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Derivations and Valuation Rings

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Abstract. A complete characterization of valuation rings closed for a holomorphic derivation is given, following an idea of Seidenberg, in dimension 2.

1 Introduction and Preliminaries

Seidenberg [6] proposed a relation between valuations (which are *contact* objects) and derivations (which are also closely related to *contact*), using the following definition which is the present paper's object of study.

Definition 1.1 Let $\mathcal{M}$ be a function field, D a derivation on $\mathcal{M}$, and $\mathcal{O}_\nu \subset \mathcal{M}$ a valuation ring. The ring $\mathcal{O}_\nu$ is *closed for D* if $D(\mathcal{O}_\nu) \subset \mathcal{O}_\nu$. We shall also say that ν *is closed for D*.

The condition can be restated as "$\nu(f) \geq 0$ implies $\nu(D(f)) \geq 0$", where ν is the valuation of $\mathcal{M}$ associated with $\mathcal{O}_\nu$. The use of valuations in the context of differential equations as in [4], has proved fruitful: see, for example [1–3].

Our aim is to describe completely the valuations that are closed for a specific derivation when $\mathcal{M}$ is the field of meromorphic functions in two variables and D corresponds to a singular holomorphic vector field on $(\mathbb{C}^2, 0)$.

From now on, we restrict ourselves to $\mathcal{M} = \mathbb{C}\{\{x, y\}\}$, the field of meromorphic functions in two variables, which is the quotient field of $\mathcal{O} = \mathbb{C}\{x, y\}$. The maximal ideal of $\mathcal{O}$ will be denoted by $\mathfrak{m}$. We fix a derivation $\mathcal{X}\colon \mathcal{O} \to \mathcal{O}$, that is, a (germ of a) holomorphic vector field at the origin. As such, it can be written

$$\mathcal{X} = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}, \tag{1.1}$$

where a, b are holomorphic. If $a(0, 0) = b(0, 0) = 0$ we shall say that $\mathcal{X}$ is *singular* at the origin. In [6], Seidenberg proved that if $\mathcal{X}$ is non-singular, then there is only one valuation, centered at $\mathcal{O}$, closed for $\mathcal{X}$ (which corresponds to the "contact" with the only invariant curve for $\mathcal{X}$ passing through $(0, 0)$).

1.1 Birational Models of Vector Fields in $(\mathbb{C}^2, 0)$

Consider a (finite or infinite) sequence of maps

$$\pi \equiv \cdots \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} (\mathbb{C}^2, 0) = X_0, \tag{1.2}$$

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where each π_i is the blowing-up centered at a closed point $P_{i-1} \in X_{i-1}$. We shall say that π is a *chain* if $P_i \in \pi_i^{-1}(P_{i-1})$ for all i.

Seidenberg [5] proved that the germ of reduced foliation associated with $\mathcal{X}$ in $(\mathbb{C}^2, 0)$ becomes, after a finite subsequence (of length, say, k) of π, *in which only singular points of the foliation are blown-up*, either regular or *simple* at P_k. This statement cannot be literally translated to our setting (vector fields). However, as we shall see, the situation is not essentially different. Let P be any point in the exceptional divisor of π_k for some k in the sequence π in (1.2).

Definition 1.2 We say that $\mathcal{X}$ is *pseudoregular at P* if there exists a holomorphic function f at P such that $f(P) = 0$ and $\mathcal{X} = f\tilde{\mathcal{X}}$, with $\tilde{\mathcal{X}}$ a non-singular holomorphic vector field at P.

If $\mathcal{X}$ is of the form $f(x, y)(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})$ with $f(P) = 0$, a, b are holomorphic and have no common factor, and $bdx - ady$ has a simple singularity at P (in the sense of [5]), then we shall say that $\mathcal{X}$ is *pseudosimple at P*.

Remark 1.3 Recall that the definition of *simple singularity* means not only that the linear part of $bdx - ady$ is not nilpotent, but also that if its eigenvalues are λ and $\mu \neq 0$, then $\lambda/\mu \notin \mathbb{Q}_{>0}$.

