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DERIVATIONS OF FRÉCHET NUCLEAR GB*-ALGEBRAS

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Abstract

It is an open question whether every derivation of a Fréchet GB*-algebra $A[\tau]$ is continuous. We give an affirmative answer for the case where $A[\tau]$ is a smooth Fréchet nuclear GB*-algebra. Motivated by this result, we give examples of smooth Fréchet nuclear GB*-algebras which are not pro-C*-algebras.

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1. Introduction

The algebras known as GB*-algebras (that is, generalised B*-algebras) are locally convex $*$-algebras which are generalisations of C*-algebras. They were introduced in [4] by Allan in 1967, and the concept was later extended by Dixon in [11] to incorporate nonlocally convex algebras. GB*-algebras are also abstract algebras of unbounded operators in the sense that the well-known Gelfand–Naimark representation theorem for C*-algebras extends to GB*-algebras [11, Theorems 7.6 and 7.11].

Algebras of unbounded operators are important in quantum physics and quantum statistical mechanics, in that the observables of a quantum mechanical system are regarded as unbounded self-adjoint operators on a Hilbert space and the time evolution of the system can be modelled by one-parameter automorphism groups of the latter algebras. Furthermore, derivations are the generators of these groups [8].

If X is a bimodule over an algebra A, then we call a linear map $\delta : A \to X$ a *derivation* if $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in A$. The derivation δ is said to be *inner* if there exists an $x \in X$ such that $\delta(a) = ax - xa$ for all $a \in A$. It is well known that the zero derivation is the only derivation of a commutative C*-algebra [23, Corollary 2.2.8]

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and that all derivations of a C*-algebra are continuous [23, Theorem 2.3.1]. Also, all derivations of a von Neumann algebra are inner [23, Theorem 2.5.3]. An abundance of automatic continuity results for derivations of Banach algebras can be found in [10].

The article of Brödel and Lassner [9] is the first article about derivations of unbounded operator algebras to appear in the literature. Much later, in 1992, Becker proved that every derivation $\delta : A \to A$ of a pro-C*-algebra $A[\tau_\Gamma]$ is continuous [5, Proposition 2]. By a pro-C*-algebra, we mean a complete topological $*$-algebra $A[\tau_\Gamma]$ for which there exists a directed family of C*-seminorms defining the topology τ_Γ [13].

Becker also proved that commutative pro-C*-algebras have no nonzero derivations [5, Corollary 3]. Other results concerning derivations of nonnormed topological $*$-algebras and unbounded operator algebras can be found in [1, 2, 6, 18, 26, 28, 30]. For a detailed survey of derivations of locally convex $*$-algebras, see [17].

All of the above, together with [17, discussion after Theorem 5.2], provide motivation for a general investigation of derivations of GB*-algebras. In [28], we proved that a complete commutative GB*-algebra having jointly continuous multiplication has no nonzero derivations [28, Theorem 3.3]. In particular, every Fréchet commutative GB*-algebra has no nonzero derivations [28, Corollary 3.4].

A *nuclear GB*-algebra* is a GB*-algebra for which $A[B_0]$ is a nuclear C*-algebra [16]. Clearly, every commutative GB*-algebra is a nuclear GB*-algebra. Some basic properties and examples of nuclear GB*-algebras can be found in [16]. For instance, [16, Theorem 6.8] is a characterisation of nuclear GB*-algebras.

If $A[\tau]$ is a locally convex algebra with a defining family of seminorms Γ and $X[\tau']$ is a locally convex A-bimodule, then we say that $X[\tau']$ is a $\tau - \tau'$ *smooth* A-bimodule if there exist defining families of seminorms $\{p_\lambda : \lambda \in \Lambda_0\}$ and $\{q_\alpha : \alpha \in \Lambda_1\}$ for $A[\tau]$ and $X[\tau']$, respectively, such that Λ_0 and Λ_1 are directed and such that, for every $\alpha \in \Lambda_1$, there exists $\lambda \in \Lambda_0$ such that $q_\alpha(ax) \le p_\lambda(a)q_\alpha(x)$ and $q_\alpha(xa) \le p_\lambda(a)q_\alpha(x)$ for all $a \in A$ and $x \in X$.

We say that the locally convex algebra $A[\tau]$ is *smooth* if there exists a defining family of seminorms $\{p_\lambda : \lambda \in \Lambda_0\}$ for $A[\tau]$ such that, for every $\lambda \in \Lambda_0$, there exists $\mu \in \Lambda_0$ such that $p_\lambda(ab) \le p_\mu(a)p_\lambda(b)$ and $p_\lambda(ba) \le p_\mu(a)p_\lambda(b)$ for all $a, b \in A$. We say that this family of seminorms is smooth. It is clear that every smooth locally convex algebra is a smooth locally convex bimodule over itself.

