

RADICALS AND SEMISIMPLE CLASSES OF Ω -GROUPS

by RAINER MLITZ
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In this paper radicals in the sense of Kuroš and Amitsur (*KA*-radicals) for Ω -groups will be studied. For the sake of simplicity these radicals will be considered on varieties, the results remaining valid for more general classes.

A *KA*-radical can be defined in the following way (see for example (1)):

a property P of Ω -groups is called a radical property if the following conditions hold:

(α) every homomorphic image of an Ω -group having P has P ;

(β) every Ω -group G in the considered variety contains a unique maximal ideal having $P : P(G)$;

(γ) $P(P/P(G))$ is the zero ideal.

It is well known that a subclass \mathfrak{R} of a variety \mathfrak{B} is a radical class (i.e. the class of all Ω -groups in \mathfrak{B} with $P(G) = G$ for some radical property P), iff it satisfies the conditions

(1) \mathfrak{R} is homomorphically closed;

(2) any sum of ideals in \mathfrak{R} of some Ω -group G belongs to \mathfrak{R} ;

(3) \mathfrak{R} is closed under extensions (i.e.: $I \triangleleft G, I \in \mathfrak{R}, G/I \in \mathfrak{R}$ implies $G \in \mathfrak{R}$).

(see for example (8)—the proofs mentioned there make use only of isomorphism theorems and are therefore valid for Ω -groups). Good characterizations of the semisimple classes (i.e. the classes of all Ω -groups in \mathfrak{B} with $P(G) = \{0\}$ for some radical property P) have been given by v. Leeuwen, Roos and Wiegandt in (5) for associative and alternative rings. Unfortunately their methods—using the hereditariness of the radicals—do not apply to arbitrary rings or to Ω -groups (see 4).

The aim of this paper is threefold: (1) to describe the semisimple classes of Ω -groups; (2) to describe the pairs of corresponding radical and semisimple classes (in analogy to Dickson's torsion theory—see for example (6)); (3) to describe the mappings assigning to each Ω -group its radical. The proof methods will be essentially based on Hoehnke's theory of M -radicals (see 2).

Let \mathfrak{B} be any variety of Ω -groups.

Definition 1. For any subclass \mathfrak{C} of \mathfrak{B} define the classes $S\mathfrak{C}$ and $R\mathfrak{C}$ by:

$$S\mathfrak{C} = \{G \in \mathfrak{B} / I \triangleleft G, I \in \mathfrak{C} \Rightarrow I = \{0\}\}$$

$$R\mathfrak{C} = \{G \in \mathfrak{B} / G/I \in \mathfrak{C} \Rightarrow I = G\}$$

($S\mathfrak{C} = \mathfrak{C}'$ and $R\mathfrak{C} = \mathfrak{C}'$ in Hoehnke's terminology).

Definition 2. A radical in the sense of Hoehnke (2) (*H-radical*) on \mathfrak{B} is defined by a mapping ρ assigning to each $G \in \mathfrak{B}$ an ideal $\rho(G)$ (the radical) of G and satisfying

- (i) $\varphi(\rho G) \subseteq \rho(\varphi G)$ for any homomorphism φ defined on G
- (ii) $\rho(G/\rho G) = \{0\}$.

Definition 3. Given a binary relation M on \mathfrak{B} , a *H-radical* is called an *M-radical* if the corresponding semisimple class \mathfrak{S} satisfies the conditions:

- (M₁) $G \in \mathfrak{S}, (I, G) \in M \Rightarrow I \notin R\mathfrak{S}$
- (M₂) If $I \notin R\mathfrak{S}$ for every I with $(I, G) \in M$, then G belongs to \mathfrak{S} .

The following results of Hoehnke (2, p. 365–366) will be used:

For *M-radicals*, the radical class \mathfrak{R} and the semisimple class \mathfrak{S} are uniquely determined each by the other: $\mathfrak{S} = S\mathfrak{R}$ and $\mathfrak{R} = R\mathfrak{S}$. It follows that $\mathfrak{S} = SR\mathfrak{S}$ and $\mathfrak{R} = RS\mathfrak{R}$ hold. A subclass \mathfrak{S} of \mathfrak{B} is the semisimple class of an *M-radical* iff \mathfrak{S} is closed under subdirect products and satisfies (M₁) and (M₂); a subclass \mathfrak{R} of \mathfrak{B} is the radical class of an *M-radical* iff it satisfies the conditions

- (N₁) \mathfrak{R} is homomorphically closed;
- (N₂) If for every homomorphic image $\varphi G \neq \{0\}$ of G there exists $I \in \mathfrak{R}$ with $(I, \varphi G) \in M$, then G belongs to \mathfrak{R} ;
- (N₃) $S\mathfrak{R}$ is closed under subdirect products;
- (N₄) For every $G \in \mathfrak{R}, G \neq \{0\}$, there is an element I in \mathfrak{R} with $(I, G) \in M$.

