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ON HOLOMORPHIC MAPS INTO A REAL LIE GROUP OF HOLOMORPHIC TRANSFORMATIONS

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1. Introduction. Let M, N be complex manifolds and G be a group of holomorphic automorphisms of N. In [3] (c.f. p. 74) W. Kaup introduced the notion of holomorphic maps into a family of holomorphic maps between complex spaces. By definition, a map $g : M \to G$ is holomorphic if and only if the induced map $\tilde{g}(x, y) := g(x)(y)$ $(x \in M,\ y \in N)$ of $M \times N$ into N is holomorphic in the usual sense. The purpose of this note is to give a description of a holomorphic map of a connected complex manifold M into G. We show first the existence of the maximum connected Lie subgroup G_0 of G which is a complex Lie transformation group of N.

We prove the following:

In the above situation, any holomorphic map $g : M \to G$ *can be written* $g = g_0 \cdot h$ *and* $g = h' \cdot g'_0$ *with suitable* $h, h' \in G$ *and holomorphic maps* $g_0, g'_0 : M \to G_0$ (Theorem 4.3).

A holomorphic map g of M into the holomorphic automorphism group of N corresponds to each holomorphic automorphism g^* of $M \times N$ with $\pi_M g^* = \pi_M$, where $\pi_M : M \times N \to M$ is the natural projection. As an application of the above result, we see

If M *is connected and* N *is a bounded domain in* C^n, *any holomorphic automorphism* h *of* $M \times N$ *with* $\pi_M h = \pi_M$ *can be written* $h(x, y) = (x, g(y))$ $(x \in M,\ y \in N)$ *with a suitable holomorphic automorphism* g *of* N (Corollary 4.6).

This result is a generalization of one side of H. Cartan's theorem in [1] which asserts that any holomorphic automorphism of $M \times N$ sufficiently near to the identity can be written as the product of the holomorphic automorphisms of M and N if M and N are both bounded domains.

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2. The maximum complex Lie subgroup of a real Lie transformation group. Let N be a complex manifold and (G, φ) an effective Lie transformation group of N, that is, G be a real Lie group, φ a real analytic map of $G \times N$ into N which defines an injective group homomorphism φ^* of G into the holomorphic automorphism group $\mathrm{Aut}(N)$ of N, where $\varphi^*(g)(y) = \varphi(g, y)$ $(g \in G,\ y \in N)$. We say a vector field X on N to be conformal if $[X, JY] = J[X, Y]$ for any vector field Y on N, where J denotes the almost complex structure of N. The set $\mathfrak{A}(N)$ of all conformal vector fields on N is a (not necessarily finite-dimensional) complex Lie algebra with the complex structure J. We know that each left invariant vector field on G corresponds to a one-parameter subgroup of G and defines canonically a vector field on N, which is conformal because $\varphi^*(g) \in \mathrm{Aut}(N)$ $(g \in G)$. Thus we have a Lie algebra isomorphism Φ of the Lie algebra $\mathfrak{g}$ of G onto a real Lie subalgebra $\mathfrak{g}^*$ of $\mathfrak{A}(N)$.

Now, we consider the Lie subalgebra $\mathfrak{g}_0^* := \mathfrak{g}^* \cap J\mathfrak{g}^*$ of $\mathfrak{g}^*$. Obviously, $J\mathfrak{g}_0^* \subset \mathfrak{g}_0^*$ and so $\mathfrak{g}_0^*$ is considered as a complex Lie subalgebra of $\mathfrak{A}(N)$. Put $\mathfrak{g}_0 = \Phi^{-1}(\mathfrak{g}_0^*)$. As is well-known, G has a uniquely determined connected Lie subgroup G_0 with the Lie algebra $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is a complex Lie algebra with the complex sturcture induced from $\mathfrak{g}_0^*$ by Φ, G_0 has a structure of a complex Lie group and, furthermore, is considered as a complex Lie transformation group of N. On the other hand, $\mathfrak{g}_0^*$ is the maximum complex Lie subalgebra of $\mathfrak{A}(N)$ which is included in $\mathfrak{g}^*$. If a connected Lie subgroup G' of G has a complex structrue with which $(G', \varphi | G' \times N)$ is a complex Lie transformation group of N, the Lie algebra of G' is necessarily included in $\mathfrak{g}_0$ and so G' is a Lie subgroup of G_0. This shows that G_0 is the maximum connected Lie subgroup of G which is a complex Lie transformation group of N.