We shall also make extensive use of the following well-known property. If P is a simple singularity for $\omega = bdx - ady$ and C_1, C_2 are two invariant curves (formal or convergent) for ω through P, then the following hold:

- The curves C_1 and C_2 are the only invariant curves for ω through P.
- Both C_1 and C_2 are non-singular at P.

With the same notation, we have the following lemma.

Lemma 1.4 *Let $\mathcal{X}$ be pseudoregular at P and let η be the blowing-up with center P. Let $P' \in \eta^{-1}(P)$. Then either $\mathcal{X}$ is regular or pseudoregular at P' or it is pseudosimple at P'. The latter happens only when P' corresponds to the tangent direction of the invariant curve of $\tilde{\mathcal{X}}$.*

Proof We only need to express $\mathcal{X}$ in local coordinates at P'. From the hypothesis, and after a local change of coordinates, we may assume that $\mathcal{X} = f(x, y)\frac{\partial}{\partial y}$. Depending on the chart, the local coordinates of η can be taken as

$$\eta \equiv \begin{cases} \tilde{x} = \frac{x}{y} + c, c \in \mathbb{C}, \\ \tilde{y} = y, \end{cases} \quad \text{or} \quad \eta \equiv \begin{cases} \tilde{x} = x, \\ \tilde{y} = \frac{y}{x}. \end{cases}$$

In the first case, $\mathcal{X}$ is

$$\mathcal{X} = f((\tilde{x} - c)\tilde{y}, \tilde{y})\Big(\frac{(c - \tilde{x})}{\tilde{y}}\frac{\partial}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{y}}\Big),$$

while in the second case

$$\mathcal{X} = f(\tilde{x}, \tilde{x}\tilde{y})\frac{1}{\tilde{x}}\frac{\partial}{\partial \tilde{y}}.$$

In both cases, the fact that $f(0, 0) = 0$ gives the result. ■

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Remark 1.5 Notice that the tangent cone of $(f = 0)$ at P is "irrelevant", *i.e.*, f only makes the field at P' holomorphic.

A similar computation gives the following lemma.

Lemma 1.6 *Assume $\mathcal{X} = f\tilde{\mathcal{X}}$ is pseudosimple at P and let η be the blowing-up with center P. Let P_1, P_1' be the points in $E = \eta^{-1}(P)$ corresponding to the eigenvectors of the linear part of $\tilde{\mathcal{X}}$ at P. Then $\mathcal{X}$ is pseudosimple at P_1 and P_1' and regular or pseudoregular at any other point in E.*

For simple singularities, the same statement holds if one removes the "pseudo" everywhere.

For chains of blowing-ups one has the following result, which guarantees that *a field becomes regular before becoming non-holomorphic* in a chain of blowing-ups.

Lemma 1.7 *Let $\mathcal{X}$ be a holomorphic vector field at $(0,0)$ and π a finite or infinite chain as in* (1.2). *Then one of the following holds:*

- *All the centers of π are singular for $\mathcal{X}$.*
- *There exists a finite (possibly empty) initial subsequence*

$$\pi'\colon X_k \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} (\mathbb{C}^2, 0) = X_0,$$

  such that the centers $P_0, \dots, P_{k-1}$ are singular for $\mathcal{X}$ and P_k is regular for $\mathcal{X}$.

In the latter case we shall say that $\mathcal{X}$ becomes regular *at P_k and that π (or π')* regularizes $\mathcal{X}$.

Proof The result is obviously true for a regular vector field, taking π' empty. Assume therefore that $\mathcal{X}$ is singular at $(\mathbb{C}^2, 0)$.

We are done if we show that $\mathcal{X}$ is holomorphic at P_i if it is singular at P_{i-1}. To this end, we just need to compute the expression of $\mathcal{X}$ at P_i from its expression at P_{i-1}. Fix coordinates (x, y) at P_{i-1} and write $\mathcal{X}$ as in (1.1). Without loss of generality (making a linear change of coordinates) we may assume that the local equations at P_i are $(\tilde{x}, \tilde{y})$ with

$$\tilde{x} = \frac{x}{y}, \quad \tilde{y} = y.$$

Thus $\mathcal{X}$ is written in these new coordinates,

$$\mathcal{X} = \Big(a(\tilde{x}\tilde{y}, \tilde{y})\frac{1}{\tilde{y}} - b(\tilde{x}\tilde{y}, \tilde{y})\frac{\tilde{x}}{\tilde{y}}\Big)\frac{\partial}{\partial \tilde{x}} + b(\tilde{x}\tilde{y}, \tilde{y})\frac{\partial}{\partial \tilde{y}},$$

which is holomorphic at P_i if $\mathcal{X}$ is singular at P_{i-1}, *i.e.*, $a(0,0) = b(0,0) = 0$. ∎

