) \ll k^{\varepsilon}.$$

In [Lu], we proved an asymptotic formula for the mean value of the linear form in $\psi_j(\cdot)$ in the level aspect. In this paper, we establish an analogous result for the weight aspect as well in the context of the quadratic field **F**, with an application to the second moment of $L(1/2, \mathrm{As}(f))$. The generalization of Theorem 1.2 to the general totally real fields is straightforward.

Theorem 1.2 *Let $b(\cdot)$ be an arbitrary complex numbers such that $b(\varepsilon^2\mu) = b(\mu)$ for $\varepsilon \in U$, and $\eta > 0$. Then for $S_k(\Gamma_0(\mathcal{C}))$, we have as $k \to \infty$,*

$$\sum_{j=1}^{J}\left|\sum_{\mu} b(\mu)\psi_j(\mu)\right|^2 \ll (N(\mathcal{C})k^2 + X)(kXN(\mathcal{C}))^{\eta}\sum_{\mu}|b(\mu)|^2,$$

*where the summation over μ's is restricted to $\mu \in O^{\times}/U^2,\ \mu > 0,\ N(\mu) \le X$, and the implicit constant only depends on the quadratic field **F** and η.*

Assume $As(f)$ is cuspidal. From [IK, p. 98], we have a series representation for the central L-value of $L(s, As(f))$,

$$L(1/2,\ As(f)) = 2\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^{1/2}}V_{1/2}\left(\frac{n}{\sqrt{D}}\right), \tag{3}$$

where

$$V_{1/2}(y) = \frac{1}{2\pi i}\int_{(2)}(4\pi^2 y)^{-u}\ \zeta(1+2u)\frac{\Gamma(1/2+u)\ \Gamma(k+u-1/2)}{\Gamma(1/2)\ \Gamma(k-1/2)}\frac{du}{u}.$$

Since

$$\frac{\Gamma(k+u-1/2)}{\Gamma(k-1/2)} \ll k^{\Re(u)}$$

by Stirling's formula, we see that $V_{1/2}(y) \ll k^{-A}$ for any $A \ge 1$, if $y > k^{1+\eta}$ for any $\eta > 0$. Thus, we have

$$L(1/2,\ As(f)) = 2\sum_{n \le k^{1+\eta}}\frac{\lambda_f(n)}{n^{1/2}}V_{1/2}\left(\frac{n}{\sqrt{D}}\right) + O(1).$$

From Theorem 1.2 and the above formula for $L(1/2, As(f))$, and by extending the orthonormal Hecke basis of $S_k(GL_2^+(O))$ to an orthonormal (Hecke) basis of $S_k(SL(2,O))$ and the positivity, we obtain the following theorem.

Theorem 1.3 *For the orthonormal Hecke basis $\{f_j\}$ of $S_k(GL_2^+(O))$ and any $\eta > 0$, we have*

$$\sum_{1\le j\le J}{}^{*} |L(1/2,\ As(f_j))|^2 \ll k^{2+\eta},$$

where the $*$ means that the summation is restricted to cuspidal Asai lifts $As(f_j)$, and the constant implicit only depends on the quadratic field $\mathbf{F}$ and η.

It remains to prove Theorem 1.2, which is the goal of the next section.

2 Proof of the Theorem 1.2

From the Poisson integral representation [GR, p. 953, (8)], we have

$$\begin{aligned} J_{k-1}(x) &= \frac{\left(\frac{x}{2}\right)^{k-1}}{\sqrt{\pi}\,\Gamma(k-1/2)} \int_{-1}^{1} (1-t^2)^{k-3/2}\cos(xt)\,dt \\ &\ll \left(\frac{ex}{2k}\right)^{k-1}, \end{aligned} \tag{4}$$

where the implicit constant is absolute.