Smooth modules were introduced in an attempt to generalise a result of Sheinberg concerning the flatness of cyclic Banach modules over Banach algebras to locally convex modules over locally convex algebras. This leads to a characterisation of the notion of amenability in the context of Fréchet algebras (see [20] in this regard). Some general information concerning smooth modules can be found in [21].

It is an open question whether every derivation of a Fréchet GB*-algebra is continuous. The purpose of this article is to give a partial answer to this question. To be more specific, we prove in Section 4 that every derivation of a smooth Fréchet nuclear GB*-algebra is continuous. In [29], we proved that every derivation of a Fréchet GB*-algebra $A[\tau]$ is continuous, for which $A[B_0]$ is a W*-algebra. It is clear that every pro-C*-algebra is a smooth GB*-algebra and, in Section 3, motivated by the main result

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in Section 4, we give examples of smooth Fréchet nuclear GB*-algebras which are not pro-C*-algebras in general. Section 2 consists of all the necessary background for understanding and proving the main results of this paper.

2. Preliminaries

All vector spaces in the paper are over the field $\mathbb{C}$ of complex numbers and all topological spaces are assumed to be Hausdorff. Moreover, all algebras are assumed to have an identity element denoted by 1.

A *topological algebra* is an algebra which is also a topological vector space such that the multiplication is separately continuous in both variables [13]. A *topological $*$-algebra* is a topological algebra endowed with a continuous involution. A topological $*$-algebra which is also a locally convex space is called a *locally convex $*$-algebra*. The symbol $A[\tau]$ will stand for a topological $*$-algebra A endowed with given topology τ.

Definition 2.1 [4]. Let $A[\tau]$ be a topological $*$-algebra and $\mathcal{B}^*$ a collection of subsets B of A with the following properties:

(i) B is absolutely convex, closed and bounded;
(ii) $1 \in B$, $B^2 \subset B$ and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of B, which is a normed algebra under the gauge function $\|\cdot\|_B$ of B. If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudo-complete*.

An element $x \in A$ is called *(Allan) bounded* if, for some nonzero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, 3, \ldots\}$ is bounded in A. We denote by A_0 the set of all bounded elements in A.

A topological $*$-algebra $A[\tau]$ is called *symmetric* if, for every $x \in A$, the element $(1 + x^*x)^{-1}$ exists and belongs to A_0.

In [11], the collection $\mathcal{B}^*$ in the definition above is defined to be the same as above, except that $B \in \mathcal{B}^*$ is no longer assumed to be absolutely convex. The notion of a bounded element is a generalisation of the concept of a bounded operator on a Banach space and was introduced by Allan in [3] in order to develop a spectral theory for general locally convex $*$-algebras.

Definition 2.2 [4]. A symmetric pseudo-complete locally convex $*$-algebra $A[\tau]$ such that the collection $\mathcal{B}^*$ has a greatest member denoted by B_0 is called a *GB*-algebra* over B_0.

Every sequentially complete locally convex algebra is pseudo-complete [3, Proposition 2.6]. In [11], Dixon extended the notion of GB*-algebras to include topological $*$-algebras which are not locally convex. In this definition, GB*-algebras are not assumed to be pseudo-complete, B_0 is the only element in $\mathcal{B}^*$ which is necessarily absolutely convex (see the paragraph before Definition 2.2) and only $A[B_0]$ is assumed to be complete with respect to the gauge function $\|\cdot\|_{B_0}$. For a survey of GB*-algebras, see [15].

Proposition 2.3 ([4, Theorem 2.6] and [7, Theorem 2]). *If $A[\tau]$ is a GB*-algebra, then the Banach ∗-algebra $A[B_0]$ is a C*-algebra sequentially dense in A and $(1 + x^*x)^{-1} \in A[B_0]$ for every $x \in A$. Furthermore, B_0 is the unit ball of $A[B_0]$.*

If A is commutative, then $A_0 = A[B_0]$ [4, page 94]. In general, A_0 is not a ∗-subalgebra of A and $A[B_0]$ contains all normal elements of A_0 [4, page 94].

It is well known that every commutative C*-algebra is topologically and algebraically ∗-isomorphic to $C(X)$ for some compact Hausdorff space (in fact, X is the maximal ideal space of A). More generally, any commutative GB*-algebra is algebraically ∗-isomorphic to an algebra of functions on a compact Hausdorff space X which are allowed to take the value infinity on at most a nowhere dense subset of X [4, Theorem 3.9]. This algebraic ∗-isomorphism extends the Gelfand isomorphism of $A[B_0]$ onto the corresponding $C(X)$.

Recall that every C*-algebra is topologically-algebraically ∗-isomorphic to a norm closed ∗-subalgebra of $B(H)$ for some Hilbert space H. In general, for every GB*-algebra $A[\tau]$, there exists a faithful ∗-representation $\pi : A \to \pi(A)$ such that $\pi(A)$ is an algebra of closed densely defined operators on a Hilbert space H with B_0 being identified with $\{x \in \pi(A) \cap B(H) : \|x\| \leq 1\}$ [11, Theorem 7.11]. Therefore, for every $a \in A$, it follows that $\|(1 + a^*a)^{-1}\| \leq 1$ (see also [4, Proposition 2.6]) and that $a(1 + a^*a)^{-1} \in A[B_0]$.