Remark 1.8 Incidentally, we have also proved that if $\mathfrak{m}_i$ is the maximal ideal at P_i and $m_i = \min(\operatorname{ord}_{\mathfrak{m}_i}(a), \operatorname{ord}_{\mathfrak{m}_i}(b))$ (assuming $\mathcal{X}$ is holomorphic at P_i), then $m_i \geq m_{i-1} - 1$ if $m_{i-1} > 0$.

The following lemma deals with the generic point in the exceptional divisor after blowing-up a non-dicritical singularity (in which case the exceptional divisor is invariant for the corresponding reduced foliation).

Lemma 1.9 *Let $\mathcal{X}$ be pseudoregular of the form $\mathcal{X} = u(x,y)x^m \frac{\partial}{\partial y}$ with $m > 0$ and u holomorphic with $u(0,0) \neq 0$. Let η denote the blowing-up with center $(0,0)$ and let P_1 be a point in the exceptional divisor $\eta^{-1}(0,0)$.*

(i) *If P_1 corresponds to the direction $(x = 0)$, then P_1 is a pseudosimple singularity.*
(ii) *Otherwise, $\mathcal{X}$ is pseudoregular (or regular) at P_1 and there are local coordinates $(\tilde{x}, \tilde{y})$ at P_1 such that*

$$\mathcal{X} = v(\tilde{x}, \tilde{y})\tilde{x}^{m-1}\frac{\partial}{\partial y},$$

with $v(0,0) \neq 0$, so that in this case $\mathcal{X}$ is regular at P_1 if and only if $m = 1$.

Proof The same proof as for Lemma 1.4 applies. The decrease of the exponent is due to the appearance of $\tilde{x}$ in the denominator after blowing-up. ■

Lemma 1.10 *Let π denote an infinite chain of blowing-ups as in* (1.2) *and assume that $\mathcal{X}$ is pseudoregular at P_k for some k. Then there exists $l \geq k$ such that $\mathcal{X}$ is either regular or pseudosimple at l.*

Proof From Lemma 1.4, the only other possibility is that $\mathcal{X}$ be always pseudoregular. However, from Remark 1.5, this would happen only if for all $l \geq k$, π_l does not follow the (only) separatrix of $\tilde{\mathcal{X}}$ at P_l. But this would give rise to an infinite sequence in case (ii) of Lemma 1.9, which is impossible. ■

From Seidenberg's reduction of singularities of holomorphic foliations [5] and Lemmas 1.4, 1.9, and 1.10 we have the following corollary.

Corollary 1.11 *Let $\mathcal{X}$ be a holomorphic vector field in $(\mathbb{C}^2, 0)$ as in* (1.1) *with a, b relatively prime. Let π be an infinite chain of blowing ups. Then either π regularizes $\mathcal{X}$ or there is a $k \geq 0$ such that $\mathcal{X}$ is holomorphic and simple or pseudosimple at P_k and P_i is singular for $\mathcal{X}$ for $i < k$.*

The following result is used systematically in the next section.

Corollary 1.12 *Let π be an infinite chain of blowing-ups. Then one of the following alternatives holds:*

- *The chain π regularizes $\mathcal{X}$.*
- *For any integer $n \geq 0$ there is $l \geq n$ such that P_l is a pseudosimple singularity.*

Proof If π does not regularize $\mathcal{X}$, then by Seidenberg's reduction of singularities, there is an n_0 such that P_{n_0} is pseudosimple. For $l \geq n_0$, whenever π_l does not follow any of the two directions corresponding to the separatrices at P_l, P_{l+1} is pseudoregular. By Lemma 1.10, there must be $k \geq l+1$ such that P_k is pseudosingular (otherwise π would regularize $\mathcal{X}$) and we are done. ■

2 Classification of Closed Valuation Rings: Nondivisorial Valuations

In the rest of the paper we assume $\mathcal{X}$ can be written at $(0,0)$ as in (1.1) with a, b relatively prime. This is the usual situation when studying singularities of vector fields/foliations.

