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Asymptotic behaviour of iterated piecewise monotone maps

JÜRGEN WILLMS
Universität Bielefeld, Fakultät für Mathematik, 4800 *Bielefeld* 1, *West Germany*

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Abstract. In this paper the asymptotic behaviour of piecewise monotone functions $f: I \to I$ with a finite number of discontinuities is studied (where $I \subseteq \mathbb{R}$ is a compact interval). It is shown that there is a finite number of f-almost-invariant subsets $C_1, \ldots, C_r, R_1, \ldots, R_s$, where each C_i is a disjoint union of closed intervals and each R_j is a Cantor-like subset of I, such that if x is a 'typical' point in I (in a topological sense) then exactly one of the following three possibilities will happen:

(1) $\{f^n(x)\}_{n\geq 0}$ eventually ends up in some C_i.

(2) $\{f^n(x)\}_{n\geq 0}$ is attracted to some R_j.

(3) $\{f^n(x): n \geq 0\}$ is contained in an open, invariant set $Z \subseteq I$, which is such that for each $n \geq 1$ f^n is monotone and continuous on each connected component of Z.

Moreover, f acts topologically transitively on each C_i and minimally on each R_j. Furthermore, it is shown how the sets $C_1, \ldots, C_r, R_1, \ldots, R_s$ can be constructed. Finally, our results are applied to some examples.

1. *Introduction*

In the last few years there has been considerable interest in the qualitative behaviour of iterates of maps on an interval into itself (see, for instance, [**2**], [**6**], [**9**], [**11**], [**12**], [**14**]). Although they are the simplest examples of non-linear (discrete) dynamical systems their asymptotic behaviour can exhibit a surprisingly complex structure. One-dimensional maps have been used as models for various systems (see, for instance, the models of density dependent population growth studied in [10]). It is known that in certain cases the asymptotic behaviour of higher-dimensional (discrete as well as continuous) dynamical systems can be, at least partly, described by iterates of maps on an interval into itself. For instance, in [**19**] the Lorenz attractor is described as the inverse limit of a semi-flow on a two-dimensional branched manifold. The Poincaré map of this semi-flow is a function on a bounded interval $[a, b]$ into itself which has a single discontinuity at $c = (a+b)/2$ and is strictly increasing on $[a, c)$ and $(c, b]$. Interval exchange transformations, Newton's method for determining the zeros of a polynomial (identifying $\bar{\mathbb{R}}$ with the unit interval) and the β-transformations discussed in [**17**] are some more examples for discrete dynamical systems on a compact interval, having a finite number of discontinuities.

In [**15**] Preston studied the asymptotic behaviour of iterates of piecewise monotone continuous functions on a compact interval I into itself, i.e. continuous functions

$f: I \to I$ with only a finite number of points at which f is not strictly monotone (an improved and much simplified version of [**15**] is contained in §§ 2, 3 and 4 of [**16**]). The present work generalizes his main result in that a finite number of discontinuities is allowed. More exactly, let $a, b \in \mathbb{R}$ with $a < b$ and let I be the closed interval with endpoints a and b. For technical reasons we consider a point ω not contained in $\mathbb{R}$ and put $I' = I \cup \{\omega\}$. We call a map $f: I' \to I'$ *piecewise monotone* on I if there exist $m \geq 1$ and $a = d_0 < d_1 < \cdots < d_m = b$ such that $f^{-1}(\{\omega\}) = \{d_0, d_1, \ldots, d_m, \omega\}$ and, for all $0 \leq k \leq m-1$, f is continuous and strictly monotone on each of the intervals (d_k, d_{k+1}). $\mathcal{N}(I)$ will denote the set of piecewise monotone maps on I. For $f \in \mathcal{N}(I)$ put $S(f) = f^{-1}(\{\omega\}) \cap I$. Now let $f \in \mathcal{N}(I)$ be fixed. We define $f^n \in \mathcal{N}(I)$, the *nth. iterate* of f, inductively by $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $x \in I'$ and $n \geq 0$. Unlike in [7] where the structure of the non-wandering set is studied (using symbolic dynamics) our aim is to analyse the asymptotic behaviour of $\{f^n(x)\}_{n\geq 0}$ for a 'typical' point x of I (in a topological sense).