DEFINITION 2.1. We shall call the connected complex Lie group G_0 constructed as the above *the maximum complex Lie subgroup of* (G, φ).

EXAMPLE 2.2. (i) A complex Lie group G is canonically considered as a complex Lie transformation group of G itself with left translations. For a real Lie subgroup H of G the maximum complex Lie subgroup H_0 of H is nothing but the maximum connected complex Lie subgroup of G which is included in H. In particular, for a maximal compact subgruop K of a connected complex Lie group G, $K_0 = \{e\}$ if and only if G is a

Stein group, i.e. the variety of G is Stein (c.f. Matsushima-Morimoto [4], p. 139).

(ii) Let N be a bounded domain in $\boldsymbol{C}^n$. We know that the holomorphic automorphism group $\mathrm{Aut}(N)$ of N with the compact-open topology is a real Lie group. In this case, the maximum complex Lie subgroup $\mathrm{Aut}(N)_0$ of $\mathrm{Aut}(N)$ consists of the identity only. In fact, for any complex one-parameter subgroup $\{g_t\}$ of G_0 and any $y \in N$, the map $\psi(t) = g_t(y)$ $(t \in \boldsymbol{C})$ of $\boldsymbol{C}$ into N is constantly equal to $\psi(0) = g_0(y) = y$ as is easily seen by the Liouville's theorem. This shows that $\mathrm{Aut}(N)_0 = \{e\}$.

3. A charactrization of holomorphic maps into a real Lie transformation group. Let (G, φ) be an effective Lie transformation group of a complex manifold N.

DEFINITION 3.1. A map g of a complex manifold M into G is called to be *holomorphic* if the map $\tilde{g} : M \times N \to N$ defined as $\tilde{g}(x, y) = \varphi(g(x), y)$ $(x \in M,\ y \in N)$ is holomorphic in the usual sense.

As is stated in the previous section, there is the canonically defined Lie algebra isomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is a Lie subalgebra of $\mathfrak{A}(N)$. On the other hand, the exponential map $\exp : \mathfrak{g} \to G$ maps diffeomorphically a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of the identity e in G. Take a continuous map g of a connected complex manifold M into G and assume that $g(x_0) = e$ for some $x_0 \in M$. We can define the map $\bar{g} = \Phi \exp^{-1} \cdot g$ of a neighborhood V of x_0 into $\mathfrak{A}(N)$ with the image in $\mathfrak{g}^*$. Put $\mathfrak{g}^*_C = \mathfrak{g}^* + J\mathfrak{g}^*$. It is a complex Lie subalgebra of $\mathfrak{A}(N)$ and $\dim_C \mathfrak{g}^*_C < +\infty$. We consider the above $\bar{g}$ as a map of V into $\mathfrak{g}^*_C$.

THEOREM 3.2. *For a continuous map $g : M \to G$ with $g(x_0) = e$, g is holomorphic in a neighborhood of x_0 in the sense of Definition* 3.1 *if and only if $\bar{g} = \Phi \cdot \exp^{-1} \cdot g$ is holomorphic at x_0 as a map with the values in the finite-dimensional complex vector space $\mathfrak{g}^*_C$.*

For the proof of Theorem 3.2, we use the following Lemma which was shown in the previous paper [2], Lemma 2.6.

LEMMA 3.3. *Let M, N, N' and H be complex manifolds and $\psi : H \times N \to N'$ a holomorphic map. Assume that N' is holomorphically separable and $\psi(t', y) = \psi(t, y)$*

for any $y \in N$ *implies* $t = t'$ $(t, t' \in H)$. *Then a map* $g : M \to H$ *is holomorphic if* $\varphi(g(x), y)$ $(x \in M,\ y \in N)$ *is a holomorphic map of* $M \times N$ *into* N'.