To prove Theorem 1.2, we may assume that μ's are chosen among their associates mod U^2 the representatives satisfying $N(\nu)^{1/2} \ll \nu^{(i)} \ll N(\nu)^{1/2},\ i = 1, 2$. We have by the Petersson formula (2),

$$\begin{aligned} &\sum_{j=1}^{J} |\sum_{\mu} b(\mu)\psi_j(\mu)|^2 \\ &= \sum_{\mu,\,\nu} b(\mu)\bar{b}(\nu) \sum_{j=1}^{J} \psi_j(\mu)\bar{\psi}_j(\nu) \\ &= \sum_{\mu} |b(\mu)|^2) D^{3/2}(k-1)^2 N(\mathfrak{C}) \\ &\quad + (k-1)^2 D N(\mathfrak{C})(2\pi)^2 \sum_{\varepsilon\in U} \sum_{c\in\mathfrak{C}^\times/U} \\ &\quad \times \frac{1}{|N(c)|} \sum_{\mu,\,\nu} b(\mu)\bar{b}(\nu) S(\nu, \mu\varepsilon^2; c)\ N J_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|) \\ &= \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.} \end{aligned}$$

We first prove Theorem 1.2 under the condition that $k^2 N(\mathfrak{C}) \ge 8(4\pi)^2 X$. In view of (4) and bound $|J_{k-1}(y)| \le 1$, we have $J_{k-1}(y) \ll \left(\frac{ey}{2k}\right)^{k-1-\eta'} \ll \left(\frac{2y}{k}\right)^{k-1-\eta'}$, for $y > 0$ and $0 \le \eta' < 1/2$, we have (choosing η' to be 0 or η, $0 < \eta < 1/2$ depending upon whether $|\varepsilon^{(i)}| \geq 1$ or not)

$$NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|)) \ll (4(4\pi)^2\sqrt{(N\mu)(N\nu)}/k^2|N(c)|)^{k-1}(k^2|N(c)|)^{\eta} \prod_{1\leq j\leq 2,\ |\varepsilon^{(j)}|\geq 1} |\varepsilon^{(j)}|^{-\eta}$$
$$\ll \left(\frac{1}{2|N(c_1)|}\right)^{k-1} (k^2|N(c)|)^{\eta} \prod_{1\leq j\leq 2,\ |\varepsilon^{(j)}|\geq 1} |\varepsilon^{(j)}|^{-\eta},$$

where we write $c = c_1\mathcal{C}$.

Also we have trivially

$$|S(\nu,\mu\varepsilon^2;c)| \leq N(c).$$

Hence, the partial sum of $\sum_2$ with the condition $*$ on U that $\varepsilon^{(0)} =: \max(|\varepsilon^{(1)}|, |\varepsilon^{(2)}|) \geq \exp(\log^2 N(\mathcal{C}))$, is bounded by

$$k^{2+2\eta}(N(\mathcal{C}))^{1+\eta} \sum_{\varepsilon\in U}{}^{*} |\varepsilon^{(0)}|^{-\eta} \sum_{c_1\in O^{\times}/U} \frac{2^{-k}X}{|N(c_1)|^{k-1-\eta}} \sum_{\mu} |b(\mu)|^2 \ll X \sum_{\mu} |b(\mu)|^2,$$

where we use the fact that the number of units ε satisfying $x \leq \log \varepsilon^{(0)} < 2x$, is $O(x)$ since U is cyclic and generated by a fundamental unit of O.

It remains to deal with the remaining sum Σ_2' with the sum over the units ε in U satisfying the condition #: $\log \varepsilon^{(0)} < \log^2 N(\mathcal{C})$. Note the above method clearly also works in this case if $N(\mathcal{C}) \leq 2^{k/2}$. Hence, we may assume $N(\mathcal{C}) > 2^{k/2}$ and thus $k \ll \log N(\mathcal{C})$. We will apply the following lemma proved in [Lu].

