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A GENERALIZATION OF WATSON'S LEMMA

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1. Introduction. Many functions, $F(z)$, have integral representations of the form

$$F(z) = \int_0^\infty f(t)e^{-zt}dt, \tag{1.1}$$

the so-called Laplace transform of $f(t)$. When $f(t)$ satisfies certain regularity conditions, it is possible to use Cauchy's theorem to deform the contour so that $F(z)$ has the integral representation

$$F(z) = \int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt, \tag{1.2}$$

where γ is a fixed real number, and the path of integration is the straight line joining $t = 0$ to $t = \infty e^{i\gamma}$, small indentations in the path of integration being allowed to avoid singularities of $f(t)$ where necessary. When the two functions defined by (1.1) and (1.2) are not the same, (1.2) provides the more general situation for a theoretical discussion of the properties of $F(z)$.

In 1918, G. N. Watson [**7**] proved an important result concerning the asymptotic behavior of functions $F(z)$ defined by (1.1). Although this result is one of the more important tools of asymptotic theory, it is known that the conditions on $f(t)$ and the path of integration are needlessly restricted. Generalizations can be found in [**2**; **5**; **8**]. A generalized version of Watson's result can be formulated as follows:

LEMMA (Generalized Watson's Lemma). *If:*

(i) $F(z)$, *as defined by* (1.2), *exists for some fixed* $z = z_0$;

(ii) *for some integer* N

$$f(t) = \sum_{n=0}^{N} a_n t^{\lambda_n - 1} + o(t^{\lambda_N - 1}), \tag{1.3}$$

as $t \to 0$ *along* $\arg t = \gamma$;

(iii) $\{a_n\}$, $0 \leqq n \leqq N$, *is a sequence of complex numbers*; *and*

(iv) $\{\lambda_n\}$, $0 \leqq n \leqq N$, *is a sequence of complex numbers satisfying* $\operatorname{Re} \lambda_0 > 0$, *and* $\operatorname{Re} \lambda_n > \operatorname{Re} \lambda_{n-1}$, $1 \leqq n \leqq N$;

then

$$F(z) = \sum_{n=0}^{N} a_n \Gamma(\lambda_n) z^{-\lambda_n} + o(z^{-\lambda_N}) \tag{1.4}$$

as $z \to \infty$ *in* $|\arg(ze^{i\gamma})| \leqq \pi/2 - \Delta$, *for any real number* Δ *in the interval* $0 < \Delta \leqq \pi/2$.

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The greater the integer N can be taken, the more detailed the information given in (1.4) becomes. From the known inversion properties of (1.2), one can be reasonably certain that the properties listed above for $f(t)$ are close to being necessary and sufficient conditions for (1.4) to be true. If this is so, significant generalization of Watson's Lemma will not be obtained without changing the nature of the function $f(t)$ in (1.2), and, at the same time, changing the form of the final result given in (1.4).

In a recent paper, Erdélyi [**1**] has studied the asymptotic behavior of a function $F(z)$, defined by (1.1), in which $f(t)$ may have a singularity of the type $t^{\lambda-1}(-\log t)^{\mu}$, a possibility that will be the subject of further discussion in the present paper. One would hope, and indeed expect, that such a generalization would retrieve the results of Watson's Lemma by placing $\mu = 0$. Although such a generalization is obtained, the hope and expectation are not easily realized, and involve an unexpected form of expansion of $f(t)$ in a neighborhood of $t = 0$.

As indicated above, generalizations will likely involve a change in the form of the asymptotic information given in (1.4). A suitable framework for discussing problems of this type is given in Erdélyi and Wyman [**3**], and a summary of the fundamental definitions involved is given in the next section.

2. Asymptotic expansions. Let R be an unbounded point set in the complex plane. A neighborhood, $U(z_0, \delta)$, of a finite point z_0, which may or may not be a point of R, is defined as a set of points z such that $z \in R$ and $|z - z_0| < \delta$. If z_0 is the point at infinity, then $U(z_0, \delta)$ is defined as the set of points z such that $z \in R$ and $|z| > \delta > 0$. With such a neighborhood system, the concepts of limit point of a set, limit, the Landau order relations O and o, all are given meaning in the usual way. It will always be assumed that z_0 is a limit point of R.

Asymptotic analysis attempts to obtain information concerning the behavior of a function, $F(z)$, in a neighborhood of a point z_0. Although such functions must be defined for points belonging to some neighborhood $U(z_0, \delta)$, they need not be defined for all points of R. Similarly, the information obtained need not retain validity for all points of R.

Throughout the paper, the symbol I will stand for one of the two following sets of integers:

$$(2.1) \qquad \begin{aligned} I &= \{0, 1, 2, \ldots, M\}, \textit{ a finite set of integers; or} \\ &= \{0, 1, 2, 3, \ldots\}, \textit{ the set of non-negative integers.} \end{aligned}$$

If convenient, summations of the form

$$(2.2) \qquad \begin{aligned} \sum_{n \in I} f_n &= \sum_{n=0}^{M} f_n, \; I \textit{ finite} \\ &= \sum_{n=0}^{\infty} f_n, \; I \textit{ infinite}, \end{aligned}$$

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may be written $\sum f_n$. When I is an infinite set, the series may be formal and need not converge.

Definition 2.1. A sequence of functions $\{\varphi_n(z)\}$, $n \in I$, is called an *asymptotic sequence* if

$$\varphi_{n+1} = o(\varphi_n), \tag{2.3}$$

as $z \to z_0$, as long as n and $n + 1$ both belong to I.

The notation $\{\varphi_n(z)\}$ will be used throughout to denote an asymptotic sequence, and when ambiguity must be avoided, the limit point z_0 will be specified in some way.

Definition 2.2. The series $\sum f_n(z)$ is called an *asymptotic expansion* of a function $F(z)$ with respect to the asymptotic sequence $\{\varphi_n(z)\}$, as $z \to z_0$, if for every fixed integer $N \in I$,

$$F(z) = \sum_{n=0}^{N} f_n(z) + o(\varphi_N), \tag{2.4}$$

as $z \to z_0$.

The notation

$$F(z) \sim \sum_{n \in I} f_n(z);\ \{\varphi_n\}, \tag{2.5}$$

as $z \to z_0$ has the meaning given in (2.4).

If for each n, $f_n(z) = a_n\varphi_n(z)$, where the a_n are constants, then the expansion (2.5) is said to be of Poincaré's type. Furthermore, if the expansion is of this type, and $\varphi_n(z) = (Z(z))^{\lambda_n}$, λ_n a complex number, the expansion is said to be of power series form.

Definition 2.3. Two functions $F(z)$ and $G(z)$ defined on some neighborhood $U(z_0, \delta)$ are said to be *asymptotically equal*, written

$$F(z) \approx G(z);\ \{\varphi_n\}, \tag{2.6}$$

as $z \to z_0$, if

$$F(z) = G(z) + o(\varphi_n), \tag{2.7}$$

as $z \to z_0$, for every fixed integer $n \in I$.

Two functions having the same asymptotic expansion are asymptotically equal, and the converse is also true.

Even this degree of generality is not sufficient to describe the asymptotic behavior of many of the known functions in mathematics. The form

$$F(z) \sim G_1(z)\left[\sum_{n \in I_1} f_n^{(1)}(z);\ \{\varphi_n^{(1)}\}\right] + G_2(z)\left[\sum_{n \in I_2} f_n^{(2)}(z);\ \{\varphi_n^{(2)}\}\right] + \dots \tag{2.8}$$

as $z \to z_0$, with the meaning

$$(2.9) \quad F(z) = G_1(z)\left[\sum_{n=0}^{N_1} f_n^{(1)}(z) + o(\varphi_{N_1}^{(1)})\right] + G_2(z)\left[\sum_{n=0}^{N_2} f_n^{(2)}(z) + o(\varphi_{N_2}^{(2)})\right] + \ldots$$

as $z \to z_0$, where $N_1, N_2, \ldots$ are arbitrary fixed integers chosen from $I_1, I_2, \ldots$, respectively, must often be used to give asymptotic information for many of the higher transcendental functions.

In discussing the asymptotic behavior of a function $F(z)$, defined by (1.2), the point set R is taken to be the sector of $S(\Delta)$, where

$$(2.10) \qquad S(\Delta): |\arg(ze^{i\gamma})| \leqq \pi/2 - \Delta,$$

where γ is a real number, and the choice of Δ is restricted to the interval $0 < \Delta \leqq \pi/2$. The point z_0 is the point at infinity, and a neighborhood, $U(z_0, \delta)$, of the point at infinity will be defined as the intersection of $S(\Delta)$ and the point set for which $|z| > \delta > 0$. By varying the choice of δ, a neighborhood system is established, and, therefore, for functions $\Phi(z)$ defined on some neighborhood of the point of infinity, the Landau order relations and the limit concept are given meaning.