Proof of Theorem 3.2. As is well-known, there is a simply connected complex Lie group $\tilde{G}$ with the Lie algebra $\mathfrak{g}_C^* = \mathfrak{g}^* + J\mathfrak{g}^*$. The exponential map $\exp_C : \mathfrak{g}_C^* \to \tilde{G}$ gives a biholomorphic map of a neighborhood $\mathfrak{U}^*$ of 0 in $\mathfrak{g}_C^*$ onto a neighborhood $\tilde{U}$ of e in $\tilde{G}$. On the other hand, since $\mathfrak{g}^* = \Phi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{g}_C^*$, we can take a symmetric neighborhood U of e in G such that there is a local isomorphism i of U onto a local Lie subgroup of $\tilde{U}$, where we may assume that $\exp : \mathfrak{g} \to G$ maps diffeomorphically a neighborhood $\mathfrak{U}$ of 0 in $\mathfrak{g}$ onto U. Then we have $i \cdot \exp = \exp_C \Phi$.

Now, take a continuous map $g : M \to G$ with $g(x_0) = e$. For our purpose, it may be assumed that $g(M) \subset U$. Suppose that $\bar{g} = \Phi \cdot \exp^{-1} \cdot g : M \to \mathfrak{g}_C^*$ is holomorphic. Then, $i \cdot g = \exp_C \cdot \bar{g} : M \to \tilde{U}$ is also holomorphic with respect to the complex structure of $\tilde{U}$ in the usual sense. We shall show first that the map $\tilde{g} : M \times N \to N$ defined as $\tilde{g}(x, y) = \varphi(g(x), y)$ $(x \in M,\ y \in N)$ is holomorphic at (x_0, y_0) for an arbitrarily given $y_0 \in N$. To this end, we recall the Lie's fundamental theorem on local Lie groups of local transformations. For each $y_0 \in N$, we can find neighborhoods $\tilde{U}'$ $(\subset \tilde{U})$ of e in $\tilde{G}$ and W, W' of y_0 in N $(W \subset W')$ such that 1) there is a holomorphic map $\varphi' : \tilde{U}' \times W \to W'$, 2) $\varphi'(t, y) = y$ for any $y \in W$ if and only if $t = e$, 3) $\varphi'(t \cdot t', y) = \varphi'(t, \varphi'(t', y))$ if $t, t' \in \tilde{U}'$, $y \in W$, $t \cdot t' \in \tilde{U}'$ and $\varphi'(t', y) \in W$, where we may assume that $\varphi'(i(t), y) = \varphi(t, y)$ for any $t \in i^{-1}(\tilde{U}')$ and $y \in W$ because of the uniqueness of local transformations. Taking a sufficiently small neighborhood V of x_0 in M, we see $\tilde{g}(x, y) = \varphi'((i \cdot g)(x), y)$ on $V \times W$. Since $i \cdot g : V \to \tilde{U}'$ and $\varphi' : \tilde{U}' \times W \to W'$ are both holomorphic, $\tilde{g}$ is holomorphic on $V \times W$. To show the holomorphy of $\tilde{g} : M \times N \to N$, take an arbitrary $x_1 \in M$ and consider the map $h = g(x_1)^{-1} \cdot g$. For a sufficiently small neighborhood V' of x_1, the map $i \cdot h : V' \to \tilde{U}$ is well-defined and holomorphic, because $i \cdot h$ is obtained by the holomorphic left translation of the holomorphic map $i \cdot g$ by $i(g(x_1))^{-1}$. Since $h(x_1) = e$, we can apply the same argument as the above to the map h. So, $\tilde{h}(x, y) = \varphi(h(x), y)$ $(x \in M,\ y \in N)$ is holomorphic in a neighborhood of (x_1, y) for any $y \in N$. It then follows that $\tilde{g}(x, y) = \varphi(g(x), y) = \varphi(g(x_1) h(x), y) = \varphi(g(x_1),\ \varphi(h(x), y))$ $(x \in M,\ y \in N)$ is also holomorphic in a neighborhood of (x_1, y). This shows that $\tilde{g} : M \times N \to N$

is holomorphic, namely, $g: M \to G$ is holomorphic in the sense of Definition 3.1.