Proposition 2.4 [19, Proposition 3.4]. *If $A[\tau]$ is a pro-C*-algebra and $X[\tau']$ is a complete locally convex A-bimodule having $\tau \times \tau' - \tau'$ jointly continuous module actions, then $X[\tau']$ is a $\tau - \tau'$ smooth A-bimodule.*

If, in particular, $A[\|\cdot\|]$ is a C*-algebra and $X[\tau']$ is a complete locally convex A-bimodule having $\|\cdot\| \times \tau' - \tau'$ jointly continuous module actions, then $X[\tau']$ is a $\|\cdot\| - \tau'$ smooth A-bimodule.

3. Examples of smooth Fréchet GB*-algebras

In this section, we give an example of a smooth Fréchet GB*-algebra (Example 3.4) and use it to construct an example of a smooth Fréchet nuclear GB*-algebra (Example 3.5). We begin by first outlining an example of a GB*-algebra given in [12].

Definition 3.1 [12, Definition 1.1]. A set $\mathcal{R}$ of bounded self-adjoint linear operators on a Hilbert space H is called a generating family if it satisfies the following conditions:

(i) $0 \leq a \leq 1$ for all $a \in \mathcal{R}$;
(ii) $ab = ba$ for all $a, b \in \mathcal{R}$;
(iii) for all $a, b \in \mathcal{R}$, there exists $c \in \mathcal{R}$ such that $a \leq c$ and $b \leq c$;
(iv) for every $a \in \mathcal{R}$, there exists $b \in \mathcal{R}$ such that $a \leq b^2$.

Observe that 1 need not be in $\mathcal{R}$. In what follows, we equip the set $\mathcal{L}_{\mathcal{R}} = \bigcup_{a\in\mathcal{R}} aH$ with the inductive limit topology [12, Definition 1.3]. Let x be a densely defined linear operator on H such that its domain contains $\mathcal{L}_{\mathcal{R}}$. Then x is called $\mathcal{R}$- *bounded* if xa is

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a bounded linear operator for every $a \in \mathcal{R}$ [12, Definition 2.1]. The vector space of all $\mathcal{R}$-bounded linear operators is denoted by $\mathcal{R}B(H)$. For every $x \in \mathcal{R}B(H)$, the restriction of x to $\mathcal{L}_{\mathcal{R}}$ is a continuous linear operator into H and, conversely, every continuous linear operator from $\mathcal{L}_{\mathcal{R}}$ into H is $\mathcal{R}$-bounded [12, remark, page 112]. Clearly, $\mathcal{R} \subset \mathcal{R}B(H)$.

On $\mathcal{R}B(H)$, we define the seminorms p_a, $a \in \mathcal{R}$, by $p_a(x) = \|xa\|$ for every $x \in \mathcal{R}B(H)$. From the previous paragraph, it follows that if $p_a(x) = 0$ for all $a \in \mathcal{R}$, then $x = 0$ [12, page 112], implying that $\mathcal{R}B(H)$ is a Hausdorff locally convex space.

If $A \subset \mathcal{R}B(H)$, then we define $A' \subset \mathcal{R}B(H)$ as follows [12, Definition 2.4]:

$$\begin{aligned} A' = \{y \in \mathcal{R}B(H) : yx \in \mathcal{R}B(H), xy \in \mathcal{R}B(H) \\ \text{and } yxa = xya \text{ for all } x \in A \text{ and } a \in \mathcal{R}\}. \end{aligned}$$

The map $x \mapsto x^+ := x^*|_{\mathcal{L}_{\mathcal{R}}}$ defines an involution on $\mathcal{R}'$ [12, Definition 3.3 and Lemma 3.4], and $p_a(x) = p_a(x^+)$ for every $a \in \mathcal{R}$ and $x \in \mathcal{R}'$ [12, Lemma 3.6(ii)].

Theorem 3.2 [12, Theorem 3.10, Lemma 3.6(i) and Corollary 3.8]. *Let $\mathcal{R}$ be a generating family of bounded linear operators on H. Then $\mathcal{R}'$ is a sequentially complete GB*-algebra with respect to the locally convex topology defined by the family of seminorms p_a, $a \in \mathcal{R}$, restricted to $\mathcal{R}'$, and with respect to the involution defined above. Furthermore, for every $a \in \mathcal{R}$, there exists $b \in \mathcal{R}$ such that $p_a(xy) \le p_b(x)p_b(y)$ for all $x, y \in \mathcal{R}'$.*

The last statement of Theorem 3.2 implies that $\mathcal{R}'$ is a GB*-algebra having jointly continuous multiplication. Observe in Theorem 3.2 that the seminorms p_a, $a \in \mathcal{R}$, are generally not smooth on $\mathcal{R}'$. However, if $\mathcal{R}$ has the additional property that for every $a \in \mathcal{R}$, there exists $q \in \mathcal{R}$ such that $a = qa$, then, for every $a \in \mathcal{R}$,

$$p_a(xy) = \|xya\| = \|xyqa\| = \|xq \cdot ya\| \le \|xq\| \, \|ya\| = p_q(x)p_a(y)$$

for all $x, y \in \mathcal{R}'$.