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Definition 2.1 A holomorphic vector field $\mathcal{X}$ as (1.1) is *reduced* at $(0,0)$ if a and b have no common irreducible factors in $\mathcal{O}_{(0,0)}$.

All the preliminaries above allow us to classify all the closed valuation rings centered at $(\mathbb{C}^2, 0)$ associated with an infinite sequence of blowing-ups, *i.e., nondivisorial* valuations. Assume that ν is such a valuation and let π_ν be its associated chain of blowing-ups (see [7], for example), with sequence of centers $(P_i)_{i=0}^{\infty}$.

If $\mathcal{X}$ is simple or pseudosimple at P_k, there are local coordinates $(\tilde{x}, \tilde{y})$ at P_k such that

$$\mathcal{X} = E\Big((\lambda\tilde{x} + a(\tilde{x},\tilde{y}))\frac{\partial}{\partial\tilde{x}} + (\mu\tilde{y} + b(\tilde{x},\tilde{y}))\frac{\partial}{\partial\tilde{y}}\Big), \tag{2.1}$$

where a, b are power series of order at least 2, $\lambda/\mu \notin \mathbb{Q}_{>0}$, and E is a holomorphic function near P_k (actually a product of powers of the local equations of the exceptional divisors at P_k if $\mathcal{X}$ is reduced at $(0,0)$).

From now on we use Spivakovsky's classification of valuations in function fields of surfaces [7]. From Equation (2.1) we obtain the following proposition.

Proposition 2.2 *Let $\mathcal{X}$ be reduced at $(0,0)$ and ν a valuation of $\mathcal{M}$ of rank 1 and rational rank 2. Then ν is closed for $\mathcal{X}$ if and only if its center is never a regular point for $\mathcal{X}$.*

Proof The rank conditions imply that, from some $k \geq 2$ on, the center of π_k is a crossing of exceptional divisors.

Assume all the centers of ν are pseudosimple for the associated foliation for $l \geq k$, for some $k \geq 0$, which is the only alternative to the regularization of $\mathcal{X}$ by Corollary 1.12, due to the previous remark. Fix some $l \geq k$. Taking local coordinates (x, y) at P_l, we may assume that $x = 0$ and $y = 0$ are both invariant for the associated foliation (they correspond to each of the exceptional divisors at P_l, which by Remark 1.3 are invariant). This means that $\mathcal{X}$ can be written as

$$\mathcal{X} = E\Big(x(\lambda + a(x,y))\frac{\partial}{\partial x} + y(\mu + b(x,y))\frac{\partial}{\partial y}\Big),$$

where E, a, b are holomorphic (at P_l) and that ν is completely determined by $\nu(x) = 1, \nu(y) = \alpha$, for some $\alpha \notin \mathbb{Q}$. This implies that

$$\nu(f(x,y)) = \mathrm{ord}_t(f(t,t^\alpha)) + \nu(E) = \min\{i + \alpha j \mid f_{ij} \neq 0\} + \nu(E),$$

where $f = \sum f_{ij}x^i y^j$, for $f \in \mathcal{O}_P$. Take $f, g \in \mathcal{O}_P$ such that $\nu(f) \geq \nu(g)$. Then

$$\nu(\mathcal{X}(f/g)) = \nu\Big(\frac{g(f_x\lambda x + f_y\mu y + \cdots) - f(g_x\lambda x + g_y\mu y + \cdots)}{g^2}\Big) + \nu(E)$$

(where subindices indicate partial differentiation), which has value at least $\nu(f) + \nu(g) - 2\nu(g) + \nu(E)$. An easy verification shows that if $\nu(f) = \nu(g)$, then the value

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of the numerator is strictly greater than $2\nu(g)$, whereas it is at least $2\nu(g)$ otherwise. In any case, ν is closed for $\mathcal{X}$.

