ON THE SPACES OF MAPPINGS ON BANACH SPACES

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Let E be a real Banach space. A mapping f of E into E is said to be *bounded* if f maps every bounded set into a bounded set.

Let $\boldsymbol{B}$ be the set of all bounded and continuous mappings of E into E. If we define the linear combination $\alpha f+\beta g$ for $f, g \in \boldsymbol{B}$ and real numbers α and β by $(\alpha f+\beta g)(x) = \alpha f(x)+\beta g(x)$ for every $x \in E$, $\boldsymbol{B}$ is a linear space. Moreover, we can define the product of f and g by

$$fg(x) = f(g(x)) \text{ for every } x \in E.$$

It is clear that the right distributive law is satisfied: $(f+g)h = fh+gh$ for every $f, g, h \in \boldsymbol{B}$, and that the left distributive law is not always true. Therefore, following the terminology of Zassenhaus [4, pp. 71—74], we may call this space $\boldsymbol{B}$ *a near-algebra*.

In this near-algebra $\boldsymbol{B}$, we can define a topology as follows. Let us consider the sets $B_n = \{x \in E \mid ||x|| \leqq n\}$ for $n = 1, 2, \cdots$, then a mapping f is bounded if and only if the semi-norms

$$||f||_n = \sup_{x \in B_n} ||f(x)||$$

are finite. Therefore,

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||f-g||_n}{1+||f-g||_n}$$

is a metric on $\boldsymbol{B}$, and with this metric $\boldsymbol{B}$ is complete. We call this topology *the uniform topology of* $\boldsymbol{B}$. Thus the space $\boldsymbol{B}$ is a Fréchet space by the uniform topology.

A mapping f of E into E is said to be *compact* if each of $f(B_n)$ $(n = 1, 2, \cdots)$ is contained in a compact set. Evidently, the set $\boldsymbol{C}$ of all compact and continuous mappings of E into E is contained in $\boldsymbol{B}$. Moreover, the set $\boldsymbol{C}$ satisfies the following conditions:

(1) *It is a linear subset.*

(2) *If f is one of its elements and $g \in B$, then fg and gf belong to it.*

(3) *It is closed under the uniform topology.*

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Generally, a non-empty subset $\boldsymbol{I}$ of $\boldsymbol{B}$ is called ***an ideal*** if it satisfies the conditions (1) and (2). An ideal is called ***the zero ideal*** if it consists of a single element 0 (the zero element of $\boldsymbol{B}$). $\boldsymbol{C}$ is a non-zero closed ideal. Since $\boldsymbol{B}$ is not an algebra, an ideal is not necessarily a kernel of a homomorphism. For a study of this fact from the algebraic point of view, we refer to [1].

When E is a Hilbert space of countable dimension, it was proved by Calkin [2] that the set of all compact and continuous linear mappings was a minimal closed ideal of the Banach algebra of all bounded linear mappings. (An ideal is said to be ***minimal*** if it is not the zero ideal and does not contain properly any ideal of the same type other than the zero ideal.) But, in the case of the near-algebra $\boldsymbol{B}$, the set $\boldsymbol{C}$ is no longer a minimal closed ideal of $\boldsymbol{B}$.

For example, let us consider the set $\boldsymbol{I}(E)$ of all constant mappings, in other words, $\boldsymbol{I}(E)$ is the set of all mappings $c_a (a \in E)$ such that $c_a(x) = a$ for every $x \in E$. It is obvious that $\boldsymbol{I}(E) \subset \boldsymbol{C} \subset \boldsymbol{B}$, and we have

$$\alpha c_a + \beta c_b = c_{\alpha a + \beta b},$$
$$c_a f = c_a, \quad f c_a = c_{f(a)} \quad \text{for every } f \in \boldsymbol{B}.$$

Therefore, $\boldsymbol{I}(E)$ is an ideal of $\boldsymbol{B}$ and, moreover, it is closed under the uniform topology. It also follows from the equality $c_a f = c_a$ that every non-zero ideal contains $\boldsymbol{I}(E)$, which means that *$\boldsymbol{I}(E)$ is a minimal (closed) ideal.*

This closed ideal does not contain the set of all linear mappings of finite rank. On the other hand, the closed ideal $\boldsymbol{C}$ contains all linear mappings of finite rank.