In Example 3.4 below, we exhibit a countable generating family $\mathcal{R}_2$ having the additional property mentioned above, thereby ensuring that $(\mathcal{R}_2)'$ is a smooth Fréchet GB*-algebra. For this example, we require Lemma 3.3 below.

In what follows, let $0 < a < 1$ be a bounded (self-adjoint) linear operator on H and let $\mathcal{R}_1 = \{a^{1/n} : n \in \mathbb{N}\} \cup \{a^n : n \in \mathbb{N}\}$. If M denotes the commutative von Neumann algebra generated by $\mathcal{R}_1$, then $r(x)$ denotes the support of $x \in M$ relative to M, that is, for every $x \in M$, $r(x)$ is the least projection in M such that $xr(x) = r(x)x = x$. It is easily seen that $r(r(x)) = r(x)$ for all $x \in M$.

Lemma 3.3.

(i) *If $b_1, b_2 \in \mathcal{R}_1$ are such that $b_1 \le b_2$, then $r(b_1) \le r(b_2)$.*
(ii) *If $x \in M$ and $0 \le x \le 1$, then $x \le r(x)$.*

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Proof. (i) Let $b_1, b_2 \in \mathcal{R}_1$ be such that $b_1 \leq b_2$. We consider the three cases below.

Case 1. $b_1 = a^n$ and $b_2 = a^k$ for some $n, k \in \mathbb{N}$, and therefore $k \leq n$. Since $\mathcal{R}_1 \subset M$ and M is commutative, it follows that

$$r(b_1) = r(a^n) = r(a^k a^{n-k}) \leq r(a^k) r(a^{n-k}) \leq r(a^k) = r(b_2).$$

Case 2. $b_1 = a^{1/k}$ and $b_2 = a^{1/m}$ for some $m, k \in \mathbb{N}$, and therefore $k \leq m$. By using a similar argument to that of Case 1, we get $r(b_1) \leq r(b_2)$.

Case 3. $b_1 = a^n$ and $b_2 = a^{1/k}$ for some $n, k \in \mathbb{N}$. Once again, by using a similar argument to that of Case 1 above, we get $r(b_1) \leq r(b_2)$.

(ii) Let $x \in M$ with $0 \leq x \leq 1$. Then $1 - x \geq 0$ and, therefore, since $r(x)x = x$, it follows for every $\xi \in H$ that

$$\begin{aligned}\langle (r(x) - x)\xi, \xi\rangle &= \langle r(x)(1 - x)\xi, \xi\rangle \\ &= \langle r(x)^* r(x)(1 - x)\xi, \xi\rangle \\ &= \langle (1 - x) r(x)\xi, r(x)\xi\rangle \\ &\geq 0,\end{aligned}$$

implying that $x \leq r(x)$. □

Example 3.4. Let a and $\mathcal{R}_1$ be as above. Then $\mathcal{R}_1$ clearly satisfies properties (i) and (ii) of Definition 3.1. For any $b_1, b_2 \in \mathcal{R}_1$, it is clear that we either have $b_1 \leq b_2$ or $b_2 \leq b_1$, so that property (iii) of Definition 3.1 holds. For any $n \in \mathbb{N}$, we have that $a^n \leq a = (a^{1/2})^2$. Observe that $a^{1/2} \in \mathcal{R}_1$. If $n \in \mathbb{N}$, then there exists $r \in \mathbb{N}$ such that $n \leq 2r$ and therefore $a^{1/n} \leq a^{1/2r} = (a^{1/4r})^2$. Note also that $a^{1/4r} \in \mathcal{R}_1$. Hence, property (iv) of Definition 3.1 holds.

Now let $\mathcal{R}_2 = \mathcal{R}_1 \cup \{r(b) : b \in \mathcal{R}_1\}$. Clearly, properties (i) and (ii) of Definition 3.1 still remain valid. Since $r(b) = r(b)^2$ for every $b \in \mathcal{R}_1$, it is clear that property (iv) of Definition 3.1 also remains valid. We now show that property (iii) of Definition 3.1 still holds. Consider $r(b_1)$ and $r(b_2)$ for some $b_1, b_2 \in \mathcal{R}_1$. As noted above, we have either $b_1 \leq b_2$ or $b_2 \leq b_1$. Without loss of generality, we may assume that $b_1 \leq b_2$. It follows from Lemma 3.3(i) that $r(b_1) \leq r(b_2)$.