After some preliminaries in § 2 we study in § 3 some basic properties of sinks and homtervals of f. § 4 is concerned with some elementary properties of topologically transitive f-cycles and f-register-shifts. The main result is stated and proved in § 5. It says that there are only a finite number of topologically transitive f-cycles $C_1, \ldots, C_r$ and f-register-shifts $R_1, \ldots, R_s$ and that for all points x lying in some residual subset of I exactly one of the following three things will happen:

(1) $\{f^n(x)\}_{n\geq 0}$ eventually ends up in some topologically transitive f-cycle C_i.

(2) $\{f^n(x)\}_{n\geq 0}$ is attracted to some f-register-shift R_j.

(3) $\{f^n(x)\}_{n\geq 0}$ eventually ends up in some sink or some homterval of f (in particular, $\{f^n(x): n \geq 0\}$ is contained in an open, invariant set $Z \subseteq I$, which is such that for each $n \geq 1$, f^n is monotone and continuous on each connected component of Z).

In § 6 we study some more properties of f-register-shifts and topologically transitive f-cycles. In particular, we show that each f-register-shift R is a Cantor-like set, that $R - Q \subseteq f(R - S(f)) \subseteq R$ where Q is a finite subset of I and that for each $x \in R$ either $\{f^n(x): n \geq 0\}$ contains a singular point of f, i.e. an element of $S(f)$, or $\{f^n(x): n \geq 0\}$ is dense in R. Moreover, we prove that each topologically transitive f-cycle is in fact strongly transitive. Finally, in § 7 we apply our results to some examples.

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This paper is a shortened and revised version of the author's Dissertation (Bielefeld 1985) with the same title.

2. *Piecewise monotone functions*

Let $a, b \in \mathbb{R}$ with $a < b$ and let ω be some point not contained in $\mathbb{R}$. Put $I = [a, b]$ and $I' = I \cup \{\omega\}$; we consider I' as a topological space by calling a subset A of I' open if there exists an open subset U of $\mathbb{R}$ such that $U \cap I = A \cap I$.

A function $f: I' \to I'$ is called *piecewise monotone* on I if there exists $m \geq 1$ and $a = d_0 < d_1 < \cdots < d_m = b$ such that

(2.1) f is continuous and strictly monotone on each of the open intervals (d_k, d_{k+1}), $k = 0, 1, \ldots, m-1$, and

(2.2) $f^{-1}(\{\omega\}) = \{d_0, d_1, \ldots, d_m, \omega\}$.

$\mathcal{N}(I)$ will denote the set of piecewise monotone functions on I. Throughout this paper we assume that f is a fixed element of $\mathcal{N}(I)$.

For $n \geq 0$ define $f^n: I' \to I'$ inductively by $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $x \in I'$. It is easy to check that the composition of two elements of $\mathcal{N}(I)$ is again an element of $\mathcal{N}(I)$; hence we have $f^n \in \mathcal{N}(I)$ for each $n \geq 1$. Let $S(f)$ denote the set of singular points of f in I, *i.e.* $S(f) = f^{-1}(\{\omega\}) \cap I$. Put $S(f^0) = \varnothing$; it is not difficult to show that for all $n, k \geq 0$ and $A \subseteq I$ we have

$$S(f^n) = \{x \in I : f^j(x) \in S(f) \text{ for some } 0 \leq j < n\}$$
$$= \{x \in I : f^n(x) = \omega\},$$
$$f^n(A - S(f^n)) = f^n(A) \cap I$$

and

$$f^n(f^k(A - S(f^k)) - S(f^n)) = f^{n+k}(A - S(f^{n+k})).$$

Let $n \geq 1$; note that f^n is continuous and monotone on an open interval $J \subseteq I$ if and only if $J \cap S(f^n) = \varnothing$ and that $J \cap S(f^n) = \varnothing$ if and only if $f^n(J) \subset I$. Note also that if $f^n(x) \in I$ for some $x \in I$ then there exists $\varepsilon > 0$ such that f^n is continuous and monotone on $(x - \varepsilon, x + \varepsilon)$; in particular, $f^n(A - S(f^n))$ and $f^{-n}(A)$ are open whenever A is an open subset of I (where $f^{-n}(A) = \{x \in I : f^n(x) \in A\}$). Moreover, if A is an open interval contained in I then $f^n(A - S(f^n))$ is a finite union of open intervals.