In some circumstances, $S(\Delta)$ is equivalent to

$$(2.11) \qquad -\pi/2 + \Delta - \gamma \leqq \arg z \leqq \pi/2 - \Delta - \gamma,$$

and in all cases $S(\Delta)$ is equivalent to

$$(2.12) \qquad (4k - 1)\pi/2 + \Delta - \gamma \leqq \arg z \leqq (4k + 1)\pi/2 - \Delta - \gamma$$

for some fixed integer k. This fact allows the particular functions $\log z$ and z^λ to be defined in the usual way by

$$(2.13) \qquad \log z = \log|z| + i \arg z$$

and

$$(2.14) \qquad z^\lambda = \exp(\lambda \log z),$$

where $\arg z$ must satisfy (2.12).

For the results to be obtained in this paper, the sequence

$$(2.15) \qquad \{\varphi_n(z) = (\log z)^{\mu_n} \cdot z^{-\lambda_n}\},$$

where $\{\mu_n\}$ and $\{\lambda_n\}$, $n \in I$, are sequences of complex numbers, plays an important role. Since

$$(2.16) \qquad \varphi_{n+1}/\varphi_n = [(\log z)^{\mu_{n+1} - \mu_n}] \cdot [z^{-(\lambda_{n+1} - \lambda_n)}]$$

does not imply $\varphi_{n+1} = o(\varphi_n)$, as $|z| \to \infty$, unless further restrictions on $\{\mu_n\}$ or

$\{\lambda_n\}$ are stated, $\{\varphi_n\}$ is not necessarily an asymptotic sequence. However, the conditions

(2.17) $\quad \operatorname{Re}\lambda_{n+1} > \operatorname{Re}\lambda_n$, μ_n arbitrary, n and $(n+1)$ both in I,

or

(2.18) $\quad \operatorname{Re}\lambda_{n+1} = \operatorname{Re}\lambda_n$, $\operatorname{Re}\mu_{n+1} < \operatorname{Re}\mu_n$, n and $(n+1)$ both in I,

will insure that $\{\varphi_n\}$, as defined by (2.15), is such a sequence. For the remainder of the paper it will be assumed that $\{\varphi_n(z)\}$ is an asymptotic sequence.

Since

$$z^\alpha(\log z)^\beta \exp(-\epsilon|z|^\delta) = o(\varphi_n), \tag{2.19}$$

as $z \to \infty$, for every $n \in I$, for arbitrary complex numbers α and β, and any positive real numbers ϵ and δ, it will be true that terms which are exponentially small in $|z|$ can be replaced by zero in an asymptotic expansion. Similarly,

$$(\log z)^\beta z^{-\alpha} = o((\log z)^{\mu_n} \cdot z^{-\lambda_n}), \tag{2.20}$$

as $|z| \to \infty$, for every $n \in I$, providing $\operatorname{Re}\alpha > \operatorname{Re}\lambda_n$ for every $n \in I$, implies

$$(\log z)^\beta z^{-\alpha} \approx 0;\ \{\varphi_n\}, \tag{2.21}$$

as $|z| \to \infty$.

It is interesting to note that the condition $\operatorname{Re}\lambda_{n+1} > \operatorname{Re}\lambda_n$ does not imply $\overline{\lim}_{n\to\infty} \operatorname{Re}\lambda_n = \infty$. Hence (2.21) may, in some instances, allow terms to be dropped which are not exponentially small.

3. Basic integrals and their asymptotic behavior. By differentiating the identity

$$\int_0^{\infty e^{i\gamma}} t^{\lambda-1}e^{-zt}dt = \Gamma(\lambda)z^{-\lambda},\ \operatorname{Re}\lambda > 0,\ |\arg(ze^{i\gamma})| < \pi/2 \tag{3.1}$$

m times with respect to λ, the result

$$\int_0^{\infty e^{i\gamma}} t^{\lambda-1}(\log t)^m e^{-zt}dt = \frac{d^m}{d\lambda^m}(\Gamma(\lambda)z^{-\lambda}) \tag{3.2}$$

is obtained. This is equivalent to

$$\int_0^{\infty e^{i\gamma}} t^{\lambda-1}(-\log t)^m e^{-zt}dt = z^{-\lambda}(\log z)^m \sum_{r=0}^{m} (-1)^r\binom{m}{r}\Gamma^{(r)}(\lambda)(\log z)^{-r}. \tag{3.3}$$

From (3.3), a reasonable conjecture might be that

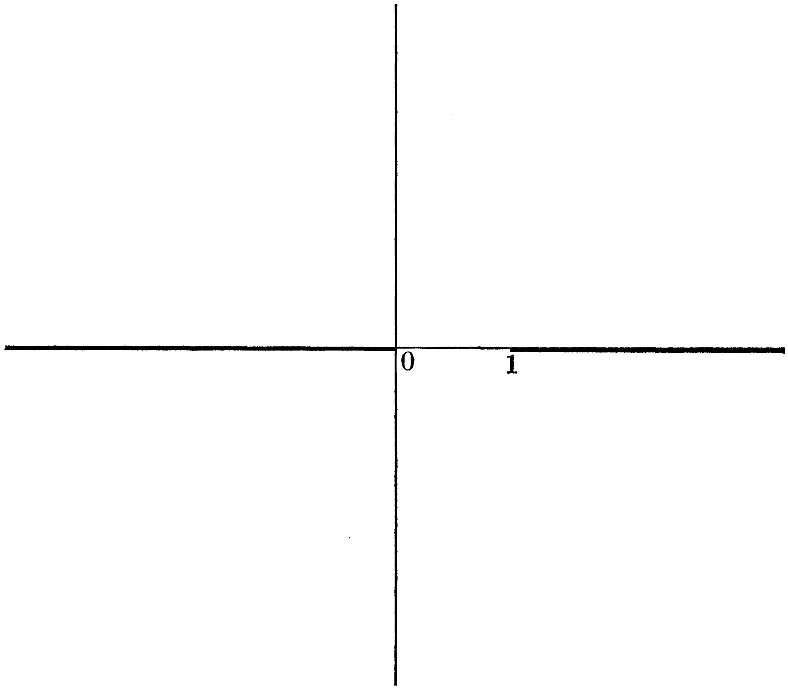
$$\int_0^{\infty e^{i\gamma}} t^{\lambda-1}(-\log t)^\mu e^{-zt}dt = (\log z)^\mu z^{-\lambda} \sum_{r=0}^{\infty} (-1)^r\binom{\mu}{r}\Gamma^{(r)}(\lambda)(\log z)^{-r}, \tag{3.4}$$

where μ is allowed to be a complex number, a conjecture that is false. The series in (3.4) diverges for all finite values of $\log z$. However, it will be shown that the weaker result obtained by replacing equality of, by an asymptotic relation between, the two sides of (3.4) does hold under suitable conditions. When this and other results are obtained, it becomes possible to use the pattern of proof of Watson's Lemma to obtain a generalization of Watson's results.

Since it is envisaged that λ of (3.4) will be a complex number, $t^{\lambda-1}$ will not be a continuous function of t unless a suitable cut is introduced into the complex t-plane. As is usual, this cut extends from $t = 0$ to $t = -\infty$ along the negative real axis, and $\arg t$ is restricted by

$$-\pi < \arg t < \pi. \tag{3.5}$$

Although $\log t$ is continuous under these conditions, a further cut is necessary in order to make $(-\log t)^\mu$ have the same property when μ is allowed to have complex values. This is accomplished by introducing a cut from $t = 1$ to $t = \infty$ along the positive real axis. Further, along the top of the cut $(-\log t)^\mu = (e^{-i\pi} \log t)^\mu = e^{-i\pi\mu}(\log t)^\mu$, $t \geqq 1$, and along the bottom of the cut $(-\log t)^\mu = (e^{i\pi} \log t)^\mu = e^{i\pi\mu}(\log t)^\mu$, $t \geqq 1$. For the purpose of discussion, the cut t-plane is as shown below:



t-plane

In discussing the asymptotic behavior of functions $F(z)$ defined by Laplace integrals of the form

$$F(z) = \int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt, \tag{3.6}$$

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it will always be assumed that the integral exists for some value $z = z_0$. The known properties of such integrals ensures the validity of the following two important properties:

(3.7) (a) $F(z)$ exists providing $\mathrm{Re}(ze^{i\gamma}) > \mathrm{Re}(z_0 e^{i\gamma})$;

$$(b) \quad \int_c^{\infty e^{i\gamma}} f(t)e^{-zt}dt = O(\exp(-\epsilon|c||z|)),$$

uniformly in arg z, as $z \to \infty$ in $S(\Delta)$, for some choice of $\epsilon > 0$, and any point $t = c$ which lies on the straight line path of integration. The result in (3.7) holds when $c = c(z)$. This degree of generality will be used in one or two proofs of the present paper. Unless otherwise stated, it will be assumed that c is fixed, and satisfies

$$(3.8) \qquad 0 < |c| < 1.$$

With such a restriction, one can delete $|c|$ in the right side of (3.7) to give

$$(3.9) \qquad \int_c^{\infty e^{i\gamma}} f(t)e^{-zt}dt = O(\exp(-\epsilon|z|)),$$

uniformly in arg z, as $z \to \infty$ in $S(\Delta)$, for some $\epsilon > 0$.