Now let $a_1 = b_1$ and $a_2 = r(b_2)$ for some $b_1, b_2 \in \mathcal{R}_1$. Once again, we have either $b_1 \leq b_2$ or $b_2 \leq b_1$. If $b_1 \leq b_2$, then it follows from Lemma 3.3(i) that $r(b_1) \leq r(b_2)$. By Lemma 3.3(ii), $b_1 \leq r(b_1)$. Therefore, $a_1 = b_1 \leq r(b_2) = a_2$. If $b_2 \leq b_1$ instead, then, by Lemma 3.3(i), $r(b_2) \leq r(b_1)$. It follows from Lemma 3.3(ii) that $b_1 \leq r(b_1)$ and hence $a_1 \leq r(b_1)$ and $a_2 \leq r(b_1)$. Therefore, property (iii) of Definition 3.1 remains valid.

We conclude from the discussion preceding Lemma 3.3 that $(\mathcal{R}_2)'$ is a smooth Fréchet GB*-algebra.

We now use Example 3.4 to construct an example of a smooth Fréchet nuclear GB*-algebra from a nuclear C*-algebra which is also a von Neumann algebra.

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EXAMPLE 3.5. Let A be a von Neumann algebra which is also a nuclear C*-algebra. Then $Z(A)$, the centre of A, is a commutative von Neumann algebra. Let $0 < a < 1$ be a bounded (self-adjoint) linear operator in $Z(A)$ and construct from this the generating family $\mathcal{R}_2$ of Example 3.4, that is,

$$\mathcal{R}_1 = \{a^{1/n} : n \in \mathbb{N}\} \cup \{a^n : n \in \mathbb{N}\}$$

and

$$\mathcal{R}_2 = \mathcal{R}_1 \cup \{r(b) : b \in \mathcal{R}_1\},$$

where, as before, for any x in the von Neumann algebra M generated by $\mathcal{R}_1$, $r(x)$ denotes the support of x in M. Now $\mathcal{R}_1 \subset Z(A)$ and so $M \subset Z(A)$. Hence, $\mathcal{R}_2 \subset Z(A)$.

It now follows that $A \subset (\mathcal{R}_2)'$: if $x \in A$, then cx and xc are in $\mathcal{R}_2 B(H)$ and $xcb = cxb$ for all $c, b \in \mathcal{R}_2$. Therefore, $A \subset (\mathcal{R}_2)'$.

This allows one to restrict the seminorms $p_c, c \in \mathcal{R}_2$, on $(\mathcal{R}_2)'$ to A, thereby obtaining a family of smooth seminorms on A (in Example 3.4, we have, from the discussion preceding Lemma 3.3, that the family of seminorms p_c, $c \in \mathcal{R}_2$, on $(\mathcal{R}_2)'$ is smooth). We use the symbol τ to denote the locally convex topology on A defined by the above family of (smooth) seminorms.

It is easily verified that the topology τ on A is weaker than the given norm topology on A. Therefore, the completion $\widetilde{A}[\tau]$ of $A[\tau]$ is a GB*-algebra over the τ-closure $\overline{\mathcal{U}(A)}$ in A of the unit ball $\mathcal{U}(A)$ of A [14, Theorem 2.1].

We prove that $\mathcal{U}(A)$ is τ-closed in A. Let $\|x\|_{bd} = \sup_{c \in \mathcal{R}_2} \|xc\|$ for all $x \in A$. We first show that $\|\cdot\|_{bd}$ is a C*-norm on A. For every $x \in A$ and $c \in \mathcal{R}_2$, we have that $\|xc\|^2 \leq \|x^+ xc\|$ (see the proof of [12, Theorem 3.10]). Therefore, for every $x \in A$,

$$\|x\|_{bd}^2 = \sup_{c \in \mathcal{R}_2} \|xc\|^2 \leq \sup_{c \in \mathcal{R}_2} \|x^+ xc\| = \|x^+ x\|_{bd}.$$

From Theorem 3.2, we have that for every $c \in \mathcal{R}_2$, there exists $b \in \mathcal{R}_2$ such that $p_c(xy) \leq p_b(x) p_b(y)$ for all $x, y \in A$. Therefore,

$$\begin{aligned} \|x^+ x\|_{bd} &= \sup_{c \in \mathcal{R}_2} p_c(x^+ x) \\ &\leq \sup_{b \in \mathcal{R}_2} p_b(x)^2 = \|x\|_{bd}^2 \end{aligned}$$

for every $x \in A$. Therefore, $\|x\|_{bd}^2 = \|x^+ x\|_{bd}$ for all $x \in A$.

