Bull. Aust. Math. Soc. **84** (2011), 372–386
doi:10.1017/S0004972711002620

ESSENTIAL CHARACTER AMENABILITY OF BANACH ALGEBRAS

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(Received 14 November 2010)

Abstract

For a Banach algebra $\mathcal{A}$ and a character ϕ on $\mathcal{A}$, we introduce and study the notion of essential ϕ-amenability of $\mathcal{A}$. We give some examples to show that the class of essentially ϕ-amenable Banach algebras is larger than that of ϕ-amenable Banach algebras introduced by Kaniuth *et al.* ['On ϕ-amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144** (2008), 85–96]. Finally, we characterize the essential ϕ-amenability of various Banach algebras related to locally compact groups.

2010 *Mathematics subject classification*: primary 43A07, 46H05; secondary 43A20.

Keywords and phrases: Banach algebra, character amenability, essential amenability, locally compact group.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$ consisting of all nonzero characters on $\mathcal{A}$. Kaniuth *et al.* [10] have recently introduced and studied the interesting notion of ϕ-amenability; see also Kaniuth *et al.* [11]. Specifically, $\mathcal{A}$ is ϕ-amenable if for every Banach $\mathcal{A}$-bimodule X with the left module action

$$a \cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X)$$

(and no restriction on the right module action), every continuous derivation from $\mathcal{A}$ into X^* is inner. This is a generalization of left amenability for Lau algebras introduced and studied by Lau [12].

More recently, Monfared [19] has introduced and studied the notion of character amenability; he called $\mathcal{A}$ character amenable if it has a bounded right approximate identity and it is ϕ-amenable for all $\phi \in \sigma(\mathcal{A})$; see also Alaghmandan *et al.* [1] and Samea [21].

Moreover, the notion of amenability for Banach algebras was introduced and studied by Johnson [9]. Several characterizations and modifications of amenability have been described by many authors; see, for example, Ghahramani *et al.* [5], Grønbæk [6], Lau *et al.* [13], and Willis [22]. In particular, the concept of essential amenability was introduced and studied by Ghahramani and Loy [4]. The Banach algebra $\mathcal{A}$ is called essentially amenable if continuous derivations from $\mathcal{A}$ into the duals of neo-unital Banach $\mathcal{A}$-bimodules are inner.

In this paper, we introduce and study the concept of essential ϕ-amenability and essential character amenability of Banach algebras. The paper is organized as follows. In Section 2 we show that for certain Banach algebras, ϕ-amenability and essential ϕ-amenability are equivalent. We then give some examples to show that essential ϕ-amenability is weaker than ϕ-amenability. In Section 3 we present a number of hereditary properties of essential ϕ-amenability. In Section 4, for two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with $\theta \in \sigma(\mathcal{B})$, we show that essential ϕ-amenability from the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{B}$ and the Lau product $\mathcal{A} \times_\theta \mathcal{B}$ transfers to $\mathcal{A}$ and $\mathcal{B}$; we also discuss the converse of this statement. Finally, in Section 5 we consider certain left introverted closed subspaces X of the dual space of the group algebra for a locally compact group, and characterize essential character amenability of the Banach algebra X^* endowed with an Arens type product.

2. Essential character amenability

Let $\mathcal{A}$ be a Banach algebra. If X is a Banach $\mathcal{A}$-bimodule, then so is the dual space X^* of X with the module actions given by

$$(a \cdot f)(x) = f(x \cdot a) \quad \text{and} \quad (f \cdot a)(x) = f(a \cdot x)$$

for all $x \in X$, $f \in X^*$ and $a \in \mathcal{A}$. A derivation is a linear map $D : \mathcal{A} \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

For $x \in X$, define $\mathrm{ad}_x : \mathcal{A} \to X$ by

$$\mathrm{ad}_x(a) = a \cdot x - x \cdot a$$

for all $a \in \mathcal{A}$. Then ad_x is a derivation; these are the inner derivations. We denote by $\mathcal{A} \cdot X$ and $X \cdot \mathcal{A}$ the sets $\{a \cdot x : a \in \mathcal{A}, x \in X\}$ and $\{x \cdot a : a \in \mathcal{A}, x \in X\}$, respectively. A Banach $\mathcal{A}$-bimodule X is neo-unital if

$$\mathcal{A} \cdot X = X \cdot \mathcal{A} = X.$$

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \sigma(\mathcal{A})$. We say that $\mathcal{A}$ is *essentially ϕ-amenable* if for every neo-unital Banach $\mathcal{A}$-bimodule X with the left module action

$$a \cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X)$$

(and no restriction on the right module action), every continuous derivation from $\mathcal{A}$ into X^* is inner. We also say that $\mathcal{A}$ is *essentially 0-amenable* if for every Banach $\mathcal{A}$-bimodule X with the zero left action such that $X \cdot \mathcal{A} = X$, every continuous derivation from $\mathcal{A}$ into X^* is inner.

We say $\mathcal{A}$ is *essentially character amenable* if it is essentially ϕ-amenable for all $\phi \in \sigma(\mathcal{A}) \cup \{0\}$.

Clearly every ϕ-amenable Banach algebra is essentially ϕ-amenable. Our first result shows that the converse is also true for certain Banach algebras.

https://doi.org/10.1017/S0004972711002620 Published online by Cambridge University Press

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra, $\phi \in \sigma(\mathcal{A})$ and let I be a closed two-sided ideal of $\mathcal{A}$ with a bounded approximate identity such that $I \not\subseteq \ker(\phi)$. Then the following statements are equivalent.*

(a) *$\mathcal{A}$ is ϕ-amenable.*
(b) *$\mathcal{A}$ is essentially ϕ-amenable.*
(c) *I is essentially $\phi|_I$-amenable.*
(d) *I is $\phi|_I$-amenable.*

Proof. That (a) implies (b) is trivial. Now let (b) hold and suppose that X is a neo-unital Banach I-bimodule such that

$$a \cdot x = \phi|_I(a)x \quad (a \in I, x \in X),$$

and $D : I \to X^*$ is a continuous derivation. Then X can be considered as a neo-unital Banach $\mathcal{A}$-bimodule with the left action

$$a \cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X),$$

for which D has an extension $\tilde{D} : \mathcal{A} \to X^*$; see [20, Proposition 2.1.6]. Since $\mathcal{A}$ is essentially ϕ-amenable, $\tilde{D}$ is inner and so D is inner. Thus I is essentially $\phi|_I$-amenable.

