ON A CLASS OF COMPLETE SIMPLE SETS

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1. Introduction. Since the solution of Post's Problem by Friedberg and Muchnik, a good deal of abstract knowledge about the semilattice of recursively enumerable degrees has been developed, especially in recent papers of G. E. Sacks. In this note, we show that one specific class of simple sets contains only sets of degree $0'$; no contribution to the general theory is claimed. One of the sets belonging to the special class considered is the original simple-but-not-hypersimple S of Post [1]. According to information received from Sacks, J. R. Myhill gave a proof, in 1953, of the completeness of S. However, as far as we know, Myhill's proof (presented in a seminar) does not exist in published form. In any case, it may be that our more general result has some interest beyond its application to the particular set S.

2. Definitions and Notation. We follow the notational conventions of [3]; regarding terminology, the necessary definitions are available in [1] and [3] with the exception of the following one:

Definition. An infinite number set A is said to be strongly effectively immune if and only if there is a partial recursive function p such that, for every n, $\omega_n \subset A \rightarrow p(n)$ is defined and $(\forall x)(x \in \omega_n \rightarrow p(n) > x)$.

Remarks. (1) It is easily seen that the adjective 'partial' may be omitted in the preceding definition without altering the class of sets defined. (2) Our notion of strong effective immunity is a modification of a concept due to R. M. Smullyan.

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Smullyan calls an infinite number set A effectively immune if and only if there exists a partial recursive function p such that if $\omega_n \subset A$ then $p(n)$ is defined and is greater than the cardinality of ω_n. The differences between our notion and Smullyan's may be summarized as follows. Sacks [2] shows the existence of a simple but not effectively simple set; modifying Sacks' argument, D. A. Martin has shown (private communication) that there exist effectively simple sets whose complements are not strongly effectively immune. J. S. Ullian has shown (private communication) that there exists an effectively immune set whose complement is also effectively immune. Finally, it is not difficult to show that the complement of a strongly effectively immune set cannot be immune; we conclude section 2 of the note with a proof of this latter fact.

PROPOSITION. Suppose A is strongly effectively immune. Then A' has an infinite r. e. (recursively enumerable) subset.

Proof. Consider the Kleene T_1-predicate, $T_1(x, y, z)$; by definition, we have $\omega_n = \{y \mid (\exists z)T_1(n, y, z)\}$. Choose ω_n to be nonrecursive, and suppose that r is a recursive function which witnesses the strong effective immunity of A. Now, there exists (by standard arguments[1)]) a recursive function t such that, for each k,

$$\omega_{t(k)} = \begin{cases} \{(\mu z)T_1(n, k, z)\}, & \text{if } (\exists z)T_1(n, k, z), \\ \phi, & \text{otherwise.} \end{cases}$$

Clearly, if $(\exists z)T_1(n, k, z)$ and $(\mu z)T_1(n, k, z) \geq rt(k)$, then $(\mu z)T_1(n, k, z) \in A'$. Also, there must be infinitely many k for which $(\exists z)T_1(n, k, z)$ and $(\mu z)T_1(n, k, z) \geq rt(k)$, since

1) See, for example, Davis, Computability and Unsolvability, Chapter 9, sections 1 and 2. The same reference applies, regarding functions used in the proof of Lemma 2 below.

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otherwise there would be an effective test for membership in ω_n. Finally, it is a well-known property of $T_1(x, y, z)$ that $T_1(x, y, z) \to z > y$. From these facts, it is clear that we may enumerate an infinite r.e. subset of A' by enumerating all numbers of the form $(\mu z)T_1(n, k, z)$ such that $(\mu z)T_1(n, k, z) \geq rt(k)$.

3. S.E.S. Sets are Hypersimple or Complete. We will, for convenience, refer to r.e. sets with strongly effectively immune complements as S.E.S. (strongly effectively simple) sets.

THEOREM. If B is a nonhypersimple S.E.S. set, then B is complete.

For our proof of this result, we require two lemmas.

LEMMA 1. Suppose B is simple but not hypersimple. Then there exist r.e. sets C, D, and E such that $B \subset C$, C is simple, $C = D \cup E$, $D \cap E = \phi$, and D is quasicreative.

Proof. The result is proved in [3] for a particular simple-but-not-hypersimple set (namely, Post's set S); the proof given in [3], with obvious modifications, serves to establish the more general claim of Lemma 1.

LEMMA 2. Suppose C and D are simple sets, $C \subset D$, and $D = A \cup B$, where $A \cap B = \phi$, A is r.e., and B is quasicreative. Then if C is S.E.S., $C \cap B$ is quasicreative.

Proof. Let φ be a recursive function such that, for every n, $\omega_{\varphi(n)} = \omega_n \cap B$. Let f be a recursive function quasiproductive for B'. Let ψ be a recursive function such that, for every n, $\omega_{\psi(n)} = \omega_n \cup \alpha_{f(n)}$; and define a recursive function $g(i, n)$ as follows:

$$g(i, 0) = i; \quad g(i, n+1) = \psi(g(i, n)).$$

Once again applying remark (1) of § 2, we let r be a recursive function witnessing strong effective immunity of C'; further, let ξ be a recursive function such that, for each n, we have

$$\omega_{\xi(n)} = \omega_n - \{x \mid x < r\varphi(n)\} .$$

Finally, let f^* be the recursive function such that

$$\alpha_{f^*(n)} = \bigcup_{k \leq r\varphi(n)+1} \alpha_{f(g(\xi(n), k))} .$$

We claim that $(C \cap B)'$ is quasiproductive relative to f^*; the easy details of the verification are left to the reader.

Remark. The assumption, in Lemma 2, that $D - B$ is r. e. is not really relevant so far as the Lemma is concerned; however, we will need this assumption in our application of Lemma 2.

Proof of the Theorem. Suppose B is S. E. S. but not hypersimple. By Lemma 1, there is a simple set C such that $B \subset C$ and $C = A \cup D$, with D quasicreative, A r. e., and $D \cap A$ empty. By Lemma 2, $B \cap D$ is quasicreative. Therefore, by [3, Theorem 2], $B \cap D$ is complete. But B is the disjoint union of the r. e. sets $B \cap D$ and $B \cap A$; it follows that the degree of B is the l. u. b. of the degrees of $B \cap D$ and $B \cap A$. Thus B must be complete.

COROLLARY. Post's simple set S is complete, together with all of its simple, nonhypersimple extensions.

Proof. S is nonhypersimple ([1]), and it is easily checked that S is S. E. S. The proof of the Corollary is completed by observing that any infinite subset of a strongly effectively immune set must be strongly effectively immune.

Remark. The above discussion does not bear on the question of completeness of S. E. S. hypersimple sets; for, it can be shown that a quasiproductive set must intersect a hypersimple set quasiproductively, so that Lemma 1 is never true in the hypersimple case.

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2. G. E. Sacks, A simple set which is not effectively simple, Proc. Amer. Math. Soc., 15 (1964), 51-55.

3. J. R. Schoenfield, Quasicreative sets, Proc. Amer. Math. Soc., 8 (1957), 964-967.

Added in proof. In a letter of December 18, 1964, D. A. Martin has communicated to the author a proof that every effectively simple set is complete. This result subsumes both the central assertion in the present note and (in view of other well-known theorems) the main content of reference [2].

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