REMARK. Let $\bar{E}$ be the conjugate space of E. Then, a mapping f is said to be *of finite rank* if there exist $a_i \in E$ $(i = 1, 2, \cdots, k)$ and $\bar{a}_i \in \bar{E}$ $(i = 1, 2, \cdots, k)$ such that $f(x) = \bar{a}_1(x)a_1 + \bar{a}_2(x)a_2 + \cdots + \bar{a}_k(x)a_k$ for every $x \in E$.

The purpose of this paper is to prove that the closed ideal $\boldsymbol{C}$ is minimal amongst all closed ideals which contains all linear mappings of finite rank.

THEOREM. *Let E be a real Banach space, $\boldsymbol{B}$ be the near-algebra of all bounded and continuous mappings and $\boldsymbol{I}$ be a closed ideal of $\boldsymbol{B}$. If $\boldsymbol{I}$ contains all linear mappings of finite rank, then $\boldsymbol{C} \subset \boldsymbol{I}$.*

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LEMMA. *For any $f \in \boldsymbol{B}$, there exists a sequence $f^{[n]} \in \boldsymbol{B}$ such that*

(1) $f^{[n]}(E) \subset \bigcup_{0 \leqq \alpha \leqq 1} \alpha f(B_{n+1})$;

(2) $\lim_{n \to \infty} d(f^{[n]}, f) = 0$;

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(3) *if* $f \in \boldsymbol{C}$, *then* $f^{[n]} \in \boldsymbol{C}$. *Moreover,* $f^{[n]}(E)$ *is contained in a compact set;*

(4) *if each of* $f(B_n)$ $(n = 1, 2, \cdots)$ *is contained in a finite-dimensional subset of* E, *then each of* $f^{[n]}(E)$ *is contained in a finite-dimensional subset.*

PROOF. Let us consider the real continuous functions:

$$\phi_n(\lambda) = \begin{cases} 1 & \text{if } 0 \leqq \lambda \leqq n; \\ -\lambda+n+1 & \text{if } n < \lambda \leqq n+1; \\ 0 & \text{if } \lambda > n+1 \end{cases}$$

and define the mappings $f^{[n]}$ by

$$f^{[n]}(x) = \phi_n(||x||)f(x) \text{ for every } x \in E.$$

Since $0 \leqq \phi_n(\lambda) \leqq 1$ for every $\lambda \geqq 0$ and $n = 1, 2, \cdots$, it is clear that $f^{[n]} \in \boldsymbol{B}$.

Proof of (1). Let us take an arbitrary $y \in f^{[n]}(E)$. Then, $y = f^{[n]}(x) = \phi_n(||x||)f(x)$ for some $x \in E$.

(i) If $x \in B_n$, then, since $||x|| \leqq n$, $\phi_n(||x||) = 1$, hence $y = \phi_n(||x||)f(x) = f(x) \in f(B_n) \subset f(B_{n+1})$.

(ii) If $x \in B_{n+1} \backslash B_n$, then, since $n < ||x|| \leqq n+1$, $0 \leqq \phi_n(||x||) = -||x||+n+1$, hence it follows that $y = \phi_n(||x||)f(x) \in \bigcup_{0 \leqq \alpha \leqq 1} \alpha f(B_{n+1})$.

(iii) If $x \notin B_{n+1}$, then, since $\phi_n(||x||) = 0$, we have $y = 0 \in 0 \cdot f(B_{n+1}) \subset \bigcup_{0 \leqq \alpha \leqq 1} \alpha f(B_{n+1})$.

Proof of (2). For any m and n, we have

$$||f^{[n]} - f||_m = \sup_{x \in B_m} ||f^{[n]}(x) - f(x)|| = \sup_{x \in B_m} (1 - \phi_n(||x||))f(x).$$

When $n \geqq m$, since $x \in B_m$ implies $||x|| \leqq m \leqq n$, we have $\phi_n(||x||) = 1$, so that $||f^{[n]} - f||_m = 0$ if $n \geqq m$. Therefore,

$$\lim_{n \to \infty} d(f^{[n]}, f) = \lim_{n \to \infty} \sum_{m=n+1}^{\infty} \frac{1}{2^m} \frac{||f^{[n]} - f||_m}{1 + ||f^{[n]} - f||_m} \leqq \lim_{n \to \infty} \frac{1}{2^n} = 0.$$