To prove that (c) implies (d), suppose that X is a Banach I-bimodule such that $b \cdot x = \phi(b)x$ for all $b \in I$ and $x \in X$. First note that $II = I$ and $I \cdot X \cdot I$ is a Banach I-bimodule by Cohen's factorization theorem. Clearly $I \cdot X \cdot I$ is neo-unital Banach I-bimodule with

$$b \cdot y = \phi(b)y$$

for all $b \in I$ and $y \in I \cdot X \cdot I$. A similar argument as in [20, Proof of Proposition 2.1.5] shows that any continuous derivation from I into X^* is inner if and only if any continuous derivation from I into $(I \cdot X \cdot I)^*$ is inner; this shows that (c) implies (d).

For the implication (d) $\Rightarrow$ (a), let X be a Banach $\mathcal{A}$-bimodule such that

$$a \cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X),$$

and let $D : \mathcal{A} \to X^*$ be a continuous derivation. Then the map $D|_I : I \to X^*$ is a continuous derivation. By assumption, there exists $f_0 \in X^*$ such that

$$D|_I(b) = b \cdot f_0 - \phi_I(b) f_0$$

for all $b \in I$. Fix $b_0 \in I$ such that $\phi|_I(b_0) = 1$ and set

$$f_1 := b_0 \cdot f_0 \in X^*.$$

Thus for each $a \in \mathcal{A}$

$$\begin{aligned} a \cdot f_1 - \phi(a) f_1 &= ab_0 \cdot f_0 - \phi(a)\phi|_I(b_0) f_0 + \phi(a)\phi|_I(b_0) f_0 - \phi(a) b_0 \cdot f_0 \\ &= D|_I(ab_0) - \phi(a) D|_I(b_0) \\ &= D(a) + a \cdot D|_I(b_0) - \phi(a) D|_I(b_0) \\ &= D(a) + a \cdot (b_0 \cdot f_0 - f_0) - \phi(a)(b_0 \cdot f_0 - f_0); \end{aligned}$$

this shows that $D(a) = a \cdot f_0 - f_0 \cdot a = \mathrm{ad}_{f_0}(a)$, which completes the proof. □

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. Then $\mathcal{A}$ is character amenable if and only if $\mathcal{A}$ is essentially character amenable.*

Proof. This follows from [20, Proposition 2.1.3] and Proposition 2.2 above. □

The following example shows that there are essentially ϕ-amenable Banach algebras which are not ϕ-amenable.

Example 2.4. Let X be a Banach space and take $\phi \in X^* \setminus \{0\}$ with $\|\phi\| \le 1$. Define a product on X by

$$ab = \phi(b)a \quad (a, b \in X).$$

With this product X is a Banach algebra, which we denote by $A_\phi(X)$. It is clear that

$$\sigma(A_\phi(X)) = \{\phi\}$$

and $A_\phi(X)$ has a right identity; indeed, every $e \in A_\phi(X)$ with $\phi(e) = 1$ is a right identity. We show that $A_\phi(X)$ is ϕ-amenable if and only if X is one-dimensional. To see this, suppose that $A_\phi(X)$ is ϕ-amenable and consider the Banach $A_\phi(X)$-bimodule $A_\phi(X)^*$ with the module actions given by

$$(a \cdot f)(b) = f(ba) = \phi(a)f(b) \quad \text{and} \quad (f \cdot a)(b) = f(ab) = f(a)\phi(b)$$

for all $a, b \in A_\phi(X)$ and $f \in A_\phi(X)^*$. Consider the quotient Banach $\mathcal{A}$-bimodule

$$Y := A_\phi(X)^*/\mathbb{C}\phi.$$

Let $F_0 \in A_\phi(X)^{**}$ be such that $F_0(\phi) = 1$. Then the image of ad_{F_0} is a subset of Y^*, and hence, by our assumption, there exists

$$F_1 \in X^* = \{F \in A_\phi(X)^{**} : F(\phi) = 0\}$$

such that $\mathrm{ad}_{F_0} = \mathrm{ad}_{F_1}$. Thus for each $a \in A_\phi(X)$ we have $a = \phi(a)(F_1 - F_0)$. It follows that $A_\phi(X)$ is one-dimensional. The converse is trivial.

Now we show that $A_\phi(X)$ is essentially ϕ-amenable. Suppose that Y is a neo-unital Banach $A_\phi(X)$-bimodule always with left action

$$a \cdot y = \phi(a)y \quad (a \in A_\phi(X), y \in Y).$$

Let $a \in A_\phi(X)$ and let $y \in Y$. Since $Y = Y \cdot A_\phi(X)$, there exist $b \in A_\phi(X)$ and $z \in Y$ such that $y = z \cdot b$. Hence,

$$y \cdot a = (z \cdot b) \cdot a = z \cdot (ba) = \phi(a)(z \cdot b) = \phi(a)y.$$

Therefore

$$a \cdot f = \phi(a)f \quad (a \in A_\phi(X), f \in Y^*).$$

https://doi.org/10.1017/S0004972711002620 Published online by Cambridge University Press

Fix $a_0 \in A_\phi(X)$ such that $\phi(a_0) = 1$. If $D : A_\phi(X) \to Y^*$ is a continuous derivation, then for each $a \in A_\phi(X)$,

$$\begin{aligned}\phi(a)D(a_0) &= D(\phi(a)a_0)\\ &= D(a_0a)\\ &= D(a_0)\cdot a + a_0 \cdot D(a)\\ &= \phi(a)D(a_0) + D(a).\end{aligned}$$

So $D = 0$, and thus $A_\phi(X)$ is essentially ϕ-amenable.