In the following, we will consider the relation M defined by:

$$(I, G) \in M \text{ iff } \{0\} \neq I \triangleleft G.$$

For this relation M , one easily checks that every subclass of \mathfrak{B} satisfying (M₁) and (M₂) is closed under subdirect products and that the conditions (N₁) and (N₂) imply (N₃) and (N₄). Therefore our *M-radicals* are then exactly the radicals in the sense of Kuroš and Amitsur (see (8), p. 10).

Notation. For any subclass \mathfrak{C} of \mathfrak{B} containing $\{0\}$ and any $G \in \mathfrak{B}$ define $(G)\mathfrak{C} = (\cap I : I \triangleleft G, G/I \in \mathfrak{C})$ and $\mathfrak{C}(G) = (\sum I : I \triangleleft G, I \in \mathfrak{C})$.

Theorem 1. For all pairs $(\mathfrak{R}, \mathfrak{S})$ of corresponding *KA-radical* and semisimple classes in \mathfrak{B} , $(G)\mathfrak{S}$ equals $\mathfrak{R}(G)$ for every $G \in \mathfrak{B}$.

Proof. \mathfrak{R} satisfies the conditions (1)–(3), \mathfrak{S} the conditions (M₁) and (M₂). Let I be any ideal of $G/\mathfrak{R}(G)$ with $I \in \mathfrak{R}$ and let K be the complete preimage of I in G . (2) and (3) for \mathfrak{R} then imply $K \in \mathfrak{R}$, i.e. $K \subseteq \mathfrak{R}(G)$. Hence I is the zero-ideal and $G/\mathfrak{R}(G)$ belongs to \mathfrak{S} by (M₂). It follows that $(G)\mathfrak{S}$ is contained in $\mathfrak{R}(G)$.

Conversely it follows from (2) that the radical defined by \mathfrak{S} is $(G)\mathfrak{S}$ for every $G \in \mathfrak{B}$. (1) and (2) for \mathfrak{R} imply that $\mathfrak{R}(G)/(G)\mathfrak{S}$ belongs to \mathfrak{R} . But $\mathfrak{R}(G)/(G)\mathfrak{S}$ is an ideal in $G/(G)\mathfrak{S} \in \mathfrak{S}$, hence by (M₁) $\mathfrak{R}(G)$ must be contained in $(G)\mathfrak{S}$.

Theorem 2. For a pair $(\mathfrak{F}, \mathfrak{X})$ of subclasses of \mathfrak{B} the following assertions are equivalent:

(2.1) $(\mathfrak{F}, \mathfrak{X})$ is the pair of corresponding semisimple class (\mathfrak{F}) and radical class (\mathfrak{X}) for some KA-radical on \mathfrak{B} ;

(2.2) $(\mathfrak{F}, \mathfrak{X})$ satisfies the conditions

(A) $\mathfrak{F} \cap \mathfrak{X} = \{0\}$,

(B) \mathfrak{X} is homomorphically closed,

(C) $G \in \mathfrak{F}$, $(I, G) \in M$ implies $I \notin \mathfrak{X}$,

(D) For any $G \in \mathfrak{B}$, there is an ideal $T(G)$ of G satisfying $T(G) \in \mathfrak{X}$ and $G/T(G) \in \mathfrak{F}$;

(2.3) $(\mathfrak{F}, \mathfrak{X})$ satisfies the conditions (A), (C), (D) and the dual to (C):

(E) $G \in \mathfrak{X}$, $G/I \neq \{0\}$ implies $G/I \notin \mathfrak{F}$.

(Assertion 2.3 is more general than 2.2 and gives a self-dual characterisation of the pairs of semisimple and radical classes; it is due to a remark of R. Wiegandt.)

Proof. Equivalence of 2.1 and 2.2:

Let $(\mathfrak{F}, \mathfrak{X})$ be a pair of corresponding semisimple and radical classes. Then $\mathfrak{F} = S\mathfrak{X}$ and $\mathfrak{X} = R\mathfrak{F}$; (A), (B) and (C) are trivially fulfilled. To prove (D) take $T(G) = (G)\mathfrak{F} = \mathfrak{X}(G)$. Then by (2) $T(G)$ belongs to T and $G/T(G)$ belongs to \mathfrak{F} as \mathfrak{F} is closed under subdirect products.