Since integrals of the type

$$\int_a^b t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt,$$

and expressions of the form

$$\sum_{r=0}^{N} (-1)^r \binom{\mu}{r} \Gamma^{(r)}(\lambda)(\log z)^{-r}$$

play important roles throughout the paper, the following notations are introduced:

$$(3.10) \qquad L(a, b, \lambda, \mu, z) = \int_a^b t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt,$$

where the path of integration is the straight line joining $t = a$ to $t = b$, with arg a = arg $b = \gamma$, and $0 \leqq |a| < |b|$;

$$(3.11) \qquad S_N(\lambda, \mu, \log z) = \sum_{r=0}^{N} (-1)^r \binom{\mu}{r} \Gamma^{(r)}(\lambda)(\log z)^{-r}.$$

Some care must be taken to insure that $L(a, b, \lambda, \mu, z)$ as given in (3.10) is a well-defined function. When μ is a non-negative integer, then $(-\log t)^{\mu}$ has no singularity at $t = 1$, and the cut from $t = 1$ to $t = \infty$ is no longer necessary. For all other values of μ, $(-\log t)^{\mu}$ will have a singularity at $t = 1$, and it may be necessary to avoid the point $t = 1$ by allowing an indentation in the path of integration, and to recognize that there are two functions which may be defined, depending on whether the path of integration includes part of the cut in the first quadrant of the t-plane or part of the cut in the fourth quadrant of the t-plane. For simplicity in the statement of our major results, it will be assumed that the path of integration in (3.10) does not include any portion of the cut in the t-plane involved when $t \geqq 1$. The situation in which the path of integration is allowed to include part of the cut from $t = 1$ to $t = \infty$ will be dealt with as a separate case.

For any non-negative integer m, (3.3) can be written as

$$L(0, \infty e^{i\gamma}, \lambda, m, z) = z^{-\lambda}(\log z)^m S_m(\lambda, m, \log z). \tag{3.12}$$

From (3.9), one obtains

$$L(c, \infty e^{i\gamma}, \lambda, m, z) = O(\exp(-\epsilon|z|)), \tag{3.13}$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$. Hence

$$L(0, c, \lambda, m, z) = (\log z)^m z^{-\lambda} S_m(\lambda, m, \log z) + O(\exp(-\epsilon|z|)), \tag{3.14}$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$. This result can be interpreted in two ways:

$$L(0, c, \lambda, m, z) \sim (\log z)^m z^{-\lambda}[S_m(\lambda, m, \log z); \{(\log z)^{-n}\}]; \tag{3.15}$$

or

$$L(0, c, \lambda, m, z) \approx (-1)^m \frac{d^m}{d\lambda^m}[\Gamma(\lambda)z^{-\lambda}]; \{\varphi_n\}, \tag{3.16}$$

providing $\{\varphi_n\}$ is an asymptotic sequence for which

$$\exp(-\epsilon|z|) \approx 0; \{\varphi_n\}, \tag{3.17}$$

as $z \to \infty$ in $S(\Delta)$. The estimates of error involved in the two interpretations are quite different, and the asymptotic information obtained by the use of (3.16) will be more detailed in nature than could be obtained by the use of (3.15).

For the more general situation $L(0, c, \lambda, \mu, z)$, Erdélyi [**1**] has given an elegant proof that

$$L(0, c, \lambda, \mu, z) \sim z^{-\lambda}(\log z)^\mu[S_\infty(\lambda, \mu, \log z); \{(\log z)^{-n}\}], \tag{3.18}$$

as $z \to \infty$ through positive real values of z, providing $0 < c < 1$, $\lambda > 0$ and μ is real. Although elegant, the proof does not seem readily adapted to extensions allowing c, λ, μ, z to be complex. This extension will be obtained by a sequence of lemmas.

LEMMA 3.1. *If $a = |z|^{-2+\delta}e^{i\gamma}$, $0 < \delta < 1$, then there will exist a number $\rho > 0$ such that*

$$L(0, a, \lambda, \mu, z) = 0(z^{-\lambda-\rho}), \tag{3.19}$$

uniformly in $\arg z$, *as* $z \to \infty$ *in* $S(\Delta)$.

Proof. For any choice of $\eta > 0$, $(-\log t)^{\mu} = o(t^{-\eta})$, as $t \to 0$. Hence

$$(3.20)\qquad |L(0, a, \lambda, \mu, z)| \leqq \int_0^a |t^{\lambda-\eta-1}dt| = O(a^{\lambda-\eta}),$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$. Since $\operatorname{Re}\lambda > 0$, $0 < \delta < 1$, and η is arbitrary, a choice of η exists such that

$$(3.21)\qquad \rho = \operatorname{Re}(\lambda(1-\delta) - (2-\delta)\eta) > 0.$$

Substituting $a = |z|^{-2+\delta}e^{i\gamma}$ into (3.20) then gives the required result.

LEMMA 3.2. *With a as in Lemma* 3.1,

$$(3.22)\qquad L(0, az, \lambda, n, 1) = O(z^{-\rho}),$$

for some $\rho > 0$, uniformly in $\arg z$, *as $z \to \infty$ in $S(\Delta)$.*

The proof is similar to that used in Lemma 3.1.

LEMMA 3.3. *With a defined as above, and b defined by $b = |z|^{-\delta}e^{i\gamma}$, then for any integer $n \geqq 0$,*

$$(3.23)\qquad L(az, bz, \lambda, n, 1) \approx (-1)^n\Gamma^{(n)}(\lambda); \{(\log z)^{-n}\}.$$

Proof.
$$\begin{aligned} L(az, bz, \lambda, n, 1) &= L(0, \infty e^{i\tau}, \lambda, n, 1) \\ &\quad - L(0, az, \lambda, n, 1) - L(bz, \infty e^{i\tau}, \lambda, n, 1) \\ &= (-1)^n\Gamma^{(n)}(\lambda) + 0(z^{-\rho}) - L(bz, \infty e^{i\tau}, \lambda, n, 1), \end{aligned}$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$, where $\tau = \arg(ze^{i\gamma})$. Since it is trivial to prove $L(bz, \infty e^{i\tau}, \lambda, n, 1) = 0(\exp(-\epsilon|z|^{1-\delta}))$, as $z \to \infty$ in $S(\Delta)$, the required result has been obtained.

These lemmas allow us to prove the following major result.

THEOREM 3.1. *For any choice of complex numbers λ and μ with* $\operatorname{Re}\lambda > 0$,

$$(3.24)\qquad L(0, c, \lambda, \mu, z) \sim z^{-\lambda}(\log z)^{\mu}[S_{\infty}(\lambda, \mu, \log z); \{(\log z)^{-n}\}]$$

uniformly in $\arg z$, *as $z \to \infty$ in $S(\Delta)$, providing $|c| < 1$.*

Proof. With a and b defined as above,

$$\begin{aligned} L(0, c, \lambda, \mu, z) &= L(a, b, \lambda, \mu, z) + L(0, a, \lambda, \mu, z) + L(b, c, \lambda, \mu, z) \\ &= L(a, b, \lambda, \mu, z) + O(z^{-\lambda-\rho}) + O(\exp(-\epsilon|z|^{1-\delta})), \end{aligned}$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$. The second order term is negligible with respect to the first, and therefore can be dropped.