If some center P_l is a regular point for $\mathcal{X}$, then the only valuation closed for $\mathcal{X}$ centered at P_l is the one associated with the separatrix at P_l (this is Seidenberg's result in [6]), which has rank and rational rank one. ■

Valuations of rank 2 correspond to either germs of analytic branches or to germs of exceptional divisors appearing after a finite number of blowing-ups of $(\mathbb{C}^2, 0)$. For ν of rank 2, let π_ν be its associated chain of blowing-ups and denote

$$\pi_\nu^0 \colon X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} (\mathbb{C}^2, 0)$$

the shortest chain of blowing-ups following centers of ν such that the curve associated with ν is "visible" in X_k. (So that if ν corresponds to a germ of analytic curve at $(0,0)$, π_ν^0 is empty.)

Theorem 2.3 *Let ν be a valuation of rank 2 and $\mathcal{X}$ a holomorphic vector field reduced at $(0,0)$. Then ν is closed for $\mathcal{X}$ if and only if its associated curve is invariant for the reduced foliation associated with $\mathcal{X}$ and no strict subsequence of π_ν^0 regularizes $\mathcal{X}$.*

Proof If a strict subsequence of π_ν^0 regularizes $\mathcal{X}$, say at P_j, then by Seidenberg's result [6] the only valuation closed for $\mathcal{X}$ centered at P_j corresponds to the separatrix of $\mathcal{X}$ through P_j, which by hypothesis *is not* the curve associated with ν (because the latter is not "visible" at P_j).

If π_ν regularizes $\mathcal{X}$, then ν is closed for $\mathcal{X}$ if and only if ν follows the trajectory of $\mathcal{X}$ at P_k (which is unique), again from Seidenberg's result [6].

Assume that π_ν does not regularize $\mathcal{X}$. Then, from Corollary 1.12 and from the reduction of singularities of analytic curves, we may assume that the center of ν at X_l, say P_l, is a pseudosimple singularity for $\mathcal{X}$, for some $l > k$, where k is the length of π_ν^0 *and that the curve associated with ν at P_l is non-singular*. Hence, we may assume that in a local system of coordinates (x, y) at P_l, the curve associated with ν is $(y = 0)$ at P_l and that $\mathcal{X}$ is pseudosimple at P_l. This means that $\mathcal{X}$ can be written as

$$\mathcal{X} = E\Big((\lambda x + a(x,y))\frac{\partial}{\partial x} + y(\mu + b(x,y))\frac{\partial}{\partial y}\Big) \tag{2.2}$$

with $\operatorname{ord}(a) \geq 2, \operatorname{ord}(b) \geq 1$. The asymmetry between x and y arises because we are not taking the equation of the exceptional divisor through P_l as the other coordinate.

The valuation is given by the following: let $f \in \mathcal{O}_{P_l}$ and write

$$f(x,y) = x^m f_1(x) + y^k f_2(x,y),$$

where $f_1, f_2 \in \mathcal{O}_{P_l}$, $f_1(0)$ depends only on x and may be 0, but $f_1(0) \neq 0$ if $f_1 \neq 0$, and $m, k \geq 0$. Then $\nu(f) = (0, m)$ if $f_1 \neq 0$. Otherwise, $\nu(f) = (k, j)$ for some non-negative integer j. The order in $\mathbb{Z}^2$ is lexicographical. If $f/g \in \mathcal{M}$ has $\nu(f/g) \geq 0$, we may assume that either $f_1 = 0$ and $g_1 \neq 0$ or that both $f_1, g_1 \neq 0$. In any case,

$$\mathcal{X}\Big(\frac{f}{g}\Big) = \frac{g\mathcal{X}(f) - f\mathcal{X}(g)}{g^2}.$$

If $f_1(x) = 0$ and $g_1(x) \neq 0$, then by (2.2) the numerator is a multiple of y, so that $\nu(\mathcal{X}(f/g)) \geq 0$. Otherwise, a simple computation using (2.2) again (which implies that $y = 0$ is invariant) gives $\nu(\mathcal{X}(f/g)) \geq 0$.