In particular, if S is a left zero semigroup with at least two elements and ϕ_S is the augmentation character on the convolution Banach algebra $\ell^1(S)$ defined by

$$\phi_S(f) = \sum_{s\in S} f(s)$$

for all $f \in \ell^1(S)$, then $f * g = \phi_S(g)f$ for all $f, g \in \ell^1(S)$ and thus $\ell^1(S)$ is not ϕ_S-amenable, but $\ell^1(S)$ is essentially ϕ_S-amenable by the above example.

Remark 2.5. Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \sigma(\mathcal{A})$. Then $\mathbb{C}$ is a neo-unital Banach $\mathcal{A}$-bimodule with the module actions

$$a\cdot z = z\cdot a = \phi(a)z \quad (a \in \mathcal{A}, z \in \mathbb{C});$$

this bimodule is denoted by $\mathbb{C}_\phi$. A derivation from $\mathcal{A}$ into $\mathbb{C}_\phi$ is a linear functional d on $\mathcal{A}$ such that

$$d(ab) = \phi(a)d(b) + d(a)\phi(b) \quad (a, b \in \mathcal{A}).$$

Such a linear functional is called a point derivation at ϕ. If $\mathcal{A}$ is an essentially ϕ-amenable Banach algebra, then any bounded point derivation at ϕ is trivial; this is because each bounded point derivation d at ϕ is a continuous derivation in $\mathbb{C}_\phi^* = \mathbb{C}_\phi$, and so $d = 0$.

3. Hereditary properties of essential character amenability

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\Theta : \mathcal{A} \to \mathcal{B}$ be a continuous epimorphism. If $\phi \in \sigma(\mathcal{A})$, then there is a unique $\phi_\Theta \in \sigma(\mathcal{B})$ with $\phi_\Theta \circ \Theta = \phi$ if and only if $\ker(\Theta) \subseteq \ker(\phi)$. In particular, if I is a closed two-sided ideal of $\mathcal{A}$, then there is a unique $\phi_q \in \sigma(\mathcal{A}/I)$ with $\phi_q \circ q = \phi$ if and only if $I \subseteq \ker(\phi)$, where $\mathcal{A}/I$ is the quotient algebra and $q : \mathcal{A} \to \mathcal{A}/I$ is the quotient map.

Proposition 3.1. *Let $\mathcal{A}$ be a Banach algebra, let I be a closed two-sided ideal of $\mathcal{A}$ and let $\phi \in \sigma(\mathcal{A})$ with $I \subseteq \ker(\phi)$. Suppose that I has a bounded right approximate identity and that $\mathcal{A}/I$ is essentially ϕ_q-amenable. Then $\mathcal{A}$ is essentially ϕ-amenable.*

Proof. Suppose that $\mathcal{A}/I$ is essentially ϕ_q-amenable. Let X be a neo-unital Banach $\mathcal{A}$-bimodule such that

$$a\cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X),$$

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and $D : \mathcal{A} \to X^*$ be a continuous derivation. Clearly X is a Banach I-bimodule with zero left action and $D|_I : I \to X^*$ is a continuous derivation. By [20, Proposition 2.1.3] there exists $f_0 \in X^*$ such that

$$D(a) = a \cdot f_0 - f_0 \cdot a \quad (a \in I).$$

As in the proof of [20, Theorem 2.3.10], set

$$\tilde{D} := D - \mathrm{ad}_{f_0}.$$

Then $\tilde{D}|_I = 0$ and thus induces a map from $\mathcal{A}/I$ into X^*, which we denote likewise by $\tilde{D}$. Let

$$Y := \{g \in X^* : a \cdot g = 0 \text{ for all } a \in I\}$$

and let X_I be the closed submodule generated by $X \cdot I$. It is easy to check that

$$Y \cong (X/X_I)^*$$

and that X/X_I is a neo-unital Banach $\mathcal{A}/I$-bimodule with left action given by

$$q(a) \cdot (x + X_I) = \phi(a)x + X_I = \phi_q(q(a))x + X_I.$$

Let $a \in I$ and $b \in \mathcal{A}$. Then

$$a \cdot \tilde{D}(b) = \tilde{D}(ab) - \tilde{D}(a) \cdot b = 0$$

because $\tilde{D}$ vanishes on I; similarly, $\tilde{D}(b) \cdot a = 0$. It follows that

$$\tilde{D}(\mathcal{A}/I) \subset Y.$$

Since $\mathcal{A}/I$ is essentially ϕ_q-amenable, there is $f_1 \in Y$ such that $\tilde{D} = \mathrm{ad}_{f_1}$. Consequently, $D = \mathrm{ad}_{f_0+f_1}$. □

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\psi \in \sigma(\mathcal{B}) \cup \{0\}$. If there is a continuous epimorphism $\Theta : \mathcal{A} \to \mathcal{B}$ and $\mathcal{A}$ is essentially $\psi \circ \Theta$-amenable, then $\mathcal{B}$ is essentially ψ-amenable.*

Proof. Suppose that $\mathcal{A}$ is essentially $\psi \circ \Theta$-amenable. Let X be a Banach $\mathcal{B}$-bimodule such that $X \cdot \mathcal{B} = X$ and

$$b \cdot x = \psi(b)x \quad (b \in \mathcal{B}, x \in X).$$

Let $Y := X$ be the Banach $\mathcal{A}$-bimodule with actions induced via Θ; note that

$$X \cdot \mathcal{A} = X \cdot \mathcal{B} = X$$

and

$$a \cdot x = \Theta(a) \cdot x = \psi(\Theta(a))x \quad (a \in \mathcal{A}, x \in X).$$