A subset $A \subseteq I$ is called *f-almost-invariant* if $f(A - S(f)) \subseteq A$. The union and the intersection of any number of f-almost-invariant subsets of I are f-almost-invariant. Furthermore, it is easy to show that if $A \subseteq I$ is f-almost-invariant then int(A) and $\bar{A}$ are both f-almost-invariant and that for each $n \geq 1$, A is also f^n-almost-invariant.

We call $A \subseteq I$ *f-biinvariant* if A is f-almost-invariant and $f^{-1}(A) \subseteq A$. Again the union and the intersection of f-biinvariant subsets of I are f-biinvariant.

3. *Sinks and homtervals*

All results and proofs in this section are almost identical with the ones in [**16**]. For the convenience of the reader we give the proofs below.

A non-empty, open interval $J \subseteq I$ is called a *sink* of f if there exists $m \geq 1$ such that $f^m(J) \subseteq J$. Note that if J is a sink of f then $f^n(J) \cap S(f) = \varnothing$ and hence $J \cap S(f^n) = \varnothing$ for all $n \geq 0$; thus for each $n \geq 0$, f^n is continuous and monotone on J.

LEMMA 3.1 (cf. [**16**, lemma 4.1]). *Let $U \subseteq I$ be a non-empty open interval such that $f^n(U) \subseteq I$ for each $n \geq 0$. If $U \cap f^m(U) \neq \varnothing$ for some $m \geq 1$ then U is contained in a sink of f.*

Proof. Put $J=\bigcup_{k\geq 0} f^{km}(U)$; then $U\subseteq J\subseteq I$ and J is a non-empty open interval with $f^m(J)\subseteq J$. □

We call a non-empty open interval $L\subseteq I$ a *homterval* of f if for each $n\geq 0$ we have $f^n(L)\subseteq I$ and $f^n(L)$ is not contained in any sink of f. Note that if L is a homterval of f then $f^n(L)\cap S(f)=\emptyset$ and hence $L\cap S(f^n)=\emptyset$ for all $n\geq 0$; thus f^n is continuous and monotone on L for each $n\geq 0$. We emphasize that by proposition 3.3 (3) our definition of a homterval is equivalent to the usual one (see for instance [**5**]). For the construction of functions having homtervals see for example [**5**] or [**16**].

PROPOSITION 3.2 (cf. [**16**, proposition 4.3]). (1) *If J_1, J_2 are sinks (resp. homtervals) of f with $J_1\cup J_2\neq\emptyset$ then $J_1\cup J_2$ is also a sink (resp. homterval) of f.*

(2) *If J is a sink (resp. homterval) of f then for each $n\geq 0$, $f^n(J)$ is also a sink (resp. homterval) of f.*

(3) *Each sink of f is contained in a maximal sink of f; if J_1 and J_2 are maximal sinks of f then either $J_1=J_2$ or $J_1\cap J_2=\emptyset$.*

(4) *Each homterval of f is contained in a maximal homterval of f; if L_1 and L_2 are maximal homtervals of f then either $L_1=L_2$ or $L_1\cap L_2=\emptyset$.*

Proof. (1) and (2) are clear.

(3) Let J be a sink of f and let U be the largest open interval with $J\subseteq U\subseteq I$ such that $f^n(U)\subseteq I$ for all $n\geq 0$. The maximality of U and lemma 3.1 ensure that U is the maximal sink of f containing J. If J_1 and J_2 are maximal sinks of f with $J_1\neq J_2$ then by (1) we have $J_1\cap J_2=\emptyset$.