Since

$$(3.25)\qquad L(a, b, \lambda, \mu, z) = \int_a^b t^{\lambda-1}(-\log t)^{\mu}e^{-zt}\,dt,$$

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the substitution $u = zt$ gives

$$(3.26)\qquad L(a, b, \lambda, \mu, z) = z^{-\lambda} \int_{az}^{bz} u^{\lambda-1}(\log z - \log u)^{\mu} e^{-u} du,$$

and, therefore,

$$(3.27)\qquad L(a, b, \lambda, \mu, z) = z^{-\lambda}(\log z)^{\mu} \int_{az}^{bz} u^{\lambda-1}\left(1 - \frac{\log u}{\log z}\right)^{\mu} e^{-u} du.$$

Since $|\log u/\log z| \leqq 1 - \delta_1$, for some choice of δ_1, $0 < \delta_1 < 1$, as $z \to \infty$ in $S(\Delta)$, then for any fixed integer $N \geqq 0$, the finite binomial expansion, with remainder, gives for all points on the path of integration that

$$(3.28)\qquad \left(1 - \frac{\log u}{\log z}\right)^{\mu} = \sum_{n=0}^{N} (-1)^n \binom{\mu}{n} \frac{(\log u)^n}{(\log z)^n} + R_N,$$

where

$$(3.29)\qquad |R_N| \leqq K \left| \frac{(\log u)^{N+1}}{(\log z)^{N+1}} \right|,$$

for some fixed $K > 0$. Hence

$$(3.30)\qquad L(a, b, \lambda, \mu, z) = z^{-\lambda}(\log z)^{\mu} \times \left[\sum_{n=0}^{N} (-1)^n \binom{\mu}{n} (\log z)^{-n} \int_{az}^{bz} u^{\lambda-1}(\log u)^n e^{-u} du + r_N \right],$$

where

$$(3.31)\qquad r_N = \int_{az}^{bz} u^{\lambda-1} e^{-u} R_N du.$$

Using the proof of Lemma 3.3 gives

$$(3.32)\qquad L(a, b, \lambda, \mu, z) = z^{-\lambda}(\log z)^{\mu} \times \left[\sum_{n=0}^{N} (-1)^n \binom{\mu}{n} \Gamma^{(n)}(\lambda)(\log z)^{-n} + O(z^{-\rho}) + r_N \right],$$

as $z \to \infty$ in $S(\Delta)$. Further,

$$(3.33)\qquad |r_N| \leqq K |\log z|^{-(N+1)} \int_{az}^{bz} |u^{\lambda-1}(\log u)^{N+1} e^{-u} du| \leqq K |\log z|^{-(N+1)} \int_{0}^{\infty e^{i\tau}} |u^{\lambda-1}(\log u)^{N+1} e^{-u} du|,$$

where $\tau = \arg(ze^{i\gamma})$.

It is trivial to show that the integral in (3.33) exists and is bounded in $\arg z$. Hence

$$(3.34)\qquad L(a, b, \lambda, \mu, z) = z^{-\lambda}(\log z)^{\mu}[S_N(\lambda, \mu, \log z) + O((\log z)^{-N-1})],$$

as $z \to \infty$ in $S(\Delta)$. Furthermore, the order relation does not depend on $\arg z$. This proves the required result.

A combination of the results given thus far yields

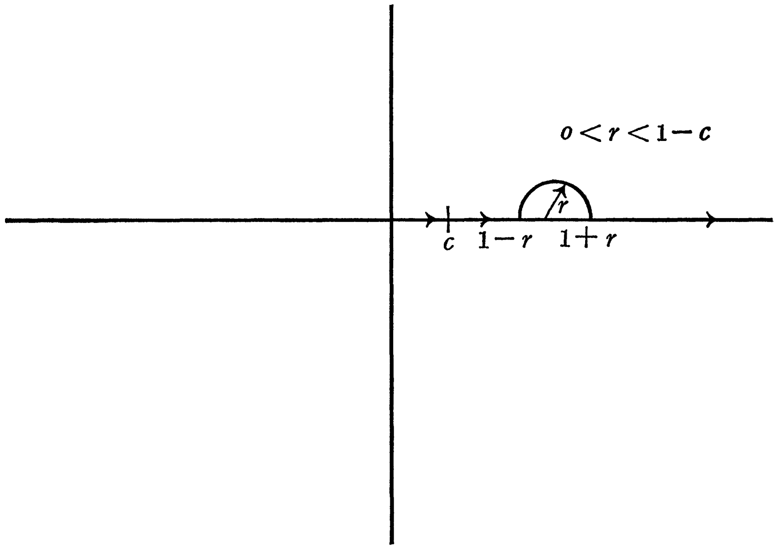
$$(3.35)\quad L(0, \infty e^{i\gamma}, \lambda, \mu, z) \sim z^{-\lambda}(\log z)^{\mu}[S_{\infty}(\lambda, \mu, \log z); \{(\log z)^{-n}\}]$$
$$\sim z^{-\lambda}(\log z)^{\mu}\left[\sum_{n=0}^{\infty}(-1)^{n}\binom{\mu}{n}\Gamma^{(n)}(\lambda)(\log z)^{-n};\ \{(\log z)^{-n}\}\right],$$

as $z \to \infty$ in $S(\Delta)$, providing $\operatorname{Re} \lambda > 0$. As indicated before, the case $\gamma = 0$ is excluded from the proof of (3.35), and validity of (3.35) when $\gamma = 0$ will now be established.

Let us consider the function $L(z)$ defined by

$$(3.36)\quad L(z) = \int_{0}^{\infty} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt,\ |\arg z| \leqq \pi/2 - \Delta,\ \operatorname{Re} \lambda > 0,$$

where the path of integration is as shown below.



t-plane

It is assumed that the path of integration is along the top of the circular cut so that $(-\log t)^{\mu} = e^{-i\pi\mu}(\log t)^{\mu}$ when $t \geqq 1 + r$. With these restrictions,

$$(3.37)\quad L(z) = \int_{0}^{c} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt + \int_{c}^{1-r} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt$$
$$+ \int_{1-r}^{1+r} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt + \int_{1+r}^{\infty} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt.$$

In the interval $c \leqq t \leqq 1 - r$, $|t^{\lambda-1}(-\log t)^{\mu}|$ is bounded, and

$$|e^{-zt}| \leqq \exp[-|z|c \sin \Delta],$$

and therefore

(3.38)
$$\int_c^{1-r} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt = O(\exp(-\epsilon|z|)),$$

as $|z| \to \infty$, for some choice of $\epsilon > 0$.

Similarly, for some choice of $r > 0$, $|t^{\lambda-1}(-\log t)^{\mu}|$ is bounded on $|t - 1| = r$, and $|\exp(-zt)| = |\exp[-z(1 + re^{i\theta})]| = O(\exp(-\epsilon|z|))$, as $|z| \to \infty$ for some choice of $\epsilon > 0$. This will imply the integral on the semi-circular path is $O(\exp(-\epsilon|z|))$ as $|z| \to \infty$, for some choice of $\epsilon > 0$. This result follows readily for the last integral of (3.37) so that

(3.39)
$$L(z) = L(0, c, \lambda, \mu, z) + O(\exp(-\epsilon|z|)),$$

as $z \to \infty$ in $|\arg(z)| \leqq \pi/2 - \Delta$, for some choice of $\epsilon > 0$. This of course implies

(3.40)
$$L(z) \sim z^{-\lambda}(\log z)^{\mu}[S_{\infty}(\lambda, \mu, \log z); \{(\log z)^{-n}\}],$$

as $z \to \infty$ in $|\arg(z)| \leqq \pi/2 - \Delta$, providing $\operatorname{Re} \lambda > 0$.

This same result holds for the function $M(z)$ defined by a path of integration in which the indentation is into the fourth quadrant, and the point joining $t = 1$ to $t = \infty$ is on the lower side of the cut. Hence (3.35) is valid when $\gamma = 0$ for both of the functions one would obtain from the two paths of integration which we have described. When $\operatorname{Re} \mu > -1$, it is not necessary to provide the indentation, and straight line paths, above and below the cut, can be used to join $t = 0$ to $t = \infty$.

With the clarification of paths of integration provided in our proof, (3.35) will be valid without placing the restriction $\gamma \neq 0$.

4. Main theorems. Turning attention to the more general case where

(4.1)
$$F(z) = \int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt,$$

it will be assumed that $F(z)$ exists for some $z = z_0$.