Assume conversely that $(\mathfrak{F}, \mathfrak{X})$ satisfies (A)–(D). Then (C) and (D) imply $S\mathfrak{X} = \mathfrak{F}$ and (A), (B) and (D) imply $R\mathfrak{F} = \mathfrak{X}$. Condition (M_1) for \mathfrak{F} is exactly (C); to prove (M_2) assume that for some $G \in \mathfrak{B}$, $(I, G) \in M$ implies $I \notin R\mathfrak{F} = \mathfrak{X}$; then by (D) $T(G) = \{0\}$ and G belongs to \mathfrak{F} . Hence \mathfrak{F} is a semisimple class and $\mathfrak{X} = R\mathfrak{F}$ the corresponding radical class of some KA-radical.

Equivalence of (B) and (E) if (A), (C) and (D) are satisfied:

(B) implies (E) by (A).

(E) implies (B) by the following argument: $G \in \mathfrak{X}$, $G/I \neq \{0\}$ implies $G/I \notin \mathfrak{F}$ by (E); $G/I/T(G/I)$ belongs to \mathfrak{F} by (D); hence $G/I/T(G/I)$ is equal to $\{0\}$; it follows that $T(G/I) = G/I$ and therefore that G/I belongs to \mathfrak{X} .

Corollary. A subclass R of \mathfrak{B} is the radical class for some KA-radical on \mathfrak{B} iff it satisfies the conditions:

(R₁) $\mathfrak{R}(G) \in \mathfrak{R}$ for every $G \in \mathfrak{B}$;

(R₂) $\mathfrak{R}(G) = (G)S\mathfrak{R}$ for every $G \in \mathfrak{B}$.

(This corollary is due to a remark of R. Wiegandt.)

Proof. (R₁) for a KA-radical class follows from (2), (R₂) is easily deduced from Theorem 1. Conversely, if (R₁) and (R₂) hold for a class \mathfrak{R} , take $\mathfrak{X} = \mathfrak{R}$ and $\mathfrak{F} = S\mathfrak{R}$; then (A) holds for $(\mathfrak{F}, \mathfrak{X})$ by construction; $G \in \mathfrak{X}$ implies $(G)S\mathfrak{R} = \mathfrak{R}(G) = G$, i.e. (E) holds; $G \in \mathfrak{F}$ implies $\mathfrak{R}(G) = (G)S\mathfrak{R} = (G)\mathfrak{F} = 0$, hence (C) and (D) are fulfilled (with $T(G) = \mathfrak{R}(G)$).

Remark. If $(\mathfrak{F}, \mathfrak{X})$ satisfies (A)–(D), $T(G)$ is uniquely determined for every $G \in \mathfrak{B}$: if we assume that (D) is satisfied for both $T(G)$ and $U(G)$, by (B) we get that $U(G)/T(G)$ belongs to \mathfrak{X} ; but $U(G)/T(G)$ is an ideal of $G/T(G) \in \mathfrak{F}$ and hence

belongs to \mathfrak{X} iff it is the zero-ideal. It follows $U(G) \subseteq T(G)$. By the same argument we get $T(G) \subseteq U(G)$.

From this remark we can deduce that for every pair $(\mathfrak{Y}, \mathfrak{X})$ of subclasses of \mathfrak{B} satisfying (A)–(D) the corresponding *KA*-radical is given for every $G \in \mathfrak{B}$ by: $T(G) = (G)\mathfrak{Y} = \mathfrak{X}(G)$.

Theorem 3. *A mapping ρ assigning to each $G \in \mathfrak{B}$ an ideal $\rho(G)$ of G defines a *KA*-radical iff it satisfies the conditions (i) and (ii) (i.e. defines a *H*-radical) and the conditions*

- (iii) *idempotence*
- (iv) *$I \triangleleft G, \rho(I) = I$ implies $I \subseteq \rho(G)$.*

Proof. Assume that ρ satisfies (i)–(iv) and take $\mathfrak{X}_\rho = \{G \in \mathfrak{B} / \rho(G) = G\}$ and $\mathfrak{Y}_\rho = \{G \in \mathfrak{B} / \rho(G) = \{0\}\}$. Then for the pair $(\mathfrak{Y}_\rho, \mathfrak{X}_\rho)$ condition (A) is fulfilled by construction, (B) follows directly from (i), (C) can be deduced from (iv) and (D) holds for $T(G) = \rho(G)$ by (ii) and (iii). Hence by Theorem 2 and the remark above follows that $(\mathfrak{Y}_\rho, \mathfrak{X}_\rho)$ is the pair of semisimple and radical class for *KA*-radical and that this radical is given by $T(G) = \rho(G)$ on every $G \in \mathfrak{B}$.