(4) Let L be a homterval of f and let U be the largest open interval with $L\subseteq U\subseteq I$ such that $f^n(U)\subseteq I$ for all $n\geq 0$. U cannot be contained in a sink of f since L is a homterval of f. Hence U is the maximal homterval of f containing L. If L_1 and L_2 are maximal homtervals of f with $L_1\neq L_2$ then by (1) we have $L_1\cap L_2=\emptyset$. □

Put

$$\mathrm{Sink}\,(f)=\{x\in I: f^m(x)\in J \text{ for some sink } J \text{ of } f \text{ and some } m\geq 0\}$$

and

$$\mathrm{Homt}\,(f)=\{x\in I: f^m(x)\in L \text{ for some homterval } L \text{ of } f \text{ and some } m\geq 0\};$$

then Sink (f) and Homt (f) are both open and by proposition 3.2 (2) we have $f(\mathrm{Sink}\,(f))\subseteq \mathrm{Sink}\,(f)$ and $f(\mathrm{Homt}\,(f))\subseteq\mathrm{Homt}\,(f)$. Moreover, it is easy to see that Sink (f) and Homt (f) are both f-biinvariant.

PROPOSITION 3.3 (cf. [**16**, proposition 4.3]). (1) $\mathrm{Sink}\,(f)\cap\mathrm{Homt}\,(f)=\emptyset$.

(2) *If $L\subseteq\mathrm{Homt}\,(f)$ is a non-empty open interval then L is a homterval of f.*

(3) *If L is a homterval of f then $f^n(L)\cap f^k(L)=\emptyset$ whenever $0\leq k<n$.*

(4) *If $x\in\mathrm{Sink}\,(f)$ then there exists $q\geq 1$ such that $\lim_{n\to\infty} f^{nq}(x)$ exists.*

Proof. (1) Suppose that $\mathrm{Sink}\,(f)\cap\mathrm{Homt}\,(f)\neq\emptyset$. Then there exist a sink J of f and a homterval L of f such that $L\cap J\neq\emptyset$. Lemma 3.1 gives us then that $J\cup L$, and thus also L, is contained in a sink of f. But this is not possible since L is a homterval of f.

(2) Let $L \subseteq \text{Homt}(f)$ be a non-empty open interval. Then for each $n \geq 0$ we have $f^n(L) \subseteq I$ and by (1) $f^n(L)$ is not contained in any sink of f. Hence L is a homterval of f.

(3) Let $0 \leq k \leq n$; applying lemma 3.1 with $U = f^k(L)$ and $m = n - k$ shows that if L is a homterval of f then $f^k(L) \cap f^n(L) = \emptyset$.

(4) Suppose that $x \in \text{Sink}(f)$; then by proposition 3.2(2) there exist $k \geq 0$, $m \geq 1$ and a sink J of f such that $f^{km}(x) \in J$ and $f^m(J) \subseteq J$. Since f^m is continuous and monotone on J this gives us that for each $n \geq k$, $f^{nm}(x) \in J$ and that the sequence $\{f^{2nm}(x)\}_{n \geq k}$ is monotone. Hence $\lim_{n \to \infty} f^{2nm}(x)$ exists. □

Put $M(f) = \{x \in I : f^m(x) = \omega \text{ for some } m \geq 0\}$. Note that $M(f)$ and thus also $\overline{M(f)}$ are f-almost-invariant and that $I - \overline{M(f)}$ is the largest open set $U \subseteq I$ such that $f^n(U) \cap S(f) = \emptyset$ for all $n \geq 0$.

Proposition 3.4 (cf. [**16**, proposition 4.2]). $I - \overline{M(f)} = \text{Sink}(f) \cup \text{Homt}(f)$.

Proof. Suppose that $x \in \text{Sink}(f) \cup \text{Homt}(f)$; then there exists $m \geq 0$ such that $f^m(x) \in J$ where J is either a sink or a homterval of f. Thus $f^m((x - \varepsilon, x + \varepsilon)) \subseteq J$ for some $\varepsilon > 0$ and hence $f^n((x - \varepsilon, x + \varepsilon)) \subseteq I$ for each $n \geq 0$. This shows that $x \in I - \overline{M(f)}$. On the other hand, let $U \subseteq I - \overline{M(f)}$ be a non-empty open interval. Then $f^n(U) \subseteq I$ for all $n \geq 0$. Thus either U is a homterval of f or there exists $m \geq 0$ such that $f^m(U)$ is contained in some sink of f. Hence $U \subseteq \text{Sink}(f) \cup \text{Homt}(f)$. □

The following lemma will be useful later.