THEOREM 4.1. *If:*

(i) *for each integer* $N \in I$

(4.2)
$$f(t) = \sum_{n=0}^{N} a_n t^{\lambda_n - 1}P_n(\log t) + o(t^{\lambda_N - 1}(\log t)^{m(N)}),$$

as $t \to 0$ *along* $\arg t = \gamma$;

(ii) $P_n(\omega)$ *is a polynomial of degree* $m = m(n)$;

(iii) $\{\lambda_n\}$ *is a sequence of complex numbers, with* $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$, $\operatorname{Re} \lambda_0 > 0$, *for all* n *such that* n *and* $n + 1$ *are both in* I;

(iv) $\{a_n\}$ *is a sequence of complex numbers;*

then as $z \to \infty$ *in* $S(\Delta)$

(4.3)
$$F(z) \sim \sum_{n \in I} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}]; \{z^{-\lambda_n}(\log z)^{m(n)}\},$$

where D_n is the operator

$$D_n = \frac{d}{d\lambda_n}.$$

The result is uniform in the approach of $z \to \infty$ in $S(\Delta)$.

Proof. Trivially, $\{z^{-\lambda_n}(\log z)^{m(n)}\}$ is an asymptotic sequence as $|z| \to \infty$ in $S(\Delta)$. For any choice of $t = c \neq 0$ on the path of integration, (3.7) implies

$$I(z, c) = \int_c^{\infty e^{i\gamma}} f(t)e^{-zt}dt \tag{4.4}$$

$$= O(\exp(-\delta|z|)) \approx 0; \{z^{-\lambda_n}(\log z)^{m(n)}\},$$

uniformly as $z \to \infty$ in $S(\Delta)$, where δ is some positive number.

Writing

$$f(t) = \sum_{n=0}^{N} a_n t^{\lambda_n - 1} P_n(\log t) + R_N, \tag{4.5}$$

gives

$$\int_0^c f(t)e^{-zt}dt = \sum_{n=0}^{N} a_n \int_0^c t^{\lambda_n-1}P_n(\log t)e^{-zt}dt + r_N, \tag{4.6}$$

where

$$r_N = \int_0^c R_N e^{-zt}dt. \tag{4.7}$$

Using (3.16) and (4.4) gives

$$\int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt = \sum_{n=0}^{N} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}] + r_N + O(\exp(-\delta|z|)), \tag{4.8}$$

for some $\delta > 0$, as $z \to \infty$ in $S(\Delta)$.

Since the choice of $t = c$ is arbitrary, it may be chosen sufficiently small so that

$$r_N = \int_0^c R_N e^{-zt}dt \tag{4.9}$$

satisfies

$$|r_N| \leqq \epsilon \int_0^c |t^{\lambda_N-1}(\log t)^{m(N)}e^{-zt}dt|. \tag{4.10}$$

Replacing zt by t gives

$$|r_N| \leqq \epsilon|(\log z)^{m(N)} \cdot z^{-\lambda_N}| \int_0^{\infty e^{i\tau}} \left| t^{\lambda_N-1}\left(1 - \frac{\log t}{\log z}\right)^{m(N)} e^{-t}dt \right|, \tag{4.11}$$

where $|\tau| \leqq \pi/2 - \Delta$. The existence of the integral, uniformly bounded in z as $z \to \infty$ in $S(\Delta)$, implies

$$r_N = o(z^{-\lambda_N}(\log z)^{m(N)}), \tag{4.12}$$

uniformly, as $z \to \infty$ in $S(\Delta)$.

These results give

$$(4.13) \qquad F(z) = \int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt$$

$$= \sum_{n=0}^{N} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}] + o(z^{-\lambda_N}(\log z)^{m(N)}),$$

uniformly, as $z \to \infty$ in $S(\Delta)$.

When $m(n) = 0$ for all $n \in I$, then $P_n(D_n)$ is a constant, and the result of Theorem 4.1 reduces to the result contained in the general form of Watson's Lemma given in § 1.

Furthermore, if

$$(4.14) \qquad f(t) \sim \sum a_n t^{\lambda_n - 1}(\log t),$$

as $t \to 0$, Theorem 4.1 gives

$$(4.15) \qquad \int_0^\infty f(t)e^{-zt}dt \sim \sum a_n \frac{d}{d\lambda_n}[\Gamma(\lambda_n)z^{-\lambda_n}]; \{z^{-\lambda_n}(\log z)\}$$

$$\sim \sum a_n\Gamma(\lambda_n)[\psi(\lambda_n) - \log z]z^{-\lambda_n}; \{z^{-\lambda_n}(\log z)\},$$

where $\psi(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda)$. This particular result is given by D. S. Jones in [**6**, p. 439].

If the integral

$$(4.16) \qquad L(0, \infty e^{i\gamma}, \lambda, \mu, z) = \int_0^{\infty e^{i\gamma}} t^{\lambda-1}(-\log t)^{\mu}e^{-zt}dt$$

is examined, it is reasonable to expect that the result contained in (3.24) can be used to advantage to discuss the asymptotic behavior of integrals of the form (4.1) with $f(t)$ now allowed to have logarithmic type singularities, as well as singularities of branch-point type. In several respects, the asymptotic behavior is quite different from what one might expect. Before proceeding, one further result is required.

LEMMA 4.1. *It is always possible to choose a complex number* $c = |c|e^{i\gamma} \neq 0$ *such that*

$$(4.17) \qquad I = \int_0^c |t^{\lambda-1}(-\log t)^{\mu}e^{-z\mu}dt| = O(z^{-\lambda}(\log z)^{\mu}),$$

as $z \to \infty$ *in* $S(\Delta)$, *where* λ, μ *are fixed complex numbers, with* $\operatorname{Re} \lambda > 0$. *The path of integration is the straight line* $\arg t = \gamma$ *joining* $t = 0$ *to* $t = c$.

Proof. Along the path of integration $t = \rho e^{i\gamma}$, $0 \leqq \rho \leqq |c|$. Hence

$$(4.18) \qquad |I| \leqq K \int_0^{|c|} \rho^{\alpha-1}|(-\log \rho - i\gamma)^{\mu}| \exp(-|z|(\sin \Delta)\rho)d\rho,$$

where $\alpha = \operatorname{Re} \lambda$. As $|c| \to 0$, $-\log \rho \to \infty$. This implies, by taking $|c|$ sufficiently small, that

$$(4.19) \qquad |I| \leqq K \int_0^{|c|} \rho^{\alpha-1}(-\log \rho)^{\beta} \exp(-|z|(\sin \Delta)\rho)d\rho$$

$$\leqq KL(0, |c|, \alpha, \beta, |z|\sin \Delta),$$

where $\beta = \operatorname{Re} \mu$. The constant K is being used as a generic symbol whose value may change from time to time in the proof. The use of (3.24) will give

$$(4.20) \qquad \begin{aligned} I &= O((|z|\sin \Delta)^{-\alpha}(\log |z|\sin \Delta)^{\beta}), \text{ as } |z| \to \infty \\ &= O(|z|^{-\alpha}(\log|z|)^{\beta}), \text{ as } |z| \to \infty \\ &= O(z^{-\alpha}(\log z)^{\beta}) = O(z^{-\lambda}(\log z)^{\mu}), \text{ as } z \to \infty \text{ in } S(\Delta). \end{aligned}$$

The result is uniform in $\arg z$ as $z \to \infty$ in $S(\Delta)$.

THEOREM 4.2. *If:*
(i) $F(z)$, *as given in* (4.1), *exists for some* $z = z_0$;
(ii) $\{t^{\lambda_n - 1}(-\log t)^{\mu_n}\}$ *is an asymptotic sequence as* $t \to 0$ *along* $\arg t = \gamma$, *where* $\{\lambda_n\}$, $\{\mu_n\}$, $n \in I$, *are both sequences of complex numbers, with* $\operatorname{Re} \lambda_0 > 0$;
(iii) $f(t) \sim \sum_{n \in I} a_n t^{\lambda_n - 1}(-\log t)^{\mu_n}$; $\{t^{\lambda_n - 1}(-\log t)^{\mu_n}\}$
as $t \to 0$ *along* $\arg t = \gamma$;

then

$$(4.21) \qquad F(z) \sim \sum_{n \in I} a_n L(0, c, \lambda_n, \mu_n, z); \{z^{-\lambda_n}(\log z)^{\mu_n}\},$$

as $z \to \infty$ *in* $S(\Delta)$, *where* $t = c$ *is some point on the path of integration with* $0 < |c| < 1$.