If $D : \mathcal{B} \to X^*$ is a continuous derivation, then $D \circ \Theta : \mathcal{A} \to Y^*$ is a continuous derivation. Thus there exists $f \in X^*$ such that $(D \circ \Theta)(a) = \mathrm{ad}_f(a)$ for all $a \in \mathcal{A}$; that is, D is inner. □

Corollary 3.3. *Let $\mathcal{A}$ be an essentially ϕ-amenable Banach algebra with $\phi \in \sigma(\mathcal{A})$ and let I be a closed two-sided ideal of $\mathcal{A}$. Then $\mathcal{A}/I$ is essentially ϕ_q-amenable.*

Corollary 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\psi \in \sigma(\mathcal{B})$. If there is a continuous epimorphism $\Theta : \mathcal{A} \to \mathcal{B}$ and* $\ker(\Theta)$ *has a bounded right approximate identity, then $\mathcal{A}$ is essentially $\psi \circ \Theta$-amenable if and only if $\mathcal{B}$ is essentially ψ-amenable.*

Proof. Suppose that $\mathcal{B}$ is essentially ψ-amenable and put $I = \ker(\Theta)$. Then I is a closed two-sided ideal of $\mathcal{A}$. It is clear that $\Theta' : \mathcal{A}/I \to \mathcal{B}$, defined by

$$\Theta'(q(a)) = \Theta(a)$$

for all $a \in \mathcal{A}$, is an injective continuous epimorphism. Thus $\mathcal{A}/I$ is essentially $\psi \circ \Theta'$-amenable. Now Proposition 3.1 shows that $\mathcal{A}$ is essentially $\psi \circ \Theta$-amenable, since

$$(\psi \circ \Theta') \circ q = \psi \circ \Theta \quad \text{and} \quad I \subseteq \ker(\psi \circ \Theta).$$

The converse follows from Theorem 3.2. □

Before we present the following result, let us recall that if $\mathcal{A}$ is a Banach algebra and $\phi \in \sigma(\mathcal{A})$, then the element $m \in \mathcal{A}^{**}$ is called a ϕ-mean on $\mathcal{A}^*$ if

$$m(\phi) = 1 \quad \text{and} \quad m(f \cdot a) = \phi(a)m(f)$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. It is shown that $\mathcal{A}$ is ϕ-amenable if and only if there exists a ϕ-mean on $\mathcal{A}^*$; see [10, Theorem 1.1].

Proposition 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\psi \in \sigma(\mathcal{B})$. Suppose that there exists a continuous homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ with $\overline{\Theta(\mathcal{A})} = \mathcal{B}$. If $\Lambda : \mathcal{A}^* \to \mathcal{B}^*$ is a continuous linear map such that $\Lambda(\psi \circ \Theta) \in \sigma(\mathcal{B})$ and $\Lambda(f \cdot a) = \Lambda(f) \cdot \Theta(a)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, then $\mathcal{A}$ is $\psi \circ \Theta$-amenable if and only if $\mathcal{B}$ is ψ-amenable.*

Proof. Suppose that $\mathcal{B}$ is ψ-amenable. Then there exists an $m \in \mathcal{B}^{**}$ such that $m(\psi) = 1$ and $m(g \cdot b) = \psi(b)m(g)$ for all $b \in \mathcal{B}$ and $g \in \mathcal{B}^*$. Since

$$\Lambda(\psi \circ \Theta) \in \sigma(\mathcal{B}) \quad \text{and} \quad \Lambda(\psi \circ \Theta) \cdot \Theta(a) = \psi(\Theta(a))\Lambda(\psi \circ \Theta)$$

for all $a \in \mathcal{A}$, it follows that $\Lambda(\psi \circ \Theta) = \psi$. The continuity of Λ implies that the functional $m \circ \Lambda : \mathcal{A}^* \to \mathbb{C}$ belongs to $\mathcal{A}^{**}$, and

$$(m \circ \Lambda)(\psi \circ \Theta) = m(\Lambda(\psi \circ \Theta)) = m(\psi) = 1.$$

Moreover, for every $f \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$,

$$\begin{aligned}(m \circ \Lambda)(f \cdot a) &= m(\Lambda(f \cdot a)) \\ &= m(\Lambda(f) \cdot \Theta(a)) \\ &= \psi(\Theta(a))m(\Lambda f) \\ &= \psi \circ \Theta(a)(m \circ \Lambda)(f).\end{aligned}$$

Hence $m \circ \Lambda(f \cdot a)$ is a $\psi \circ \Theta$-mean on $\mathcal{A}^*$. The converse follows from [10, Proposition 3.5]. □

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COROLLARY 3.6. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \sigma(\mathcal{A})$ and let I be a closed two-sided ideal of $\mathcal{A}$ such that $I \subseteq \ker(\phi)$. If $\Lambda : \mathcal{A}^* \to (\mathcal{A}/I)^*$ is a continuous linear map such that $\Lambda(\phi) \in \sigma(\mathcal{A}/I)$ and $\Lambda(f \cdot a) = \Lambda(f) \cdot q(a)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, then $\mathcal{A}$ is ϕ-amenable if and only if $\mathcal{A}/I$ is ϕ_q-amenable.*

Note that if $\mathcal{A}$ is a closed right ideal of a Banach algebra $\mathcal{B}$, then $\mathcal{A}$ is a Banach right $\mathcal{B}$-module. Thus the dual Banach left $\mathcal{B}$-module $\mathcal{A}^*$ is well defined through $(b \cdot f)(a) = f(ab)$ for all $b \in \mathcal{B}$, $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Before we give the following result, recall that for a closed ideal I of Banach algebra $\mathcal{A}$, we denote by $I^\perp$ the set

$$\{f \in \mathcal{A}^* : f|_I = 0\}.$$

We identify $I^\perp$ with $(\mathcal{A}/I)^*$ via $f \mapsto \overline{f}$, where $\overline{f} \circ q = f$.