Lemma 3.5. *If $U \subseteq \overline{M(f)}$ is non-empty, open and f-almost-invariant then $U \cap S(f) \neq \emptyset$.*

Proof. Suppose that $U \subseteq \overline{M(f)}$ is non-empty, open and f-almost-invariant. Since $U \cap M(f) \neq \emptyset$ we have $\varphi \in f^n(U) = f^n(U - S(f^n)) \subseteq U$ for some $\varphi \in S(f)$ and $n \geq 0$. □

4. *Topologically transitive cycles and register-shifts*

This section is concerned with the definitions and some elementary properties of topologically transitive f-cycles and f-register-shifts.

We call $C \subseteq I$ an *f-cycle* if C is f-almost-invariant and is the disjoint union of non-trivial closed intervals $B_1, \ldots, B_m$ $(m \geq 1)$ such that whenever $1 \leq j \leq m$ and $U \subseteq C$ is non-empty and open then $f^n(U) \cap B_j \neq \emptyset$ for some $n \geq 1$. $B_1, \ldots, B_m$ are then called the *components* of C.

Clearly, I is an f-cycle. Note that C being an f-cycle does not necessarily mean that f cyclically permutes or even permutes the components of C. But if $f_{|C}$ is really 'continuous' in that there exists a continuous function $g: C \to C$ with $g(x) = f(x)$ for all $x \in I - S(f)$, then we can label the components $B_1, \ldots, B_m$ of the f-cycle C in such a way that $f(B_i - S(f)) \subseteq B_{i+1}$ for $i = 1, \ldots, m-1$ and $f(B_m - S(f)) \subseteq B_1$. In general however, the situation is much more complicated since, for a component B of C, $f(B - S(f))$ is not necessarily contained in a single component of C.

For an f-cycle C let $A(C, f)$ denote the set of points in I which eventually 'end up' in the interior of C, i.e.

$$A(C, f) = \{x \in I : f^m(x) \in \text{int}(C) \text{ for some } m \geq 0\}.$$

Clearly, $A(C, f)$ is open and f-biinvariant.

LEMMA 4.1. *Let C and K be f-cycles. Then*:

(1) *If* $\operatorname{int}(B \cap K) = \emptyset$ *for some component B of C then* $\operatorname{int}(C \cap K) = \emptyset$.

(2) $A(C, f) \cap A(K, f) = \emptyset$ *if and only if* $\operatorname{int}(C \cap K) = \emptyset$.

Proof. (1) Let B be a component of C. If $\operatorname{int}(C \cap K) \neq \emptyset$ then by definition we have $\emptyset \neq f^n(\operatorname{int}(C \cap K)) \cap B = f^n(\operatorname{int}(C \cap K) - S(f^n)) \cap B$ for some $n \geq 1$. But since for each $m \geq 1$, $f^m(\operatorname{int}(C \cap K) - S(f^m))$ is an open subset of K it follows that $\operatorname{int}(B \cap K) \neq \emptyset$.

(2) Let $x \in A(C, f) \cap A(K, f)$; then $f^n(x) \in \operatorname{int}(C)$ and $f^m(x) \in \operatorname{int}(K)$ for some $n, m \geq 0$. Putting $k = \max\{n, m\}$ we have $f^k(x) \in \operatorname{int}(C \cap K)$. The proof of the converse is trivial since $\emptyset \neq \operatorname{int}(K') \subseteq A(K', f)$ for each f-cycle K'. □

In order to analyse the structure of $\overline{M(f)}$ we will be interested in f-cycles contained in $\overline{M(f)}$. Looking for f-cycles which are 'minimal' leads us to the following definition. We say that an f-cycle C is *topologically transitive* if whenever F is a closed, f-almost-invariant subset of C then either $F = C$ or $\operatorname{int}(F) = \emptyset$.

For $x \in I$ put $O_f(x) = \{f^n(x) : n \geq 0\}$; $O_f(x)$ is called the *orbit* of x (under f).