Proof. For any choice of $c = |c|e^{i\gamma} \neq 0$, no matter how small,

$$(4.22) \qquad F(z) = \int_0^c f(t)e^{-zt}dt + \int_c^{\infty e^{i\gamma}} f(t)e^{-zt}dt.$$

From (4.4),

$$(4.23) \qquad F(z) = \int_0^c f(t)e^{-zt}dt + O(\exp(-\delta|z|)),$$

for some fixed $\delta > 0$, as $z \to \infty$ in $S(\Delta)$.

For any fixed $N \in I$,

$$(4.24) \qquad f(t) = \sum_{n=0}^{N} a_n t^{\lambda_n - 1}(-\log t)^{\mu_n} + R_N,$$

where for any given $\epsilon > 0$, there will exist a complex number $c \neq 0$ such that

$$(4.25) \qquad |R_N| \leqq \epsilon|t^{\lambda_N - 1}(-\log t)^{\mu_N}|, \arg t = \gamma, |t| \leqq |c|.$$

Hence

$$(4.26)\qquad F(z) = \sum_{n=0}^{N} a_n L(0, c, \lambda_n, \mu_n, z) + r_N + O(\exp(-\delta|z|)),$$

as $z \to \infty$ in $S(\Delta)$, where

$$(4.27)\qquad |r_N| \leqq \epsilon \int_0^c |t^{\lambda_N - 1}(-\log t)^{\mu_N} e^{-zt} dt| \leqq K \cdot \epsilon \cdot |z^{-\lambda_N}(\log z)^{\mu_N}|,$$

by Lemma 4.1.

From these results,

$$(4.28)\qquad F(z) = \sum_{n=0}^{N} a_n L(0, c, \lambda_n, \mu_n, z) + o(z^{-\lambda_N}(\log z)^{\mu_N},$$

as $z \to \infty$ in $S(\Delta)$, and therefore

$$(4.29)\qquad F(z) \sim \sum_{n \in I} a_n L(0, c, \lambda_n, \mu_n, z); \{z^{-\lambda_n}(\log z)^{\mu_n}\},$$

as $z \to \infty$ in $S(\Delta)$.

The Erdélyi result, namely Theorem 4.2, may not be as useful in some circumstances as one might expect. To illustrate, consider Theorem 4.2 with $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$ whenever n and $n + 1$ are both in I. It will also be assumed $a_0 = a_1 = 1$ in (4.21). From Theorem 3.1, it follows that

$$(4.30)\qquad L(0, c, \lambda_0, \mu_0, z) \sim z^{-\lambda_0}(\log z)^{\mu_0}\left[\sum_{n=0}^{\infty} (-1)^n \binom{\mu_0}{n} \Gamma^{(n)}(\lambda_0)(\log z)^{-n}; \{(\log z)^{-n}\}\right],$$

and

$$(4.31)\qquad L(0, c, \lambda_1, \mu_1, z) \sim z^{-\lambda_1}(\log z)^{\mu_1}\left[\sum_{n=0}^{\infty} (-1)^n \binom{\mu_1}{n} \Gamma^{(n)}(\lambda_1)(\log z)^{-n}; \{(\log z)^{-n}\}\right],$$

as $z \to \infty$ in $S(\Delta)$. Hence for any fixed integers $N_0 > 0$ and $N_1 > 0$,

$$(4.32)\qquad F(z) = z^{-\lambda_0}(\log z)^{\mu_0}\left[\sum_{n=0}^{N_0} (-1)^n \binom{\mu_0}{n} \Gamma^{(n)}(\lambda_0)(\log z)^{-n} + o((\log z)^{-N_0})\right] + z^{-\lambda_1}(\log z)^{\mu_1}\left[\sum_{n=0}^{N_0} (-1)^n \binom{\mu_1}{n} \Gamma^{(n)}(\lambda_1)(\log z)^{-n} + o((\log z)^{-N_1})\right] + o(z^{-\lambda_1}(\log z)^{\mu_1}),$$

as $z \to \infty$ in $S(\Delta)$. If the term $z^{-\lambda_0}(\log z)^{\mu_0}$ is factored from the expressions in (4.32), one obtains

$$(4.33) \quad F(z) = z^{-\lambda_0}(\log z)^{\mu_0}\left[\sum_{n=0}^{N_0} (-1)^n \binom{\mu_0}{n} \Gamma^{(n)}(\lambda_0)(\log z)^{-n} + o((\log z)^{-N_0}) + O(z^{\lambda_0-\lambda_1}(\log z)^{\mu_1-\mu_0})\right],$$

as $z \to \infty$ in $S(\Delta)$. Since $\operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_0$,

$$(4.34) \qquad z^{\lambda_0-\lambda_1}(\log z)^{\mu_1-\mu_0} = o((\log z)^{-N_0}),$$

as $z \to \infty$ in $S(\Delta)$. Hence

$$(4.35) \quad F(z) = z^{-\lambda_0}(\log z)^{\mu_0}\left[\sum_{n=0}^{N_0} (-1)^n \binom{\mu_0}{n} \Gamma^{(n)}(\lambda_0)(\log z)^{-n} + o((\log z)^{-N_0})\right]$$

or

$$(4.36) \qquad F(z) \sim z^{-\lambda_0}(\log z)^{\mu_0}[S_\infty(\lambda_0, \mu_0, \log z); \{(\log z)^{-n}\}]$$

as $z \to \infty$ in $S(\Delta)$.

If μ_0 is not a non-negative integer, then $S_\infty(\lambda_0, \mu_0, \log z)$ has an infinite number of terms, each of which is larger than every term in

$$z^{-\lambda_1}(\log z)^{\mu_1} S_\infty(\lambda_1, \mu_1, \log z).$$

From a pragmatic point of view the first term of (iii) gives the complete asymptotic expansion of $F(z)$. This is a somewhat surprising result because the situation when the μ_n are non-negative integers is quite different. In such a case, every term of (iii) gives a contribution to a much more accurate form of asymptotic expansion.

Thus, if the Erdélyi form of (4.21) is reduced to its natural pragmatic form of (4.36), it is clear that the result is no longer a generalization of Watson's Lemma except insofar as the first term is concerned. It is therefore natural to ask whether such a generalization does exist in which the powers of $(-\log t)$ can take values which are not non-negative integers. It is possible, for example, to ask for conditions on $f(t)$ in

$$(4.37) \qquad F(z) = \int_0^{\infty e^{i\gamma}} f(t) e^{-zt} dt,$$

which will ensure that

$$(4.38) \qquad F(z) \sim \sum a_n z^{-\lambda_n}(\log z)^{\mu_n}; \{z^{-\lambda_n}(\log z)^{\mu_n}\},$$

as $z \to \infty$ in $S(\Delta)$, where $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$, whenever n and $n + 1$ are in I, and μ_n is an arbitrary complex number. This would then be a generalization of Watson's Lemma, with the result of this lemma being obtained when $\mu_n = 0$.

In order to answer this question, we shall digress to discuss briefly the function $\mu(t, \beta, \alpha)$ defined by

$$(4.39)\qquad \mu(t, \beta, \alpha) = \int_0^\infty \frac{t^{\alpha+x}x^\beta}{\Gamma(\beta+1)\Gamma(\alpha+x+1)}\,dx.$$

This function has the asymptotic expansion (see [**4**] or [**9**])

$$(4.40)\qquad \mu(t, \beta, \alpha) \sim t^\alpha(-\log t)^{-\beta-1} \times\left[\sum_{n=0}^{\infty}(-1)^n\frac{(\beta+1)_n}{n!}\mu(1, -n-1, \alpha)(-\log t)^{-n}; \{(-\log t)^{-n}\}\right],$$

as $t \to 0$. Further,

$$(4.41)\qquad \mu(1, -n-1, \alpha) = (-1)^n\frac{d^n}{d\alpha^n}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right]\Bigg|_{t=1}.$$