PROPOSITION 3.7. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \sigma(\mathcal{A})$ and let I be a closed two-sided ideal of $\mathcal{A}$ with $I \subseteq \ker(\phi)$. Suppose that $\mathcal{A}$ is a closed right ideal of Banach algebra $\mathcal{B}$ for which ϕ has an extension $\tilde{\phi} \in \sigma(\mathcal{B})$. Suppose that there exists $b_0 \notin \ker(\tilde{\phi})$ such that $b_0\mathcal{A}^* \subseteq I^\perp$. If $\mathcal{A}/I$ is ϕ_q-amenable, then $\mathcal{A}$ is ϕ-amenable.*

PROOF. Without loss of generality we can assume that $\tilde{\phi}(b_0) = 1$. It is clear that for each $f \in I^\perp$, there exists $\overline{f} \in (\mathcal{A}/I)^*$ such that

$$\overline{f}(q(a)) = f(a)$$

for all $a \in \mathcal{A}$. Thus the map $\Lambda_{b_0} : \mathcal{A}^* \to (\mathcal{A}/I)^*$ defined by $\Lambda_{b_0}(f) = \overline{b_0 \cdot f}$ is a continuous linear map and, for each $a \in \mathcal{A}$,

$$\begin{aligned}(\overline{b_0 \cdot \phi})(q(a)) &= (b_0 \cdot \phi)(a) = \phi(ab_0) \\ &= \tilde{\phi}(ab_0) = \tilde{\phi}(a)\tilde{\phi}(b_0) \\ &= \phi(a) = \phi_q(q(a)).\end{aligned}$$

Also, for each $x \in \mathcal{A}$,

$$\begin{aligned}(\overline{b_0 \cdot f \cdot a})(q(x)) &= (b_0 \cdot (f \cdot a))(x) = (b_0 \cdot f)(ax) \\ &= (\overline{b_0 \cdot f})(q(ax)) = (\overline{b_0 \cdot f})(q(a)q(x)) \\ &= (\overline{b_0 \cdot f} \cdot q(a))(q(x)),\end{aligned}$$

and hence $\Lambda_{b_0}(f \cdot a) = \Lambda_{b_0}(f) \cdot q(a)$. Thus $\mathcal{A}$ is ϕ-amenable by Corollary 3.6. □

COROLLARY 3.8. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \sigma(\mathcal{A})$ and let I be a closed two-sided ideal of $\mathcal{A}$ such that $I \subseteq \ker(\phi)$. Suppose that there exists $a_0 \notin \ker(\phi)$ such that $a_0\mathcal{A}^* \subseteq I^\perp$. If $\mathcal{A}/I$ is ϕ_q-amenable, then $\mathcal{A}$ is ϕ-amenable.*

Let $\mathcal{A}$ be a Banach algebra and let F be a subspace of $\mathcal{A}^*$. F is called *left invariant* (respectively, *right invariant*) if $F \cdot \mathcal{A} \subseteq F$ (respectively, $\mathcal{A} \cdot F \subseteq F$); it is called *invariant* if it is left and right invariant.

In this case, it is clear that $I_F = \{a \in \mathcal{A} : f(a) = 0 \text{ for all } f \in F\}$ is a closed two-sided ideal in $\mathcal{A}$ and it is easy to check that

$$\overline{F}^{w^*} = I_F^{\perp},$$

where $\overline{F}^{w^*}$ is the closure of F in the weak* topology of $\mathcal{A}^{**}$. Recall that an element $a \in \mathcal{A}$, is called *central* if $ab = ba$ for all $b \in \mathcal{A}$. An element $\mathbf{e} \in \mathcal{A}$ is called an *idempotent* if $\mathbf{e}^2 = \mathbf{e}$.

Lemma 3.9. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \sigma(\mathcal{A})$. Suppose that $\mathcal{A}$ is a closed right ideal of Banach algebra $\mathcal{B}$ for which ϕ has an extension $\tilde{\phi} \in \sigma(\mathcal{B})$. If $\mathcal{B}$ has a central idempotent $\mathbf{e} \notin \ker(\tilde{\phi})$, then $\mathbf{e}\mathcal{A}^*$ is an invariant closed subspace of $\mathcal{A}^*$ and*

$$\mathcal{A}/I_{\mathbf{e}\mathcal{A}^*} \cong \mathcal{A}\mathbf{e}.$$

Proof. Set $F := \mathbf{e}\mathcal{A}^*$. Since $\mathbf{e}$ is central, $F \cdot \mathcal{A} \cup \mathcal{A} \cdot F \subseteq F$ and hence F is invariant. It is easy to check that F is a subspace of $\mathcal{A}^*$ and $\overline{F}^{w^*} = F$. Since $\mathbf{e} \notin \ker(\tilde{\phi})$, it follows that $\tilde{\phi}(\mathbf{e}) = 1$. So, for each $a \in \mathcal{A}$,

$$\phi(a) = \tilde{\phi}(a)\tilde{\phi}(\mathbf{e}) = \tilde{\phi}(a\mathbf{e}) = (\mathbf{e} \cdot \phi)(a)$$

and hence $\phi = \mathbf{e} \cdot \phi$. But

$$\begin{aligned} I_F &= \{a \in \mathcal{A} : (\mathbf{e} \cdot f)(a) = 0 \ \text{ for all } f \in \mathcal{A}^*\} \\ &= \{a \in \mathcal{A} : a\mathbf{e} = 0\}, \end{aligned}$$

and hence $\mathcal{A}/I_F \cong \mathcal{A}\mathbf{e}$. □

Corollary 3.10. *Let $\mathcal{A}$ be an essentially ϕ-amenable Banach algebra with $\phi \in \sigma(\mathcal{A})$. If $\mathcal{A}$ has a central idempotent $\mathbf{e} \notin \ker(\phi)$, then $\mathcal{A}$ is ϕ-amenable.*