PROPOSITION 4.2 (cf. [18, theorem 5.8]. *Let $C \subseteq I$ be an f-cycle. Then the following are equivalent*:

(1) *C is a topologically transitive f-cycle.*

(2) *If $U \subseteq C$ is non-empty, open and f-almost-invariant then $\bar{U} = C$.*

(3) *If $U, V \subseteq C$ are non-empty and open then $f^n(U) \cap V \neq \emptyset$ for some $n \geq 0$.*

(4) *If $U, V \subseteq C$ are non-empty and open then $f^{-n}(U) \cap V \neq \emptyset$ for some $n \geq 0$.*

(5) *$\{x \in C : \overline{O_f(x)} \neq C\}$ is of the first category.*

(6) *$\overline{O_f(x)} = C$ for some $x \in C$.*

Proof. (1)⇒(2) Suppose that $U \subseteq C$ is non-empty, open and f-almost-invariant. Then $\bar{U} \subseteq C$ is closed and f-almost-invariant; hence since C is topologically transitive we have $\bar{U} = C$.

(2)⇒(3) Suppose that $U, V \subseteq C$ are non-empty and open. Put $A = \bigcup_{n \geq 0} f^n(U - S(f^n))$. Then A is a non-empty, open and f-almost-invariant subset of C, and thus by (2) $\bar{A} = C$. Hence $f^n(U) \cap V \neq \emptyset$ for some $n \geq 0$.

(3)⇒(4) This is clear.

(4)⇒(5) Let $U_1, U_2, \ldots$ be a countable base for the relative topology on C and let $x \in C$. Then $\overline{O_f(x)} \neq C$ if and only if $O_f(x) \cap U_n = \emptyset$ for some $n \geq 1$. Thus

$$\{x \in C : \overline{O_f(x)} \neq C\} = C \cap \bigcup_{n \geq 1} \bigcap_{m \geq 0} f^{-m}(I' - U_n)$$
$$\subseteq \bigcup_{n \geq 1} \left(I - \left(\bigcup_{m \geq 0} f^{-m}(U_n) \cap \operatorname{int}(C) \right) \right) \cap C.$$

Since $\bigcup_{m \geq 0} f^{-m}(U_n) \cap \operatorname{int}(C)$ is open and by (4) dense in C the set of points $\{x \in C : \overline{O_f(x)} \neq C\}$ is of the first category.

(5)⇒(6) This is clear since C is a set of the second category.

(6)⇒(1) Let $x \in C$ with $\overline{O_f(x)} = C$ and let $F \subseteq C$ be closed and f-almost-invariant with $\operatorname{int}(F) \neq \emptyset$. Then we have $f^k(x) \in F$ for some $k \geq 0$, and thus $\{f^n(x) : n \geq k\} \subseteq F$. Since C has no isolated points it follows that $C = \overline{O_f(x)} = \overline{\{f^n(x) : n \geq k\}} = F$. □

PROPOSITION 4.3 (cf. [**16**, proposition 2.1]). *Let C and K be topologically transitive f-cycles. Then*:

(1) $A(C,f)\cap(\mathrm{Sink}\,(f)\cup\mathrm{Homt}\,(f))=\varnothing$.

(2) $C\subseteq\overline{M(f)}$.

(3) $\mathrm{int}\,(C)\cap S(f)\cap(a,b)\neq\varnothing$.

(4) *Either* $C=K$ *or* $A(C,f)\cap A(K,f)=\varnothing$.

Proof. (1) Suppose that $A(C,f)\cap\mathrm{Homt}\,(f)\neq\varnothing$. Then there exists a homterval L of f with $L\subseteq C$. By proposition 3.3(3) we would then have $L\cap f^n(f(L))=\varnothing$ for all $n\geq 0$ which by proposition 4.2 is not possible.

Suppose next that $A(C,f)\cap\mathrm{Sink}\,(f)\neq\varnothing$. Then $\mathrm{int}\,(C)\cap\mathrm{Sink}\,(f)\neq\varnothing$ and by proposition 3.3(4) we would have

$$\mathrm{int}\,(C)\cap\mathrm{Sink}\,(f)\subseteq\{x\in C\colon\overline{O_f(x)}\neq C\}.$$

Again this contradicts proposition 4.2.

(2) By (1) and proposition 3.4 we have

$$A(C,f)\subseteq I-(\mathrm{Sink}\,(f)\cup\mathrm{Homt}\,(f))=\overline{M(f)}.$$

Hence $C\subseteq\overline{M(f)}$.