Proof of (3). Let us take an arbitrary sequence $(y_k) \subset f^{[n]}(E)$. Then, by (1), we can find numbers α_k and elements x_k such that $0 \leqq \alpha_k \leqq 1$, $x_k \in B_{n+1}$ and $y_k = \alpha_k f(x_k)$. Since $f(B_{n+1})$ is contained in a compact set, there exists a subsequence (x_{k_i}) such that $\lim_{i \to \infty} f(x_{k_i}) = y_0$ for some $y_0 \in E$. Since (α_{k_i}) is a bounded sequence, there exists a subsequence $(\alpha_j) \subset (\alpha_{k_i})$ such that $\lim_{j \to \infty} \alpha_j = \alpha_0$ for some α_0. Thus, $\lim_{j \to \infty} y_j = \lim_{j \to \infty} \alpha_j f(x_j) = \alpha_0 y_0$, which means that $f^{[n]}(E)$ is relatively compact.

Proof of (4). By the assumption, for each n, there exists a finite-dimensional subspace E_n such that $f(B_{n+1}) \subset E_n$. Since E_n is linear, $\alpha f(B_{n+1}) \subset E_n$ for every α, which implies that $f^{[n]}(E) \subset \bigcup_{0 \leqq \alpha \leqq 1} \alpha f(B_{n+1}) \subset E_n$.

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Proof of Theorem. Let us take an arbitrary $f \in \boldsymbol{C}$. By (2) and (3) of the above lemma, $\lim_{n\to\infty} d(f^{[n]}, f) = 0$ and each of $f^{[n]}(E)$ is contained in a compact set. Therefore, since $\boldsymbol{I}$ is assumed to be closed, we have only to prove that, *if* $f(E)$ *is contained in a compact set*, $f \in \boldsymbol{I}$. Let us assume that $f(E)$ is contained in a compact set. Following the method first used by Leray and Schauder [3, p. 51] we can construct a sequence f_n such that $\lim_{n\to\infty} d(f_n, f) = 0$ and each $f_n(E)$ is contained in a finite-dimensional subspace of E as follows. Since $f(E)$ is totally bounded, for each n, there exists a finite number of elements $y_1, y_2, \cdots, y_{k(n)}$ such that

$$f(E) \subset \bigcup_{i=1}^{k(n)} \{y \in E \,/\, \|y - y_i\| < 1/n\}.$$

Let E_n be a finite-dimensional subspace which is spanned by $y_1, y_2, \cdots, y_{k(n)}$, then the mapping f_n is defined by

$$f_n(x) = \sum_{i=1}^{k(n)} \mu_i(x) y_i \Big/ \sum_{i=1}^{k(n)} \mu_i(x)$$

where

$$\mu_i(x) = \begin{cases} 1/n - \|f(x) - y_i\| & \text{if } \|f(x) - y_i\| \leqq 1/n; \\ 0 & \text{if } \|f(x) - y_i\| \geqq 1/n. \end{cases}$$

It is clear that $f_n \in \boldsymbol{B}$ and $\|f(x) - f_n(x)\| < 1/n$ for every $x \in E$, from which it follows that $\lim_{n\to\infty} d(f, f_n) \leqq \lim_{n\to\infty} 1/(n+1) = 0$. Therefore, we have only to prove that, if $f(E)$ is contained in a finite-dimensional subspace, then $f \in \boldsymbol{I}$. Let us assume that $f(E) \subset E_0$, where E_0 is a k-dimensional subspace. Let $e_1, e_2, \cdots, e_k$ be a base of E and $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k$ be elements of $\bar{E}$ such that $\bar{e}_i(e_j) = 1$ if $i = j$; $= 0$ if $i \neq j$. Then, we have

$$f(x) = \bar{e}_1(f(x))e_1 + \cdots + \bar{e}_k(f(x))e_k \quad \text{for every } x \in E.$$

Now, let us consider the mapping

$$g(x) = \bar{a}(f(x))a \qquad (x \in E)$$

where $a \in E$ and $\bar{a} \in \bar{E}$. If we put $(a \otimes \bar{a})(x) = \bar{a}(x)a$, then the mapping $a \otimes \bar{a}$ is a linear mapping of finite rank, and $g = (a \otimes \bar{a})f$. Therefore, since $a \otimes \bar{a} \in \boldsymbol{I}$, we have $g \in \boldsymbol{I}$, and, since f is a linear combination of $(e_i \otimes \bar{e}_i)f$, we have $f \in \boldsymbol{I}$. Thus, the proof is completed.

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