Proof. Suppose that $\mathcal{A}$ is essentially ϕ-amenable. Then $\mathcal{A}/I_{\mathbf{e}\mathcal{A}^*}$ is essentially ϕ_q-amenable by Corollary 3.3. But

$$\mathcal{A}\mathbf{e} \cong \mathcal{A}/I_{\mathbf{e}\mathcal{A}^*}$$

by Lemma 3.9 and $\mathcal{A}\mathbf{e}$ has the identity $\mathbf{e}$. Therefore by Proposition 2.2, $\mathcal{A}/I_{\mathbf{e}\mathcal{A}^*}$ is ϕ_q-amenable. An application of Corollary 3.8 to $I = I_{\mathbf{e}\mathcal{A}^*}$ and $a_0 = \mathbf{e}$ completes the proof. □

Proposition 3.11. *Let $\mathcal{A}$ be an essentially ϕ-amenable Banach algebra with $\phi \in \sigma(\mathcal{A})$. If $\mathcal{A}^* \cdot \mathcal{A} = \mathcal{A}^*$, then $\mathcal{A}$ is ϕ-amenable.*

Proof. Suppose that $\mathcal{A}$ is essentially ϕ-amenable. Consider the Banach $\mathcal{A}$-bimodule $\mathcal{A}^*$ with the left action given by

$$a \cdot f = \phi(a)f \quad (a \in \mathcal{A}, f \in \mathcal{A}^*),$$

and the dual right module action. Since $\mathcal{A}^* \cdot \mathcal{A} = \mathcal{A}^*$, it follows that $\mathcal{A}^*$ is a neo-unital Banach $\mathcal{A}$-bimodule. Consider the quotient neo-unital Banach $\mathcal{A}$-bimodule $X := \mathcal{A}^*/\mathbb{C}\phi$ and choose $m_0 \in \mathcal{A}^{**}$ with $m_0(\phi) = 1$. Then the image of ad_{m_0} is a subset

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of X^*, and hence, by our assumption, there exists

$$m_1 \in X^* = \{n \in \mathcal{A}^{**} : n(\phi) = 0\}$$

such that $\mathrm{ad}_{m_0} = \mathrm{ad}_{m_1}$. Therefore if we set $m := m_0 - m_1$, then $m(\phi) = 1$ and

$$m(f \cdot a) = \phi(a)m(f)$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Thus m is a ϕ-mean on $\mathcal{A}^*$ and so $\mathcal{A}$ is ϕ-amenable. □

Corollary 3.12. *Let $\mathcal{A}$ be an essentially ϕ-amenable Banach algebra with $\phi \in \sigma(\mathcal{A})$ and let I be a closed two-sided ideal of $\mathcal{A}$ with $I \subseteq \ker(\phi)$ such that $I^\perp = I^\perp \cdot \mathcal{A}$. Then $\mathcal{A}/I$ is ϕ_q-amenable.*

Proof. Suppose that $f \in (\mathcal{A}/I)^*$ and $a \in \mathcal{A}$. Clearly $(f \cdot q(a)) \circ q = (f \circ q) \cdot a$ for all $a \in \mathcal{A}$. Thus

$$I^\perp \cdot (\mathcal{A}/I) = I^\perp.$$

By Proposition 3.11 and Corollary 3.3, the proof is complete. □

Corollary 3.13. *Let $\mathcal{A}$ be an essentially ϕ-amenable Banach algebra with a left identity and let $\phi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is ϕ-amenable.*

Proof. By Proposition 3.11, we only need to note that

$$\mathcal{A}^* = \mathcal{A}^*\mathbf{e} \subseteq \mathcal{A}^* \cdot \mathcal{A} \subseteq \mathcal{A}^*$$

for any left identity $\mathbf{e}$ of $\mathcal{A}$. □

4. Essential character amenability of tensor product and Lau product

As usual, denote by $\mathcal{A} \widehat{\otimes} \mathcal{B}$ the projective tensor product of Banach algebras $\mathcal{A}$ and $\mathcal{B}$; moreover, for $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$, denote by $f \otimes g$ the element of $(\mathcal{A} \widehat{\otimes} \mathcal{B})^*$ satisfying

$$(f \otimes g)(a \otimes b) = f(a)g(b)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and note that

$$\sigma(\mathcal{A} \widehat{\otimes} \mathcal{B}) = \{\phi \otimes \psi : \phi \in \sigma(\mathcal{A}), \psi \in \sigma(\mathcal{B})\}.$$

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\phi \in \sigma(\mathcal{A})$ and $\psi \in \sigma(\mathcal{B})$. If $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is essentially $\phi \otimes \psi$-amenable, then $\mathcal{A}$ is essentially ϕ-amenable and $\mathcal{B}$ is essentially ψ-amenable.*

Proof. It is clear that the map $\Theta : \mathcal{A} \widehat{\otimes} \mathcal{B} \to \mathcal{A}$, defined by $\Theta(a \otimes b) = \psi(b)a$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, determines an epimorphism, and that $\phi \circ \Theta = \phi \otimes \psi$. So the proof is complete by Theorem 3.2. □

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We do not know if the converse of Theorem 4.1 is true.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\sigma(\mathcal{B}) \neq \emptyset$, and $\theta \in \sigma(\mathcal{B})$. Then the θ-Lau product, denoted by $\mathcal{A} \times_\theta \mathcal{B}$, is defined as the set $\mathcal{A} \times \mathcal{B}$ equipped with the multiplication

$$(a, b)(a', b') = (aa' + \theta(b)a' + \theta(b')a, bb'),$$

and the norm $\|(a, b)\| = \|a\| + \|b\|$ for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. Then $\mathcal{A}$ is a closed two-sided ideal of $\mathcal{A} \times_\theta \mathcal{B}$. We note that in the special case where $\mathcal{B}$ is the complex numbers $\mathbb{C}$ and θ is the identity map on $\mathbb{C}$, then $\mathcal{A} \times_\theta \mathcal{B}$ is the unitization $\mathcal{A}^\sharp$ of $\mathcal{A}$. Lau products have been studied in [12, 18]. In particular, it is shown in [18, Proposition 2.4], that

$$\sigma(\mathcal{A} \times_\theta \mathcal{B}) = \sigma(\mathcal{A}) \times \{\theta\} \cup \{0\} \times \sigma(\mathcal{B}).$$