We now show that $\|x^* x\|_{bd} = \|x^+ x\|_{bd}$ for all $x \in A$. For any $c \in \mathcal{R}_2$,

$$\begin{aligned} \|x^* xc\| = \sup_{\|\xi\|=1} \|x^* xc\xi\| &= \sup_{\|\xi\|=1} \|(x^* x)^* c\xi\| \\ &= \sup_{\|\xi\|=1} \|(x^* x)^+ c\xi\| = \sup_{\|\xi\|=1} \|x^+ (x^*)^+ c\xi\| \\ &= \sup_{\|\xi\|=1} \|x^+ (x^*)^* c\xi\| = \sup_{\|\xi\|=1} \|x^+ xc\xi\| \\ &= \|x^+ xc\|. \end{aligned}$$

Therefore, $\|x^* x\|_{bd} = \sup_{c \in \mathcal{R}_2} \|x^* xc\| = \sup_{c \in \mathcal{R}_2} \|x^+ xc\| = \|x^+ x\|_{bd}$ for all $x \in A$.

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It follows that $\|\cdot\|_{\mathrm{bd}}$ is a C*-norm on A and, therefore, by [25, Corollary I.5.4], $\|x\| = \|x\|_{\mathrm{bd}}$ for every $x \in A$. Finally, to show that $\mathcal{U}(A)$ is τ-closed in A, we consider a net (x_α) in $\mathcal{U}(A)$ with $x_\alpha \underset{\tau}{\to} x \in A$. Therefore, $\|x_\alpha c\| \le \|x_\alpha\|\,\|c\| \le 1$ for all α and $c \in \mathcal{R}_2$. Since $x_\alpha \underset{\tau}{\to} x \in A$, we have $\|x_\alpha c\| \to \|xc\|$ for all $c \in \mathcal{R}_2$, implying that $\|xc\| \le 1$ for all $c \in \mathcal{R}_2$. Therefore, $\|x\| = \|x\|_{\mathrm{bd}} = \sup_{c\in\mathcal{R}_2} \|xc\| \le 1$. Hence, $x \in \mathcal{U}(A)$, and therefore $\mathcal{U}(A)$ is τ-closed in A.

It follows that $\widetilde{A}[\tau]$ is a GB*-algebra over $\mathcal{U}(A)$, which yields $(\widetilde{A}[\tau])[B_0] = A$, and therefore $\widetilde{A}[\tau]$ is a Fréchet nuclear GB*-algebra. Since $A[\tau]$ is smooth (with respect to the seminorms $p_c, c \in \mathcal{R}_2$) and $\widetilde{A}[\tau]$ has jointly continuous multiplication, it follows that $\widetilde{A}[\tau]$ is a smooth Fréchet nuclear GB*-algebra (the extensions of the seminorms $p_c, c \in \mathcal{R}_2$, of $A[\tau]$ to $\widetilde{A}[\tau]$ are seminorms which define the topology τ on $\widetilde{A}[\tau]$).

4. Main results

For the proof of the following theorem, we establish the convention that if $X[\|\cdot\|]$ is a Banach space, then $\|\cdot\|$ will also be used to denote the norms on X^* and X^{**}.

Let $X[\|\cdot\|_0]$ be a normed space which is also a bimodule over a Banach algebra $A[\|\cdot\|]$. If the module actions of X over A are $\|\cdot\| \times \|\cdot\|_0 - \|\cdot\|_0$ jointly continuous, then it is well known that there exists a norm $\|\cdot\|_1$ equivalent to $\|\cdot\|_0$ (that is, defining the same topology on X as $\|\cdot\|_0$) such that $\|ax\|_1 \le \|a\|\,\|x\|_1$ and $\|xa\|_1 \le \|a\|\,\|x\|_1$ for every $a \in A$ and $x \in X$. Therefore, X is a normed A-bimodule. This will also be needed in the proof of the following theorem.

Theorem 4.1. *Let $A[\|\cdot\|]$ be a nuclear C*-algebra and τ a locally convex topology on A such that τ is weaker than the topology defined by $\|\cdot\|$. If $X[\tau']$ is a complete $(\tau - \tau')$ smooth locally convex A-bimodule and $\delta : A \to X$ is a derivation, then δ is $\tau - \tau'$ continuous.*

Proof. By hypothesis, there exist defining families of seminorms $\{p_\lambda : \lambda \in \Lambda_0\}$ and $\{q_\alpha : \alpha \in \Lambda_1\}$ for $A[\tau]$ and $X[\tau']$, respectively, such that, for every $\alpha \in \Lambda_1$, there exists $\lambda \in \Lambda_0$ such that $q_\alpha(ax) \le p_\lambda(a) q_\alpha(x)$ and $q_\alpha(xa) \le p_\lambda(a) q_\alpha(x)$ for all $a \in A$ and $x \in X$. Furthermore, Λ_0 and Λ_1 are directed. It follows, for every $\alpha \in \Lambda_1$, that $X_\alpha \equiv X/N_\alpha$ is an A-bimodule with respect to the following (well-defined) module actions:

$$a \cdot (x + N_\alpha) = ax + N_\alpha \quad \text{and} \quad (x + N_\alpha) \cdot a = xa + N_\alpha,$$

where $N_\alpha = \{x \in X : q_\alpha(x) = 0\}$ for every $\alpha \in \Lambda_1$. Then X is isomorphic to $\varprojlim \widetilde{X_\alpha}$ as locally convex spaces, where $\widetilde{X_\alpha}$ is the completion of $X_\alpha = X/N_\alpha$ with respect to the topology induced by the norm $\overline{q}_\alpha(x + N_\alpha) = q_\alpha(x)$, $x \in X$, on X_α.