This gives

$$(4.42)\qquad \mu(t, \beta, \alpha) = \frac{t^\alpha(-\log t)^{-\beta-1}}{\Gamma(\alpha+1)}\left[1+O\left(\frac{1}{\log t}\right)\right],$$

as $t \to 0$. Hence

$$(4.43)\qquad \mu(t, -\mu_n-1, \lambda_n-1) = \frac{t^{\lambda_n-1}(-\log t)^{\mu_n}}{\Gamma(\lambda_n)}\left[1+O\left(\frac{1}{\log t}\right)\right],$$

as $t \to 0$. The sequences $\{\mu(t, -\mu_n-1, \lambda_n-1)\}$ and $\{t^{\lambda_n-1}(-\log t)^{\mu_n}\}$ as $t \to 0$ along $\arg t = \gamma$ and $\{z^{-\lambda_n}(\log z)^{\mu_n}\}$ as $z \to \infty$ are all asymptotic sequences for the same conditions on $\{\lambda_n\}$ and $\{\mu_n\}$. It is assumed that such conditions are met, and all three are asymptotic sequences. In such a situation one can study functions $f(t)$ which have asymptotic expansions of the form

$$(4.44)\qquad f(t) \sim \sum_{n\in I} a_n t^{\lambda_n-1}(-\log t)^{\mu_n}; \{t^{\lambda_n-1}(-\log t)^{\mu_n}\},$$

as $t \to 0$, which has already been accomplished, or one can discuss the possibility that

$$(4.45)\qquad f(t) \sim \sum_{n\in I} a_n\mu(t, -\mu_n-1, \lambda_n-1); \{\mu(t, -\mu_n-1, \lambda_n-1)\},$$

as $t \to 0$. If (4.45) exists then the constants a_n are given by

$$(4.46)\qquad \begin{aligned} a_0 &= \lim_{t\to 0} f(t)/\mu(t, -\mu_0-1, \lambda_0-1) \\ &= \lim_{t\to 0} f(t)\Gamma(\lambda_0)/t^{\lambda_0-1}(-\log t)^{\mu_0}, \end{aligned}$$

$$\begin{aligned} a_K &= \lim_{t\to 0}\left[f(t) - \sum_{n=0}^{k-1} a_n\mu(t, -\mu_n-1, \lambda_n-1)\right]\Big/\mu(t, -\mu_k-1, \lambda_k-1) \\ &= \lim_{t\to 0}\Gamma(\lambda_k)\left[f(t) - \sum_{n=0}^{k-1} a_n\mu(t, -\mu_n-1, \lambda_n-1)\right]\Big/t^{\lambda_k-1}(-\log t)^{\mu_k}. \end{aligned}$$

Although these formulae give an explicit determination of these constants, the formulae are not useful in specific determinations. The problem of determining conditions on $f(t)$ for which (4.45) is valid is interesting, but is not a problem which will be considered in this paper.

THEOREM 4.3 *If:*

(i) $F(z) = \int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt,$

exists for some fixed $z = z_0$;

(ii) $\{t^{\lambda_n - 1}(-\log t)^{\mu_n}\}$ *is an asymptotic sequence as* $t \to 0$ *along* $\arg t = \gamma$ *with* $\operatorname{Re} \lambda_0 > 0$;

(iii) $f(t) \sim \sum_{n \in I} a_n \mu(t, -\mu_n - 1, \lambda_n - 1); \{t^{\lambda_n - 1}(-\log t)^{\mu_n}\},$

as $t \to 0$ *along* $\arg t = \gamma$;

then

$$F(z) \sim \sum_{n \in I} a_n z^{-\lambda_n}(\log z)^{\mu_n}; \{z^{-\lambda_n}(\log z)^{\mu_n}\}, \tag{4.47}$$

as $z \to \infty$ *in* $S(\Delta)$. *The result holds uniformly in the approach of* $z \to \infty$.

Proof. For any choice of $c = |c|e^{i\gamma} \neq 0$, it is true that

$$\begin{aligned} F(z) &= \int_0^c f(t)e^{-zt}dt + \int_c^{\infty e^{i\gamma}} f(t)e^{-zt}dt \\ &= \int_0^c f(t)e^{-zt}dt + O(\exp(-\delta|z|)) \end{aligned} \tag{4.48}$$

for some fixed $\delta > 0$, as $z \to \infty$ in $S(\Delta)$. From (iii), it follows that for any fixed integer $N \in I$,

$$f(t) = \sum_{n=0}^{N} a_n \mu(t, -\mu_n - 1, \lambda_n - 1) + R_N, \tag{4.49}$$

where for every $\epsilon > 0$, there will exist a number $|c|$ such that

$$|R_N| \leqq \epsilon |t^{\lambda_N - 1}(-\log t)^{\mu_N}|, \tag{4.50}$$

providing $|t| \leqq |c|$. Since the choice in (4.48) is arbitrary, there is no loss of generality in identifying $|c|$ in (4.48) with that in (4.50), and assuming $|c|$ is small if so desired. These results give

$$F(z) = \sum_{n=0}^{N} a_n \int_0^c \mu(t, -\mu_n - 1, \lambda_n - 1)e^{-zt}dt + r_N + O(\exp(-\delta|z|)), \tag{4.51}$$

as $z \to \infty$ in $S(\Delta)$, where

$$|r_N| \leqq \epsilon \int_0^c |t^{\lambda_N - 1}(-\log t)^{\mu_N} e^{-zt}dt|, \tag{4.52}$$

which as before, (4.27), means

$$r_N = o(z^{-\lambda_N}(\log z)^{\mu_N}), \tag{4.53}$$

as $z \to \infty$ in $S(\Delta)$.

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Further,

$$(4.54)\qquad \int_0^c \mu(t, -\mu_n - 1, \lambda_n - 1)e^{-zt}dt = \int_0^{\infty e^{i\gamma}} \mu(t, -\mu_n - 1, \lambda_n - 1)e^{-zt}dt + O(\exp(-\delta|z|)),$$

for some fixed $\delta > 0$, as $z \to \infty$ in $S(\Delta)$. Hence the well-known result [4, p. 222],

$$(4.55)\qquad \int_0^{\infty e^{i\gamma}} \mu(t, -\mu_n - 1, \lambda_n - 1)e^{-zt}dt = z^{-\lambda_n}(\log z)^{\mu_n},$$

coupled with the results given above yields

$$(4.56)\qquad F(z) = \sum_{n=0}^{N} a_n z^{-\lambda_n}(\log z)^{\mu_n} + o(z^{-\lambda_N}(\log z)^{\mu_N}),$$

as $z \to \infty$ in $S(\Delta)$, and the order relation is independent of z. This of course proves that

$$(4.57)\qquad F(z) \sim \sum_{n\in I} a_n z^{-\lambda_n}(\log z)^{\mu_n}; \{z^{-\lambda_n}(\log z)^{\mu_n}\},$$

uniformly, as $z \to \infty$ in $S(\Delta)$.

Since

$$(4.58)\qquad \mu(t, -m, \alpha) = (-1)^{m-1}\frac{d^{m-1}}{d\alpha^{m-1}}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right),$$

and

$$(4.59)\qquad \mu(t, -1, \lambda_n - 1) = \frac{t^{\lambda_n - 1}}{\Gamma(\lambda_n)},$$

this result shows that $\mu_n = 0$ will yield Watson's Lemma, and Theorem 4.3 is a true generalization of this latter result. In Theorem 4.3, μ_n is an arbitrary fixed complex number.

Returning to Theorem 4.2, the conditions under which

$$(4.60)\qquad \operatorname{Re}\lambda_n = \operatorname{Re}\lambda_{n+1},\ \operatorname{Re}\mu_{n+1} < \operatorname{Re}\mu_n$$

whenever n and $n + 1$ are members of I, leads to a result that one might reasonably expect to hold. In this instance,

$$\begin{aligned}(4.61)\quad F(z) = a_0 z^{-\lambda_0}\Big[&\Gamma(\lambda_0)(\log z)^{\mu_0} - \binom{\mu_0}{1}\Gamma^{(1)}(\lambda_0)(\log z)^{\mu_0-1} + \ldots \\ &+ (-1)^{r_0}\binom{\mu_0}{r_0}\Gamma^{(r_0)}(\lambda_0)(\log z)^{\mu_0-r_0} + 0((\log z)^{\mu_0-r_0})\Big] \\ + a_1 z^{-\lambda_1}\Big[&\Gamma(\lambda_1)(\log z)^{\mu_1} - \binom{\mu_1}{1}\Gamma^{(1)}(\lambda_1)(\log z)^{\mu_1-1} + \ldots \\ &+ (-1)^{r_1}\binom{\mu_1}{r_1}\Gamma^{(r_1)}(\lambda_1)(\log z)^{\mu_1-r_1} + 0((\log z)^{\mu_1-r_1})\Big] + \ldots \\ + a_M z^{-\lambda_M}\Big[&\Gamma(\lambda_M)(\log z)^{\mu_M} - \binom{\mu_M}{1}\Gamma^{(1)}(\lambda_M)(\log z)^{\mu_M-1} + \ldots \\ &+ (-1)^{r_M}\binom{\mu_M}{r_M}\Gamma^{(r_M)}(\lambda_M)(\log z)^{\mu_M-r_M} + 0((\log z)^{\mu_M-r_M})\Big] + \ldots .\end{aligned}$$