Conversely, suppose that a map $g: M \to G$ is holomorphic in the sense of Definition 3.1 and $g(x_0) = e$. Take an arbitrary $y_0 \in N$, neighborhoods $\tilde{U}'$ of e in $\tilde{G}$ and W, W' of y_0 in N having the properties stated in the above argument, where we choose W' so small as to be holomorphically separable. Without loss of generality, it may be assumed that $(i \cdot g)(M) \subset \tilde{U}'$. Then $\tilde{g}(x, y) = \varphi(g(x), y) = \varphi'((i \cdot g)(x), y)$ is a holomorphic map of $M \times W$ into W'. Putting $M := M$, $N := W$, $N' := W'$, $H := U'$ and $\psi := \varphi'$, we can apply Lemma 3.3. The map $i \cdot g: M \to \tilde{U}'$ is holomorphic in the usual sense. Therefore, $\bar{g} = \Phi \cdot \exp^{-1} \cdot g = \exp_C^{-1}(i \cdot g) : M \to \mathfrak{g}_C^*$ is also holomorphic.

REMARK. As is easily seen, if a map $g: M \to G$ is holomorphic, $g \cdot h$ and $h \cdot g$ are both holomorphic for any $h \in G$. In particular, $g: M \to G$ is holomorphic in a neighborhood of $x_0 \in M$ if the map $h(x) = g(x)g(x_0)^{-1}$ or $h'(x) = g(x_0)^{-1}g(x)$ $(h(x_0) = h'(x_0) = e)$ is holomorphic in a neighborhood of x_0.

4. The main theorems and their applications. Now, we give the following main theorems.

THEOREM 4.1. *Let (G, φ) be an effective Lie transformation group of N and G_0 be the maximum complex Lie subgroup of (G, φ). Take a holomorphic map g of a connected complex manifold M into G. If $g(x_0) \in G_0$ for some $x_0 \in M$, then $g(x) \in G_0$ for any $x \in M$.*

For the proof, we give

LEMMA 4.2. *Let E be a finite-dimensional complex vector space with the complex structure J and F be a real vector subspace of E with the property that $F \cap JF = (0)$. If a holomorphic map g of a connected complex manifold M into E has the image in F, then g is necessarily a constant function on M.*

Proof of Lemma 4.2. By the assumption we can take a base $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_m, Je_1, \cdots, Je_m\}$ $(m = \dim_C E)$ of E over $\boldsymbol{R}$ such that $\{e_1, \cdots, e_k\}$ generates the real vector subspace F. Then $\{e_1, \cdots, e_m\}$ may be considered as a base of E over $\boldsymbol{C}$. Writing $g(x) = g_1(x)e_1 + \cdots + g_m(x)e_m (x \in M)$, we have holomorphic functions $g_1, \cdots, g_m$ on M which satisfy the conditions that for any $x \in M$ $\operatorname{Im}(g_i(x)) = 0$ if $1 \leqq i \leqq k$ and $g_j(x) = 0$ if $k+1 \leqq j \leqq m$. We know that any real-valued holomorphic function is necessarily a constant. This concludes Lemma 4.2.

Proof of Theorem 4.1. Firstly, we shall show that there is a neighborhood U of x_0 with the property $g(U) \subset G_0$ under the restricted assumption $g(x_0) = e$. Using the same notations as in the previous sections, we know that the map $\bar{g} = \Phi \cdot \exp^{-1} \cdot g$ of a sufficiently small neighborhood U of x_0 into $\mathfrak{g}_C^*$ is holomorphic in virtue of Theorem 3.2. Consider the complex Lie subalgebra $\mathfrak{g}_0^* = \mathfrak{g}^* \cap J\mathfrak{g}^*$ of $\mathfrak{g}_C^*$ and the quotient complex vector space $E := \mathfrak{g}_C^*/\mathfrak{g}_0^*$. The vector subspace $F := \mathfrak{g}^*/\mathfrak{g}_0^*$ of E have the property $F \cap \bar{J}F = (0)$, where $\bar{J} : E \rightarrow E$ is the complex structure of E induced from J. For the projection $p : \mathfrak{g}_C^* \rightarrow E$, the map $p \cdot \bar{g} : U \rightarrow E$ satisfies the conditions in Lemma 4.2. Since $\bar{g}(x_0) = 0$ in $\mathfrak{g}_C^*$, $p \cdot \bar{g}$ is constantly equal to zero and so $\bar{g}(x) \in \mathfrak{g}_0^*$ for any $x \in U$. Then $g = \exp \cdot \Phi^{-1} \cdot \bar{g} : U \rightarrow G$ has the image in $\exp(\Phi^{-1}(\mathfrak{g}_0^*)) \subset G_0$.