Lemma *Let c_1, $c_2 > 0$ be constants, $X \geq 1$, $d(\cdot)$ arbitrary complex numbers, and $c \in O$. Then we have*

$$\sum_{a \pmod c} \left|\sum_{N(\nu)\leq X,\ \nu\in O}{}' d(\nu)e\left(\frac{\nu a}{c}\right)\right|^2 = (|N(c)| + O(X)) \sum_{N(\nu)\leq X,\ \nu\in O}{}' |d(\nu)|^2,$$

where "$\prime$" means that the summation is restricted to those ν's such that $\nu > 0$, $c_1 N(\nu)^{1/2} \leq \nu^{(i)} \leq c_2 N(\nu)^{1/2}$.

Using the Mellin–Barnes integral representation [MOS, Section 3.6.3, p. 82],

$$J_{k-1}\left(\frac{4\pi\sqrt{\mu^{(i)}\nu^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|}\right)$$
$$=\frac{1}{4\pi i}\int_{(2+\eta)} \left(\frac{2\pi\sqrt{\mu^{(i)}\nu^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|}\right)^s \Gamma\left(\frac{k-1}{2}-\frac{s}{2}\right)\left[\Gamma\left(1+\frac{k-1}{2}+\frac{s}{2}\right)\right]^{-1} ds,$$

opening the Kloosterman sum, and by Cauchy's inequality, we infer that for $c \in \mathcal{C}^{\times}/U$ and with $s_i = 2+\eta+\sqrt{-1}t_i$ $(i=1,2)$ and $0 < \eta < 1/2$,

$$\sum_{\mu,\nu} b(\mu)\bar{b}(\nu)S(\nu,\mu\varepsilon^2;c)\ NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|)$$
$$\ll \int_{(2+\eta)} |ds_1| \int_{(2+\eta)} |ds_2| \left|\frac{\Gamma\left(\frac{k-1}{2}-\frac{s_1}{2}\right)}{\Gamma\left(1+\frac{k-1}{2}+\frac{s_1}{2}\right)}\right| \cdot \left|\frac{\Gamma\left(\frac{k-1}{2}-\frac{s_2}{2}\right)}{\Gamma\left(1+\frac{k-1}{2}+\frac{s_2}{2}\right)}\right|$$

$$\times \max_{s_1,s_2} \sum_{h \ (\mathrm{mod}\ c)} \left| \sum_{\mu,\,\nu} b(\mu)\,\bar{b}(\nu)\left(4\pi^2\sqrt{N(\mu)N(\nu)}/|N(c)|\right)^{2+\eta} \prod_{i=1}^{2} (\sqrt{\mu^{(i)}\nu^{(i)}})^{\sqrt{-1}t_i} e\left(\frac{\mu h}{c}\right)\right|$$

$$\ll N(c)^{-(2+\eta)} \int_{(2+\eta)} \frac{|ds_1|}{k+|s_1|} \int_{(2+\eta)} \frac{|ds_2|}{k+|s_2|} \left|\frac{\Gamma\left(\frac{3}{2}-\frac{s_1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{s_1}{2}\right)}\right| \left|\frac{\Gamma\left(\frac{3}{2}-\frac{s_2}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{s_2}{2}\right)}\right|$$

$$\times \max_{s_1,s_2} \sum_{h \ (\mathrm{mod}\ c)} \left| \sum_{\mu} b(\mu)\,(N(\mu))^{1+\eta/2} \prod_{i=1}^{2} (\mu^{(i)})^{\sqrt{-1}t_i/2} e\left(\frac{\mu h}{c}\right)\right|^2$$

$$\ll N(c_1)^{-(2+\eta)}\,(|N(c)|+X)\,(N(\mathcal{C}))^{\eta} \sum_{\mu} |b(\mu)|^2,$$

since $k \ll \log N(\mathcal{C})$, where as before, we write $c = c_1\mathcal{C}$.

Thus the partial sum Σ_2' is bounded by

$$\begin{aligned} &k^2(N(\mathcal{C}))^{\eta} \sum_{\varepsilon\in U}^{\#} \sum_{c_1\in O^{\times}/U} \frac{1}{|N(c_1)|^{2+\eta}} (|N(c_1\mathcal{C})|+X) \sum_{\mu} |b(\mu)|^2 \\ \ll &(N(\mathcal{C})+X)N(\mathcal{C})^{\eta} \sum_{\mu} |b(\mu)|^2, \end{aligned}$$

since

$$\sum_{\varepsilon\in U}^{\#} 1 \ll \log^2 N(\mathcal{C}).$$

Hence, Theorem 1.2 is true if $k^2N(\mathcal{C}) \geq 8(4\pi)^2X$.