Theorem 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\theta \in \sigma(\mathcal{B})$. Then the following statements hold.*

(a) *For each $\phi \in \sigma(\mathcal{A})$, if $\mathcal{A} \times_\theta \mathcal{B}$ is essentially (ϕ, θ)-amenable, then $\mathcal{A}$ is essentially ϕ-amenable.*

(b) *For each $\psi \in \sigma(\mathcal{B}) \cup \{0\}$, if $\mathcal{A} \times_\theta \mathcal{B}$ is essentially $(0, \psi)$-amenable, then $\mathcal{B}$ is essentially ψ-amenable.*

(c) *If $\mathcal{A} \times_\theta \mathcal{B}$ is essentially $(0, \theta)$-amenable, then $\mathcal{A}$ is essentially 0-amenable and $\mathcal{B}$ is essentially θ-amenable.*

Proof. (a) Suppose that $\mathcal{A} \times_\theta \mathcal{B}$ is essentially (ϕ, θ)-amenable. Let X be a neo-unital Banach $\mathcal{A}$-bimodule with the left module action

$$a \cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X),$$

and $D : \mathcal{A} \to X^*$ be a continuous derivation. Clearly X is a neo-unital Banach $\mathcal{A} \times_\theta \mathcal{B}$-bimodule with the actions

$$(a, b) \cdot x = (\phi, \theta)(a, b)x, \quad x \cdot (a, b) = x \cdot a + \theta(b)x \quad (a \in \mathcal{A}, b \in \mathcal{B}, x \in X).$$

Now if we define $\tilde{D} : \mathcal{A} \times_\theta \mathcal{B} \to X^*$ by

$$\tilde{D}(a, b) = Da$$

for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$, then we prove that $\tilde{D}$ is a continuous derivation. In fact, for every $(a, b), (a', b') \in \mathcal{A} \times_\theta \mathcal{B}$,

$$\begin{aligned}\tilde{D}((a, b)(a', b')) &= D(aa' + \theta(b')a + \theta(b)a') \\ &= D(aa') + \theta(b')D(a) + \theta(b)D(a').\end{aligned}$$

On the other hand,

$$\tilde{D}((a, b)) \cdot (a', b') = \phi(a')D(a) + \theta(b')D(a)$$

and

$$(a, b) \cdot \tilde{D}((a', b')) = a \cdot D(a') + \theta(b)D(a').$$

Thus, we conclude that $\tilde{D}$ is a continuous derivation. By essential (ϕ, θ)-amenability of $\mathcal{A} \times_\theta \mathcal{B}$, we have that $\tilde{D}$ is inner. Since $\tilde{D}|_{\mathcal{A}} = D$, it follows that D is inner.

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(b) It is trivial that the map $\Theta : \mathcal{A} \times_\theta \mathcal{B} \to \mathcal{B}$, defined by

$$\Theta((a, b)) = b$$

for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$, determines a continuous epimorphism. Since $\psi \circ \Theta = (0, \psi)$, it follows that $\mathcal{B}$ is essentially ψ-amenable if $\mathcal{A} \times_\theta \mathcal{B}$ is essentially $(0, \psi)$-amenable by Theorem 3.2.

(c) Suppose that $\mathcal{A} \times_\theta \mathcal{B}$ is essentially $(0, \theta)$-amenable. That $\mathcal{B}$ is essentially θ-amenable follows from (b). The proof of essentially 0-amenability of $\mathcal{A}$ is similar to the proof of (a). □

COROLLARY 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\theta \in \sigma(\mathcal{B})$. Then essential character amenability of $\mathcal{A} \times_\theta \mathcal{B}$ implies essential character amenability of $\mathcal{A}$ and $\mathcal{B}$.*

The converse of the above corollary is not true; for example, let $\mathcal{A}$ be an essentially character amenable Banach algebra which is not character amenable. Since $\mathcal{A}^\sharp$ is character amenable if and only if $\mathcal{A}$ is character amenable by [19, Theorem 2.6(iv)], it follows from Proposition 2.2 that $\mathcal{A}^\sharp$ is not essentially character amenable.

5. Essential character amenability of group algebras

Let G be a locally compact group with left Haar measure λ_G and let $L^1(G)$ be the group algebra of G as defined in [7] endowed with the norm $\|\cdot\|_1$ and the convolution product $*$. Let $L^\infty(G)$ be the usual Lebesgue space with the essentially supremum norm $\|\cdot\|_\infty$, and let $M(G)$ be the measure algebra of G as defined in [7]. We recall that a closed left invariant subspace X of $L^\infty(G)$ is called left introverted if $F \cdot f \in X$ for all $F \in L^\infty(G)$ and $f \in X$, where

$$(F \cdot f)(a) = F(f \cdot a)$$

for all $a \in L^1(G)$. In this case, X^* is a Banach algebra with the multiplication induced by the first Arens product $\odot$ on X^*, defined by

$$(E \odot F)(f) = E(F \cdot f)$$

for all $E, F \in X^*$ and $f \in X$. Examples of closed left introverted subspaces of $L^\infty(G)$ include the space $AP(G)$ of all almost periodic functions on G, the space $WAP(G)$ of all weakly almost periodic functions on G, and the space $LUC(G)$ of all left uniformly continuous functions on G; see [7] for more details.

In the following, we prove an essential character amenability version of [8, Theorem 3.9] for the class of maximally almost periodic groups, which contains all abelian groups and compact groups.