Note that X_α is a normed A-bimodule with respect to the above module actions: let (a_n) be a sequence in A such that $a_n \underset{\|\cdot\|}{\to} a \in A$, and let $(x_n + N_\alpha)$ be a sequence in X_α such that $x_n + N_\alpha \to x + N_\alpha \in X_\alpha$ with respect to $\overline{q}_\alpha$. Then $a_n \underset{\tau}{\to} a$ (since τ is weaker

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than the topology defined by $\|\cdot\|$ on A) and so

$$\begin{aligned}&\overline{q}_\alpha(a_n(x_n+N_\alpha)-a(x+N_\alpha))\\&\quad\le p_\lambda(a_n)\overline{q}_\alpha((x_n+N_\alpha)-(x+N_\alpha))+p_\lambda(a_n-a)\overline{q}_\alpha(x+N_\alpha)\\&\quad\to 0,\end{aligned}$$

that is, the module actions of X_α over A are jointly continuous. Therefore, X_α is a normed A-bimodule for every α.

The normed A-bimodule $\widetilde{X_\alpha}$ is therefore a Banach A-bimodule. For every α, let $\rho_\alpha : X \longrightarrow \widetilde{X_\alpha}$ be the αth projection map, and let $c_\alpha : \widetilde{X_\alpha} \longrightarrow (\widetilde{X_\alpha})^{**}$ be the embedding into the second dual of $\widetilde{X_\alpha}$, which is a module map. Let δ_α be defined as

$$c_\alpha\circ\rho_\alpha\circ\delta : A\longrightarrow(\widetilde{X_\alpha})^{**}$$

for every α. Then δ_α is a derivation. This follows from the definitions of the A-module actions with respect to which $\widetilde{X_\alpha}$ and $(\widetilde{X_\alpha})^{**}$ turn into Banach A-bimodules. From [22, Theorem 2], δ_α is $\|\cdot\|-\overline{q}_\alpha$ continuous for every $\alpha\in\Lambda_1$. Since A is a nuclear C*-algebra and therefore amenable, it follows that, for every $\alpha\in\Lambda_1$, δ_α is inner, that is, there is an element $z_\alpha\in(\widetilde{X_\alpha})^{**}$ such that $\delta_\alpha(b)=z_\alpha b-bz_\alpha$ for all $b\in A$. It follows easily from the hypothesis that, for every α, the module actions of X_α over A are $\tau-\overline{q}_\alpha$ smooth. Therefore, the module actions of $\widetilde{X_\alpha}$ over A are $\tau-\overline{q}_\alpha$ smooth, that is, there exists $\lambda\in\Lambda_0$ such that $\overline{\overline{q_\alpha}}(ax_\alpha)\le p_\lambda(a)\overline{\overline{q_\alpha}}(x_\alpha)$ and $\overline{\overline{q_\alpha}}(x_\alpha a)\le p_\lambda(a)\overline{\overline{q_\alpha}}(x_\alpha)$ for all $a\in A$ and $x_\alpha\in\widetilde{X_\alpha}$. As can be easily seen, these inequalities extend to the case of the module actions of $(\widetilde{X_\alpha})^{**}$ over A, which are proven therefore to be $\tau\times\overline{q}_\alpha-\overline{q}_\alpha$ continuous. Hence, δ_α is $\tau-\overline{q}_\alpha$ continuous for all α. Therefore, δ is $\tau-\tau'$ continuous. □

Lemma 4.2. *Let $A[\tau]$ be a complete GB*-algebra having jointly continuous multiplication. If (a_α) is a net in A with $a_\alpha \underset{\tau}{\to} 0$, then $(1+a_\alpha^*a_\alpha)^{-1}\underset{\tau}{\to}1$.*

Proof. Let $\{p_\lambda:\lambda\in\Lambda\}$ be a family of seminorms defining the topology τ on A. Since τ is weaker than the topology defined by $\|\cdot\|$ on $A[B_0]$, A is an $A[B_0]$-bimodule with $\|\cdot\|\times\tau-\tau$ jointly continuous module actions (the module actions being the multiplication defined on A). By Proposition 2.4, A is a $\|\cdot\|-\tau$ smooth $A[B_0]$-bimodule. Therefore, for every $\lambda\in\Lambda$,

$$\begin{aligned}0&\le p_\lambda((1+a_\alpha^*a_\alpha)^{-1}-1)\\&=p_\lambda(-(1+a_\alpha^*a_\alpha)^{-1}a_\alpha^*a_\alpha)\\&=p_\lambda((1+a_\alpha^*a_\alpha)^{-1}a_\alpha^*a_\alpha)\\&\le\|(1+a_\alpha^*a_\alpha)^{-1}\|p_\lambda(a_\alpha^*a_\alpha)\\&\le p_\lambda(a_\alpha^*a_\alpha).\end{aligned}$$