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Under the stated conditions, the ratio $z^{-\lambda_n}/z^{-\lambda_0}$ is bounded as $z \to \infty$. In (4.61), this will mean that none of the terms in (4.61) becomes ≈ 0 with respect to an asymptotic sequence composed of powers of $(\log z)^{-1}$. It is possible to regroup the terms of (4.61) so that $F(z)$ will exhibit an asymptotic expansion of the form

$$F(z) \sim z^{-\lambda_0}[\textstyle\sum b_n(z)(\log z)^{-\tau_n};\ \{(\log z)^{-\tau_n}\}], \tag{4.62}$$

uniformly, as $z \to \infty$ in $S(\Delta)$, where the $b_n(z)$ are all bounded as $z \to \infty$ in $S(\Delta)$, and the sequence of fixed complex numbers $\{\tau_n\}$ satisfies $\tau_0 = -\mu_0$, $\operatorname{Re} \tau_{n+1} > \operatorname{Re} \tau_n$. Although the explicit expression of $b_n(z)$ can be obtained, it is of such complexity that it is hardly worthwhile stating the formulae involved. In the situation just described, every term of the expansion of $f(t)$ contributes to the asymptotic expansion of $F(z)$. This is merely in keeping with what might be expected from an examination of the conditions and result given in Watson's Lemma.

As an illustration of this form of asymptotic expansion, the function $F(z)$ defined by

$$F(z) = \int_0^1 \frac{t^{\lambda-1}\exp(-zt)}{(1-\log t)^{\mu}}\,dt, \quad \operatorname{Re} \lambda > 0 \tag{4.63}$$

will be considered. The general procedures of the present paper may be used in two different ways. The function $(1 - \log t)^{-\mu}$ has the convergent expansion

$$(1-\log t)^{-\mu} = \sum_{n=0}^{\infty} \binom{-\mu}{n} (-\log t)^{-\mu-n}, \tag{4.64}$$

providing $|t| < e^{-1}$. The conditions of Theorem 4.2 are then trivially satisfied and

$$F(z) \sim \sum_{n=0}^{\infty} \binom{-\mu}{n} L(0, 1, \lambda, -\mu - n, z);\ \{z^{-\lambda}(\log z)^{-\mu-n}\}, \tag{4.65}$$

uniformly, as $z \to \infty$ in $S(\Delta)$. As before

$$L(0,1,\lambda,-\mu-n,z) \sim z^{-\lambda}\left[\sum_{m=0}^{\infty} (-1)^m \binom{-\mu-n}{m} \Gamma^{(m)}(\lambda)(\log z)^{-\mu-n-m};\ \{(\log z)^{-\mu-n-m}\}\right], \tag{4.66}$$

uniformly, as $z \to \infty$ in $S(\Delta)$. Regrouping terms will then give

$$F(z) \sim z^{-\lambda}(\log z)^{-\mu}\left[\sum_{n=0}^{\infty} a_n(\log z)^{-n};\ \{(\log z)^{-n}\}\right], \tag{4.67}$$

uniformly, as $z \to \infty$ in $S(\Delta)$, where

$$a_n = \sum_{m=0}^{n} (-1)^m \binom{-\mu}{n-m}\binom{-\mu-n+m}{m}\Gamma^{(m)}(\lambda). \tag{4.68}$$

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There does however exist a much simpler asymptotic form of expansion of $F(z)$. In (4.63), replace t by et. Hence

$$F(z) = e^{\lambda} \int_0^{e^{-1}} \frac{t^{\lambda-1} \exp[-(ze)t]}{(-\log t)^{\mu}} dt. \tag{4.69}$$

Directly, one therefore has

$$F(z) \sim z^{-\lambda}(\log(ze))^{-\mu}\left[\sum_{n=0}^{\infty} (-1)^n \binom{-\mu}{n} \Gamma^{(n)}(\lambda)(\log(ze))^{-n}; \{(\log(ze))^{-n}\}\right], \tag{4.70}$$

uniformly, as $z \to \infty$ in $S(\Delta)$, or

$$F(z) \sim z^{-\lambda}(1 + \log z)^{-\mu}\left[\sum_{n=0}^{\infty} (-1)^n \binom{-\mu}{n} \Gamma^{(n)}(\lambda)(1 + \log z)^{-n}; \{(\log z)^{-n}\}\right], \tag{4.71}$$

uniformly, as $z \to \infty$ in $S(\Delta)$. Because of the Poincaré nature of these expansions, (4.67) can be obtained from (4.71) by writing

$$(1 + \log z)^{-\mu-n} = (\log z)^{-\mu-n}\left[1 + \frac{1}{\log z}\right]^{-\mu-n},$$

and then expanding

$$\left[1 + \frac{1}{\log z}\right]^{-\mu-n}$$

in powers of $(\log z)^{-1}$.

5. A concluding remark. Although quite general theorems have been established by means of which Watson's Lemma has been generalized, no pretence is made that necessary and sufficient conditions for the validity of the results have been found. This point is stressed to emphasize the fact that it is the pattern, not the detail, of proof which is important. The pattern of proof need not be abandoned in a specific example just because one or more of the conditions of validity of a particular theorem does not happen to be true. As an illustration of this remark, we consider

$$F(z) = \int_0^c t^{\lambda-1} \log\left(\log \frac{1}{t}\right) e^{-zt} dt, \tag{5.1}$$

where it is assumed that $\operatorname{Re} \lambda > 0$ and $0 < c < 1$. Although the singularity of the integrand is not of the type considered in this paper, the pattern of procedure outlined in § 3 will yield the asymptotic behavior of $F(z)$ as $z \to \infty$. For the sake of simplicity we restrict ourselves to real z.

Consider

$$f(t) = \log \log \frac{1}{t}. \tag{5.2}$$

Clearly,

$$f(uz^{-1}) = \log(\log z - \log u) = \log\log z + \log\left(1 - \frac{\log u}{\log z}\right). \tag{5.3}$$

For every fixed positive integer $N \geqq 0$, we have

$$\log\left(1 - \frac{\log u}{\log z}\right) = -\sum_{n=1}^{N} \frac{1}{n} \frac{(\log u)^n}{(\log z)^n} + O\left(\frac{(\log u)^{N+1}}{(\log z)^{N+1}}\right), \tag{5.4}$$

as $z \to \infty$, providing

$$\left|\frac{\log u}{\log z}\right| \leqq 1 - \delta, \tag{5.5}$$

or equivalently,

$$\frac{1}{z^{1-\delta}} \leqq u \leqq z^{1-\delta} \tag{5.6}$$

for some fixed $\delta > 0$. The substitution $u = tz$ gives

$$a \leqq t \leqq b, \tag{5.7}$$

where $a = z^{-2+\delta}$ and $b = z^{-\delta}$. Therefore the use of the approximation (5.4) must exclude $t = 0$.

One now proves exactly as in Lemma 3.1 that a fixed $\rho > 0$ must exist such that

$$\int_0^a t^{\lambda-1} f(t) e^{-zt} dt = O(z^{-\lambda-\rho}), \tag{5.8}$$

as $z \to \infty$. Further, the proof of

$$\int_b^c t^{\lambda-1} f(t) e^{-zt} dt = O(\exp(-z^{1-\delta})), \tag{5.9}$$

as $z \to \infty$, is easily obtained. These two results coupled together give

$$F(z) = \int_a^b t^{\lambda-1} f(t) e^{-zt} dt + O(z^{-\lambda-\rho}), \tag{5.10}$$

as $z \to \infty$. From here on the pattern of procedure follows closely that of Theorem 3.1. The result obtained is

$$z^\lambda F(z) = (\log\log z)\{\Gamma(\lambda) + O(z^{-\eta})\} + \left[\sum_{n=1}^{N} \left(-\frac{1}{n}\right)(\log z)^{-n}\Gamma^{(n)}(\lambda) + 0((\log z)^{-N-1})\right], \tag{5.11}$$

as $z \to \infty$, for some fixed $\eta > 0$, which will of course imply

$$\{z^\lambda F(z) - \Gamma(\lambda)\log\log z\} \sim \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)(\log z)^{-n}\Gamma^{(n)}(\lambda); \{(\log z)^{-n}\}, \tag{5.12}$$

as $z \to \infty$.

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