BULLETIN **13** (1), 1970

PROBLEMS FOR SOLUTION

P.164. For $n=1, 2, \ldots$ define e_n by $e_1=1, e_{2n}=e_n, e_{2n+1}=-e_n$. Show that

$$\sum_{i=1}^{2^n-1} e_i i^k = \begin{cases} 0 & \text{for } 1 \le k \le n-1 \\ (-1)^{n+1} n!\, 2^{\binom{n}{2}} & \text{for } k = n. \end{cases}$$

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SOLUTIONS

P.159. Let M be a metric space, M_0 a compact subset and $T: M \to M$ an isometry. Then if $TM_0 \subset M_0$ or $TM_0 \supset M_0$ we have $TM_0 = M_0$.

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Solution by C. Davis, University of Toronto, Toronto, Ontario. Assume $TM_0 \not\subset M_0$. By compactness, there exists $x \in M_0 \backslash TM_0$ for which $d = \sup \{d(x, TM_0) : x \in M_0\} > 0$ is attained; that is, $d(x, TM_0) = d$. Because T is an isometry, $d(T^n x, T^{n+1} M_0) = d$, $n = 0, 1, \ldots$. But this means in particular that for $m > n$, $d(T^n x, T^m x) \ge d > 0$. Thus the sequence $(T^n x)_{n=0}^{\infty}$ in M_0 cannot contain any convergent subsequence, which contradicts compactness.

Also solved by L. Cooper, D. Ž. Djoković, D. Lind, J. Marsden, S. Reich, P. Smith, and the proposer.

P.161. For any positive integer n and any n numbers $c_1, \ldots, c_n$, let further numbers $c_{n+1}, c_{n+2}, \ldots$ be defined as continued fractions

$$c_{n+1} = 1 - c_n/1 - c_{n-1}/1 - \cdots c_2/1 - c_1,$$
$$c_{n+2} = 1 - c_{n+1}/1 - c_n/1 - \cdots c_3/1 - c_2,$$

and so on. Prove that the sequence c_i is periodic with period $n+3$; that is, $c_{n+4} = c_1$, $c_{n+5} = c_2$, and so on.

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Solution by S. Reich, Israel Institute of Technology. Suppose we prove the following lemma. Given $c_1, \ldots, c_n$, if $c_{n+1}, c_{n+2}, \ldots$ are defined (when possible) as in the problem then $c_{n+4} = c_1$.

Then we are done because when we apply the lemma to $c_2, \ldots, c_{n+1}$ we obtain $c_{n+5} = c_2$, and so on.

The lemma itself will be established by induction on n. For $n \leq 4$ the result is checked directly. Suppose now the lemma holds for some $n \geq 4$ and let us consider any sequence $c_1, c_2, \ldots, c_n, a$ of length $n+1$, which we write also as $d_1, \ldots, d_{n+1}$. The numbers $d_{n+2}, d_{n+3}, \ldots$ are defined by the sequence $d_1 (=c_1), \ldots, d_{n+1}(=a)$ and the numbers $c_{n+1}, c_{n+2}, \ldots$ are defined by the sequence $c_1, \ldots, c_n$ of length n. Let f_1, f_2, f_3, f_4 be defined as continued fractions

$$\begin{aligned} f_1 &= 1-c_n/1-c_{n-1}/1-\cdots-c_2/1-c_1, \\ f_2 &= 1-c_n/1-c_{n-1}/1-\cdots-c_3/1-c_2, \\ f_3 &= 1-c_n/1-c_{n-1}/1-\cdots-c_4/1-c_3, \\ f_4 &= 1-c_n/1-c_{n-1}/1-\cdots-c_5/1-c_4. \end{aligned}$$

From the definition we find that

$$\begin{aligned} c_{n+1} &= f_1, \\ c_{n+2} &= 1-f_1/f_2, \\ c_{n+3} &= f_1(f_3-f_2)/f_2(f_3-f_1), \\ c_{n+4} &= (f_4-f_3)(f_2-f_1)/(f_3-f_1)(f_4-f_2). \end{aligned}$$

By the inductive hypothesis the last expression equals c_1. Again from the definition we see that

$$\begin{aligned} d_{n+2} &= 1-a/f_1, \\ d_{n+3} &= a(f_2-f_1)/f_1(f_2-a), \\ d_{n+4} &= (f_3-f_2)(f_1-a)/(f_2-a)(f_3-f_1), \\ d_{n+5} &= (f_4-f_3)(f_2-f_1)/(f_3-f_1)(f_4-f_2), \end{aligned}$$

whence $d_{n+5}=c_{n+4}=c_1$, and the desired result follows.

Also solved by L. Cooper and the proposer.