THEOREM 5.1. *Let G be a maximally almost periodic locally compact group and let X be a left introverted subspace of $L^\infty(G)$ containing $AP(G)$. Then X^* is essentially character amenable if and only if G is finite.*

PROOF. The 'if' part is trivial. To prove the converse, suppose that X^* is essentially character amenable. Then $AP(G)^*$ is also essentially character amenable; this follows

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from 3.2 together with the fact that the restriction map

$$X^* \to AP(G)^*, \quad f \to f|_{AP(G)},$$

is a continuous epimorphism. Let bG be the Bohr compactification of G and note that

$$M(bG) \cong AP(G)^*$$

is essentially character amenable; since $M(bG)$ has an identity, it is character amenable. By [19, Corollary 2.5], bG must be discrete, and hence it is finite. Since, for a maximally almost periodic group G, the canonical homomorphism from G into bG is injective, it follows that G is finite. □

Theorem 5.1 is applicable to the spaces $L^\infty(G)$ and $LUC(G)$; our next result improves Theorem 5.1 to all locally compact groups for these two spaces.

Proposition 5.2. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *$LUC(G)^*$ is essentially character amenable.*
(b) *$L^1(G)^{**}$ is essentially character amenable.*
(c) *G is finite.*

Proof. (a) ⇒ (b). Suppose that $LUC(G)^*$ is essentially character amenable and note that the restriction map $\Theta : LUC(G)^* \to M(G)$ is a continuous epimorphism. This, together with Theorem 3.2, implies that $M(G)$ is essentially character amenable. Since $M(G)$ has an identity, it is character amenable by Corollary 2.3 and so G is discrete and amenable by [19, Corollary 2.5]. Thus $L^\infty(G) = LUC(G)$.

(b) ⇒ (c). Suppose that $L^1(G)^{**}$ is essentially character amenable. Then by Remark 2.5, $L^1(G)^{**}$ does not have any nonzero continuous point derivation corresponding to any character $\phi \in \sigma(L^1(G)^{**})$. It follows from [3, Theorem 11.17] that G is finite.

(c) ⇒ (a). This is trivial. □

Proposition 5.2 leads us to the conjecture that Theorem 5.1 is true for all locally compact groups. Here, we consider another subspace of $L^\infty(G)$; that is, the space $L_0^\infty(G)$ of all $f \in L^\infty(G)$ which vanish at infinity; in fact,

$$L_0^\infty(G) = \{f \in L^\infty(G) : \text{for } K \text{ compact}, \|f\chi_{G\backslash K}\|_\infty \to 0 \text{ as } K \uparrow G\}.$$

This space was introduced and studied extensively by Lau and Pym [14]; see also [2, 15–17].

Now let $\widehat{G}$ denote the dual group of G consisting of all continuous homomorphisms ρ from G into the circle group $\mathbb{T}$, and define $\phi_\rho \in \sigma(L^1(G))$ to be the character induced by ρ on $L^1(G)$; that is,

$$\phi_\rho(a) = \int_G \overline{\rho(x)} a(x)\, d\lambda_G(x) \quad (a \in L^1(G)).$$

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On the other hand, there is no other character on $L^1(G)$. that is,

$$\sigma(L^1(G)) = \{\phi_\rho : \rho \in \widehat{G}\};$$

see for example [7, Theorem 23.7]. It is known from [14] that $L_0^\infty(G)$ is a closed left introverted subspace of $L^\infty(G)$ and $L^1(G)$ is a closed two-sided ideal in $L_0^\infty(G)^*$, the dual Banach algebra endowed with the first Arens product $\odot$ defined at the beginning of the section. Thus for each $\rho \in \widehat{G}$ the induced character ϕ_ρ on $L^1(G)$ has the unique extension $\tilde{\phi}_\rho \in \sigma(L_0^\infty(G)^*)$ defined by

$$\tilde{\phi}_\rho(F) = \phi_\rho(F \odot a_0)$$

for all $F \in L_0^\infty(G)^*$, where $a_0 \in L^1(G)$ with $\phi_\rho(a_0) = 1$. Note that $L_0^\infty(G)^*$ has a bounded approximate identity if and only if G is discrete; see [17, Proposition 3.1].

Proposition 5.3. *Let G be a locally compact group and let $\rho \in \widehat{G}$. Then $L_0^\infty(G)^*$ is essentially $\tilde{\phi}_\rho$-amenable if and only if G is amenable.*

Proof. The result follows from [1, Corollary 3.4] and Proposition 2.2, together with the fact that $L^1(G)$ always has a bounded approximate identity and is a closed two-sided ideal in $L_0^\infty(G)^*$. □

Proposition 5.4. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *$L_0^\infty(G)^*$ is character amenable.*
(b) *$L_0^\infty(G)^*$ is essentially character amenable.*
(c) *G is discrete and amenable.*

Proof. That (a) implies (b) is trivial. Suppose that (b) holds. By [14, Theorem 2.11], for each right identity E of $L^\infty(G)^*$ with norm one, $E \odot L_0^\infty(G)^*$ is isometrically isomorphic with $M(G)$. Also the map $F \mapsto E \odot F$, for each $F \in L_0^\infty(G)^*$, is a continuous epimorphism from $L_0^\infty(G)^*$ onto $M(G)$, and so $M(G)$ is essentially character amenable by Theorem 3.2. Since $M(G)$ has an identity, it is character amenable by Corollary 2.3 and consequently G is discrete and amenable by [19, Corollary 2.5].

Now suppose that G is discrete and amenable. Then

$$L_0^\infty(G)^* = M(G) = \ell^1(G),$$

and so $L_0^\infty(G)^*$ is amenable; see, for example, [20, Theorem 2.1.8]. In particular, $L_0^\infty(G)^*$ is character amenable. □

Acknowledgements

The authors would like to thank the referee for the careful reading of this paper and helpful suggestions. The authors thank the Center of Excellence for Mathematics at the Isfahan University of Technology.

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