(3) Since $\mathrm{int}\,(C)\cap(a,b)$ is an f-almost-invariant subset of $\overline{M(f)}$, lemma 3.5 gives us that $\mathrm{int}\,(C)\cap S(f)\cap(a,b)\neq\varnothing$.

(4) Suppose that $A(C,f)\cap A(K,f)\neq\varnothing$. Then by lemma 4.1(2) we have $\mathrm{int}\,(C\cap K)\neq\varnothing$. Since $C\cap K$ is closed and f-almost-invariant it follows from the topological transitivity of C and K that $C=C\cap K=K$. □

Examples of piecewise monotone functions having topologically transitive cycles are easily found. For instance, let $g\in\mathcal{N}(I)$ be given by $g(x)=2x-a$ if $a<x<(a+b)/2$ and $g(x)=2x-b$ if $(a+b)/2<x<b$. As it is shown in corollary 7.2, I is then a topologically transitive g-cycle. In general we cannot describe the asymptotic behaviour of elements of $\mathcal{N}(I)$ in terms of sinks, homtervals and topologically cycles alone. For $\mu\in(0,4)$ let $f_\mu\in\mathcal{N}([0,1])$ be given by $f_\mu(x)=\mu x(1-x)$ for all $x\in(0,1)-\{\frac{1}{2}\}$. Then for a certain value of μ ($\mu\approx 3.56994$) we have $\mathrm{Sink}\,(f_\mu)=\mathrm{Homt}\,(f_\mu)=\varnothing$ and there is no topologically transitive f_μ-cycle. (See for instance [**6**] and [**3**, theorem 2.6]). In [**16**, § 7] a function $g\in\mathcal{N}(I)$ is explicitly constructed having neither sinks, homtervals nor topologically transitive g-cycles. In both cases there exists a decreasing sequence of cycles $\{K_n\}_{n\geq1}$ such that $\mathrm{int}\,(\bigcap_{n\geq1}K_n)=\varnothing$. This suggests the following definition.

We call $R\subseteq I$ an *f-register-shift* if $\mathrm{int}\,(R)=\varnothing$ and if there exists a decreasing sequence $\{K_n\}_{n\geq1}$ of f-cycles K_n contained in $\overline{M(f)}$ such that $R=\bigcap_{n\geq1}K_n$ ($\{K_n\}_{n\geq1}$ is said to be decreasing if $K_{n+1}\subseteq K_n$ for each $n\geq1$); we then say that $\{K_n\}_{n\geq1}$ is a *generator* for the f-register-shift R.

PROPOSITION 4.4. *Let $\{K_n\}_{n\geq1}$ be a generator for some f-register-shift R and let $\{K'_n\}_{n\geq1}$ be a generator for some f-register-shift R'. Then*:

(1) *If C is a topologically transitive f-cycle then $A(C,f)\cap A(K_m,f)=\varnothing$ for some $m\geq1$.*

(2) $S(f)\cap(a,b)\cap\bigcap_{n\geq1}\mathrm{int}\,(K_n)\neq\varnothing$.

(3) $R\neq R'$ *if and only if* $A(K_m,f)\cap A(K'_m,f)=\varnothing$ *for some* $m\geq1$.

Proof. (1) Suppose that C is a topologically transitive f-cycle and assume that $A(C,f)\cap A(K_n,f)\neq\varnothing$ for all $n\geq 1$. Then by lemma 4.1(2) we would have $\operatorname{int}(C\cap K_n)\neq\varnothing$ and thus $C\subseteq K_n$ for all $n\geq 1$ (because C is topologically transitive). But this is not possible since $\operatorname{int}(\bigcap_{n\geq 1} K_n)=\varnothing$.

(2) Since $\operatorname{int}(K_n)\cap(a,b)$ is an f-almost-invariant subset of $\overline{M(f)}$ we have by lemma 3.5, $\operatorname{int}(K_n)\cap S(f)\cap(a,b)\neq\varnothing$ for each $n\geq 1$. Hence

$$S(f)\cap(a,b)\cap\bigcap_{n\geq 1}\operatorname{int}(K_n)\neq\varnothing$$

(because $S(f)$ is finite).