The last inequality follows from the fact that $\|(1+a_\alpha^*a_\alpha)^{-1}\|\le 1$. Since $a_\alpha\underset{\tau}{\to}0$, it follows that $p_\lambda(a_\alpha^*a_\alpha)\to 0$. Hence, the result follows. □

The proof of the following proposition is similar to that of [27, Proposition 3.1]. We give the proof for completeness, for which we need Lemma 4.2.

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Proposition 4.3. *Let $A[\tau]$ be a Fréchet GB*-algebra. Let $\delta : A \to A$ be a derivation such that $\delta|_{A[B_0]}$ is $\tau - \tau$ closable. Then δ is $\tau - \tau$ continuous. Conversely, if δ is $\tau - \tau$ continuous, then $\delta|_{A[B_0]}$ is $\tau - \tau$ closable.*

Proof. The converse statement is trivial. Therefore, consider the case where $\delta|_{A[B_0]}$ is $\tau - \tau$ closable. Since $\delta = \delta_1 + i\delta_2$, where δ_i are $*$-derivations, that is, $\delta_i(a^*) = \delta_i(a)^*$ for all $a \in A$, $i = 1, 2$, we may assume without loss of generality that δ is a $*$-derivation. Take a sequence (a_n) in A such that $a_n \underset{\tau}{\to} 0$ and $\delta(a_n) \underset{\tau}{\to} y$.

Observe that $a_n(1 + a_n^* a_n)^{-1} \in A[B_0]$ for all $n \in \mathbb{N}$. Since $A[\tau]$ is a Fréchet GB*-algebra, multiplication of A is jointly continuous. By Lemma 4.2, it follows that $(1 + a_n^* a_n)^{-1} \underset{\tau}{\to} 1$ and therefore $a_n(1 + a_n^* a_n)^{-1} \underset{\tau}{\to} 0$. If $b \in A$ is invertible, then it follows easily that $\delta(b^{-1}) = -b^{-1}\delta(b)b^{-1}$ and hence

$$\begin{aligned}
\delta((1 + a_n^* a_n)^{-1}) &= -(1 + a_n^* a_n)^{-1}(\delta(1 + a_n^* a_n))(1 + a_n^* a_n)^{-1} \\
&= -(1 + a_n^* a_n)^{-1}(\delta(a_n^* a_n))(1 + a_n^* a_n)^{-1} \\
&= -(1 + a_n^* a_n)^{-1}(a_n^*\delta(a_n) + \delta(a_n)^* a_n)(1 + a_n^* a_n)^{-1} \\
&\underset{\tau}{\to} -1(0 \cdot y + y^* \cdot 0)1 \\
&= 0.
\end{aligned}$$

Hence, $\delta(a_n(1 + a_n^* a_n)^{-1}) \underset{\tau}{\to} y$. Let $b_n = a_n(1 + a_n^* a_n)^{-1}$ for every $n \in \mathbb{N}$. Then (b_n) is a sequence in $A[B_0]$ with $b_n \underset{\tau}{\to} 0$ and $\delta(b_n) \underset{\tau}{\to} y$. Since $\delta|_{A[B_0]}$ is $\tau - \tau$ closable, $y = 0$. Therefore, by the closed graph theorem, δ is $\tau - \tau$ continuous. □

A consequence of Theorem 4.1 is that if $A[\tau]$ is a smooth Fréchet nuclear GB*-algebra, then every derivation $\delta : A \to A$ has the property that $\delta|_{A[B_0]}$ is $\tau - \tau$ closable. One therefore obtains the following result from Proposition 4.3, which is a partial answer to the question of whether every derivation of a Fréchet GB*-algebra is continuous.

Corollary 4.4. *Every derivation of a smooth Fréchet nuclear GB*-algebra is continuous.*

If $A[\tau]$ is a commutative Fréchet GB*-algebra, then it is a Fréchet nuclear GB*-algebra with no nonzero derivations [28, Corollary 3.4], implying that $A[\tau]$ has only continuous derivations. Therefore, the assumption that $A[\tau]$ is smooth can be dropped from the hypothesis of Corollary 4.4 if $A[\tau]$ is, in addition, commutative.

A *type I* C*-algebra is a C*-algebra A such that for every representation π of A, the weak operator closure of $\pi(A)$ is a type I von Neumann algebra. Every type I C*-algebra is nuclear [24, Theorem 3].

Corollary 4.5. *If $A[\tau]$ is a smooth Fréchet GB*-algebra such that $A[B_0]$ is a type I C*-algebra, then every derivation of A is continuous.*

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