Now, we consider the set $M^* = \{x \in M;\ g(x) \in G_0\}$, which is not empty by the assumption. For any $x_1 \in M$, $h = g(x_1)^{-1} \cdot g$ is also a holomorphic map of M into G and satisfies the condition $h(x_1) = e$. By the above argument, h maps a neighborhood U of x_1 into G_0. Therefore, if $g(x_1) \in G_0$, $g(x) = g(x_1)h(x) \in G_0$ and, if not, $g(x) \notin G_0$ for any $x \in U$. This shows that M^* is open and closed in M. We conclude $M^* = M$ because of the connectivity of M.

THEOREM 4.3. *Let (G, φ) be an effective Lie transformation group of N and G_0 the maximum complex Lie subgroup of (G, φ). Then any holomorphic map of a connected complex manifold M into G can be written $g = g_0 \cdot h$ and $g = h' \cdot g_0'$ with suitable $h, h' \in G$ and maps $g_0, g_0' : M \rightarrow G_0$ which are holomorphic with respect to the complex structrue of G_0 in the usual sense.*

Proof. For an arbitrarily fixed $x_0 \in M$, we write $g(x) = (g(x)g(x_0)^{-1})g(x_0) = g(x_0)(g(x_0)^{-1}g(x))$. For our purpose, it suffices to take $h = h' = g(x_0) \in G$ and $g_0(x) = g(x)g(x_0)^{-1}$, $g_0'(x) = g(x_0)^{-1}g(x)$. In fact, since $g_0(x_0) = g_0'(x_0) = e \in G_0$, $g_0(M)$ and $g_0'(M)$ are both included in G_0 by Theorem 4.1. On the other hand, we know that any holomorphic map of M into a complex Lie transformation group G_0 of N in the sense of Definition 3.1 is holomorphic in the usual sense ([2], (2.4)). So, g_0 and $g_0' : M \rightarrow G_0$ are both holomorphic.

In particular, if $G_0 = \{e\}$, then $g_0(x) \equiv g_0'(x) \equiv e$ in the above. We have

COROLLARY 4.4. *Let G be a Stein group. If a holomorphic map g of a connected complex manifold M into G has the image in a compact subgroup K of G, then g is necessarily a constant.*

Proof. We can regard g as a map of M into K which is holomorphic in the sense of Definition 3.1. Corollary 4.4 is a direcr result of Theorem 4.3 and Example 2.2, (i).

Assume that the holomorphic automorphism group $\mathrm{Aut}(N)$ of N has a structure of a Lie transformation group of N. Any map $g: M \to \mathrm{Aut}(N)$ defines a bijective map $g^*(x, y) = (x, g(x)(y))$ $(x \in M,\ y \in N)$ of $M \times N$ into itself with the property $\pi_M g^* = \pi_M$, where $\pi_M: M \times N \to M$ denotes the natural projection. By definition, g is holomorphic if and only if g^* is holomorphic. As is well-known, the inverse map of a bijective holomorphic map is also holomorphic. So, g^* is a holomorphic automorphism of $M \times N$. Conversely, each holomorphic automorphism g^* of $M \times N$ with $\pi_M g^* = \pi_M$ defines a holomorphic map $g: M \to \mathrm{Aut}(N)$ with the property $g^*(x, y) = (x, g(x)(y))$. As an application of Theorem 4.1, we see

THEOREM 4.5. *In the above situation, let $\mathrm{Aut}(N)_0$ be the maximum complex Lie subgroup of $\mathrm{Aut}(N)$. For a connected complex manifold M any holomorphic automorphism h of $M \times N$ with the property $\pi_M h = \pi_M$ can be written $h(x, y) = (x, g_0(x)g(y))$ and $h(x, y) = (x, g'(g_0'(x)y))$ $(x \in M,\ y \in N)$ with suitable $g, g' \in \mathrm{Aut}(N)$ and holomorphic maps $g_0, g_0': M \to \mathrm{Aut}(N)_0$.*

COROLLARY 4.6. *Let M be an arbitrary connected complex manifold and N a bounded domain in C^n. Then any holomorphic automorphism h of $M \times N$ with the property $\pi_M h = \pi_M$ can be written $h = 1_M \times g$ with some $g \in \mathrm{Aut}(N)$, where $1_M: M \to M$ is the identity map.*

This is an immediate consequence of Theorem 4.5 and Example 2.2, (ii).

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