In the case $k^2N(\mathcal{C}) < 8(4\pi)^2X$, we reduce it to the previous case by the famous embedding trick of Iwaniec. Choosing a prime ideal $\mathcal{P} \subset O$ such that $N(\mathcal{P})k^2N(\mathcal{C}) \asymp X$ and $N(\mathcal{P})k^2N(\mathcal{C}) \geq 8(4\pi)^2X$. Note that $[\Gamma_0(\mathcal{C}) : \Gamma_0(\mathcal{P}\mathcal{C})] \leq N(\mathcal{P})+1$. Let $H_k(\mathcal{C})$ denote an orthonormal basis of $S_{2k}(\Gamma_0(\mathcal{C}))$, and write

$$S_{\mathcal{C}}(b) = \sum_{f\in H_k(\mathcal{C})} |\sum_{\mu} b(\mu)\psi_f(\mu)|^2.$$

We deduce that

$$\begin{aligned} S_{\mathcal{C}}(b) &\leq (1+N(\mathcal{P})^{-1})S_{\mathcal{P}\mathcal{C}}(b) \\ &\ll (N(\mathcal{P}\mathcal{C})k^2+X)(kXN(\mathcal{C}))^{\eta} \sum_{\mu} |b(\mu)|^2 \\ &\ll X(kXN(\mathcal{C}))^{\eta} \sum_{\mu} |b(\mu)|^2, \end{aligned}$$

and this completes our proof.

Acknowledgment The author wishes to thank the referee for careful reading of the paper and for the valuable comments.

References

[As] T. Asai, *On certain Dirichlet series associated with Hilbert modular forms and Rankin's method.* Math. Ann. **226**(1977), no. 1, 81–94.

[Ga] P. B. Garrett, *Holomorphic Hilbert modular forms*, Wadsworth Inc., Monterey, California, 1990.

[GR] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products*, Academic Press, New York, 1965.

[HL] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero, appendix by D. Goldfeld, J. Hoffstein, and D. Lieman, an effective zero free region*. Ann. Math. **140**(1994), 161–181.

[Iw] H. Iwaniec, *Small eigenvalues of Laplacian for $\Gamma_0(N)$*. Acta Arith. **56**(1990), no. 1, 65–82.

[IK] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, 53, American Mathematical Society, Providence, RI, 2004.

[Kr] M. Krishnamurthy, *The Asai transfer to GL_4 via the Langlands–Shahidi method*. Int. Math. Res. Not. IMRN **2003**(2003), no. 41, 2221–2254.

[Lu] W. Luo, *Poincaré series and Hilbert modular forms. Rankin memorial issues*. Ramanujan J. **7**(2003), nos. 1–3, 129–140.

[MOS] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer, New York, 1966.

[PR] D. Prasad and D. Ramakrishnan, *On the cuspidality criterion for the Asai transfer to $GL(4)$, Appendix A to the paper 'Determination of cusp forms on $GL(2)$ by coefficients restricted to quadratic subfields' by M. Krishnamurthy*. J. Number Theory **132**(2012), no. 6, 1376–1383.

[Ra] D. Ramakrishnan, *Modularity of solvable Artin representations of $GO(4)$-type*. Int. Math. Res. Not. IMRN **2002**(2002), no. 1, 1–54.

[Sh] H. Shimizu, *On discontinuous groups acting on a product of upper half planes*. Ann. of Math. **77**(1963), 33–71.

[Ta] R. Taylor, *On Galois representations associated to Hilbert modular forms*. Invent. Math. **98**(1989), 265–280.

Department of Mathematics, The Ohio State University, Columbus, OH 43210, United States
e-mail: luo.43@osu.edu