Another proof of this implication can be found in (3).

Conversely, take any *KA*-radical on \mathfrak{B} ; the pair of the corresponding semisimple class \mathfrak{Y} and radical class \mathfrak{X} satisfies (A)–(D) and the radical is given by $T(G)$ for every $G \in \mathfrak{B}$. Since the *KA*-radicals are *M*-radicals, the mapping $G \rightarrow T(G)$ satisfies (i) and (ii); (iii) follows directly from (D) and the fact that $T(G)$ is uniquely determined. To prove (iv) consider $I \triangleleft G$ with $T(I) = I$, i.e. with $I \in \mathfrak{X}$; from (B) follows that $I/T(G)$ belongs to \mathfrak{X} , but $I/T(G)$ is an ideal in $G/T(G) \in \mathfrak{Y}$; hence by (C) we get $I \subseteq T(G)$.

Theorem 4. *For a subclass \mathfrak{C} of \mathfrak{B} the following assertions are equivalent:*

- (4.1) *\mathfrak{C} is the semisimple class of some *KA*-radical on \mathfrak{B}*
- (4.2) *\mathfrak{C} has the properties*
 - (a) *closed under subdirect products*
 - (b) *(M_1)*
 - (c) *$((G)\mathfrak{C})\mathfrak{C} = (G)\mathfrak{C}$ for every $G \in \mathfrak{B}$.*
- (4.3) *\mathfrak{C} has the properties (a), (b),*
 - (d) *closed under extensions, i.e.: $I \triangleleft G, I \in \mathfrak{C}, G/I \in \mathfrak{C}$ implies $G \in \mathfrak{C}$*
 - (e) *$((G)\mathfrak{C})\mathfrak{C}$ is an ideal in G for every $G \in \mathfrak{B}$.*

Proof. (4.1) \Rightarrow (4.2): As the *KA*-radicals are *M*-radicals, (a) and (b) hold; by Theorem 2 the considered radical is given by the mapping $G \rightarrow (G)\mathfrak{C}$ and by Theorem 3 this mapping is idempotent; hence (c) holds.

(4.2) \Rightarrow (4.3): (e) is a trivial consequence of (c); (d) follows from the following argument: for every ideal I of G with $G/I \in \mathfrak{C}$, $(G)\mathfrak{C}$ is contained in I by construction and hence is an ideal of I ; $I \in \mathfrak{C}$ then implies $(G)\mathfrak{C} = \{0\}$ or $(G)\mathfrak{C} \notin RS$ by (M_1) . Since the second assertion is a contradiction to (c), we get $(G)\mathfrak{C} = \{0\}$, i.e. $G \in \mathfrak{C}$ by (a).

(4.3) \Rightarrow (4.1): (M_1) is satisfied by (b), hence we only have to prove (M_2) for \mathfrak{C} . Assume that for $G (I, G) \in M$ implies $I \notin R\mathfrak{C}$, i.e. $(I)\mathfrak{C} \neq I$; it follows that for $G \notin \mathfrak{C}$ (i.e.

$(G)\mathfrak{C} \neq \{0\}$, $((G)\mathfrak{C})\mathfrak{C} \neq (G)\mathfrak{C}$; by (e) $((G)\mathfrak{C})\mathfrak{C}$ is an ideal of G and we get

$$G/((G)\mathfrak{C})\mathfrak{C} \mid (G)\mathfrak{C} \mid ((G)\mathfrak{C})\mathfrak{C} - G/(G)\mathfrak{C};$$

since by (a) $G/(G)\mathfrak{C}$ and $(G)\mathfrak{C} \mid ((G)\mathfrak{C})\mathfrak{C}$ are elements of \mathfrak{C} , we can apply (d) to get $G/((G)\mathfrak{C})\mathfrak{C} \in \mathfrak{C}$ and hence $(G)\mathfrak{C} = ((G)\mathfrak{C})\mathfrak{C}$ in contradiction to our assumptions for $G \notin \mathfrak{C}$.

Remark. The equivalence of (4.1) and (4.3) of Theorem 4 has been independently shown by a slightly different approach by v. Leeuwen and Wiegandt (7).

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INSTITUT FÜR ANGEWANDTE MATHEMATIK
 TU WIEN
 A-1040 WIEN
 GUSSHAUSSTR. 27–29