We only need to prove the reciprocal when π_ν does not regularize $\mathcal{X}$ (the other cases are already dealt with), so that we may assume as before that P_l is a pseudosimple singularity and that $\mathcal{X}$ can be written as (2.2) with $\lambda/\mu \notin \mathbb{Q}_{>0}$, etc. We may also assume that the curve associated with ν is non-singular and has equation $x + y = 0$, after performing a local change of coordinates at P_l, *i.e.,* it is transverse to the two separatrices at P_l. Taking $f = x^k$ and $g = x + y + x^{k+2+\nu(E)}$, one gets

$$\mathcal{X}\Big(\frac{f}{g}\Big) = E\frac{(x+y+x^k)(k\lambda x^k + \dots) - x^k(\lambda x + \mu y + \dots)}{(x+y+x^{k+2+\nu(E)})^2},$$

whose value is < 0 because the second term in the numerator has contact at most $k+1$ with $x + y = 0$. (Notice that the condition $\lambda/\mu \notin \mathbb{Q}_{>0}$ is essential.) ■

An argument similar to the one used in the second case above (taking $f = x + y + y^{k+1} + \dots$) proves the following.

Theorem 2.4 *If ν is the contact with a formal non-convergent branch $\hat{f} = 0$, then ν is closed for $\mathcal{X}$ if and only if $\hat{f} = 0$ is invariant for $\mathcal{X}$.*

Finally, valuations with an infinite number of Puiseux pairs are never closed for any analytic vector field.

Theorem 2.5 *Let ν be a valuation with an infinite number of Puiseux pairs. Then ν is not closed for $\mathcal{X}$.*

Proof If π_ν regularizes $\mathcal{X}$, then we are done by Seidenberg's result [6], so that we may assume π_ν does not regularize $\mathcal{X}$ and hence that P_l is a pseudosimple singularity for $\mathcal{X}$ for some l and, as ν has an infinite number of Puiseux pairs, we may also assume P_l is a crossing of invariant exceptional divisors (by Remark 1.3). Then

$$\mathcal{X} = E\Big((\lambda x + xa(x,y))\frac{\partial}{\partial x} + (\mu y + yb(x,y))\frac{\partial}{\partial y}\Big).$$

At this point, we may reason using the same argument as in the reciprocal of Theorem 2.3 to end the proof (there is a linear combination of $cx + dy$ such that $\nu(cx + dy) \gg 0$, etc.). ■

3 The Divisorial Case

The result for divisorial valuations is in stark contrast with the corresponding result in [4] (where the author shows that a divisorial valuation is L'Hôpital if and only if it corresponds to a dicritical divisor of the foliation).

Proposition 3.1 *A divisorial valuation ν is closed for a vector field $\mathcal{X}$ reduced at* $(0,0)$ *if and only if its associated sequence of blowing-ups does not regularize $\mathcal{X}$.*

Proof If π_ν regularizes $\mathcal{X}$, say at P_k, then the only valuation centered at P_k closed for $\mathcal{X}$ would be the one associated with the separatrix through P_k, so it cannot be ν. This proves the necessity of the condition.

Assume that π_ν does not regularize $\mathcal{X}$ and let P_k be the last center in π_ν. This means that if $(\mathcal{O}_k, \mathfrak{m}_k)$ is the local ring at P_k, then ν is given by $\nu(f) = \operatorname{ord}_{\mathfrak{m}_k}(f)$ for $f \in \mathcal{O}_k$.

As

$$\nu(\mathcal{X}(h/g)) = \nu\Big(\frac{g\mathcal{X}(h) - h\mathcal{X}(g)}{g^2}\Big) + \nu(E),$$

an elementary verification shows that if $\nu(f) \geq \nu(g)$, then $\nu(\mathcal{X}(f/g)) \geq 0$. ∎

We say that a finite chain of blowing-ups π as (1.2) is *included* in the infinitely near singularities of a singular holomorphic foliation on $(\mathbb{C}^2, 0)$ if all the centers P_k of π are singularities of the corresponding reduced foliation in X_k.

Corollary 3.2 *Any divisorial valuation ν whose associated chain of blowing-ups π_ν is included in the infinitely near singularities of the reduced foliation associated with a holomorphic vector field $\mathcal{X}$ is closed for $\mathcal{X}$.*

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