(3) Let $x\in R-R'$; then $x\in R-K_j'$ for some $j\geq 1$. For $n\geq 1$ let B_n be the unique component of K_n with $x\in B_n$. Since $\operatorname{int}(R)=\varnothing$ we have $\{x\}=\bigcap_{n\geq 1} B_n$; hence $B_m\cap K_j'=\varnothing$ for some $m\geq j$. Now lemma 4.1 gives that

$$\operatorname{int}(K_m\cap K_m')\subseteq\operatorname{int}(K_m\cap K_j')=\varnothing,$$

and thus $A(K_m,f)\cap A(K_m',f)=\varnothing$. Conversely, if $A(K_m,f)\cap A(K_m',f)=\varnothing$ for some $m\geq 1$ then $\operatorname{int}(K_m)\cap\operatorname{int}(K_m')=\varnothing$, and thus by (2) $R\neq R'$. □

Let R be an f-register-shift; put $A(R,f)=\bigcap_{C\in\mathscr{C}(R)} A(C,f)$ where $\mathscr{C}(R)$ is the set of f-cycles C with $R\subseteq C$. Let $\{K_n\}_{n\geq 1}$ be a generator for R. One can show that in general $A(R,f)=\bigcap_{n\geq 1} A(K_n,f)$ is not true. However, we will prove in § 5 (theorem 5.9(2)) that there exists a generator $\{K_n\}_{n\geq 1}$ for R such that $A(R,f)=\bigcap_{n\geq 1} A(K_n,f)$. In particular, this shows that $A(R,f)$ is a G_δ-set (i.e. $A(R,f)$ can be written as a countable intersection of open sets).

In § 6 we will study some more properties of f-register-shifts. We will show that each f-register-shift R is a Cantor-like set, that $R-Q\subseteq f(R-S(f))\subseteq R$ where Q is a finite set of points in I, that the orbit of each point in $R-M(f)$ is dense in R and that each element of $A(R,f)$ is attracted to R.

5. *The main result*

We now come to the main result.

THEOREM 5.1. *Let $C_1,\ldots,C_r$ be the topologically transitive f-cycles, let $R_1,\ldots,R_s$ be the f-register-shifts and for $1\leq i\leq s$ let $\{K_n^{(i)}\}_{n\geq 1}$ be a generator for R_i. Then*

$$r+s\leq\operatorname{card}(S(f)\cap(a,b)),$$

and there exists $m\geq 1$ such that for each $n\geq m$ the open and f-biinvariant sets $A(C_1,f),\ldots,A(C_r,f)$, $A(K_n^{(1)},f),\ldots,A(K_n^{(s)},f)$, $\operatorname{Sink}(f)$, $\operatorname{Homt}(f)$ are disjoint and their union is dense in I.

Proof. The proof of theorem 5.1, which is based on a couple of lemmas and propositions, can be found at the end of this section.

Let $C_1,\ldots,C_r$ be the topologically transitive f-cycles and let $R_1,\ldots,R_s$ be the f-register-shifts. Put

$$G(f)=A(C_1,f)\cup\ldots\cup A(C_r,f)\cup A(R_1,f)\cup\ldots\cup A(R_s,f)\cup\operatorname{Sink}(f)\cup\operatorname{Homt}(f).$$

Theorem 5.9(2) will show that for each $1\leq i\leq s$ there exists a generator $\{K_n^{(i)}\}_{n\geq 1}$ for R_i such that $A(R_i,f)=\bigcap_{n\geq 1} A(K_n^{(i)},f)$. Hence by theorem 5.1 and the Baire category theorem the G_δ-set $G(f)$ is dense in I.

The following example shows that $I-G(f)$ (which is f-biinvariant) can have a quite complicated structure. Consider Newton's Method for determining the zeros of a polynomial p. By identifying $\bar{\mathbb{R}}$ with I we obtain a discrete dynamical system on I represented by some $g \in \mathcal{N}(I)$. If p is an n-th degree polynomial with $n \geq 4$ having real roots, then it follows from a result by Barna [**1**] that Sink (g) is dense in I (and thus there are neither homtervals of g, topologically transitive g-cycles nor g-register-shifts) and that $I-G(g)$ contains a Cantor-like set. Furthermore, it should be noted that the action of f restricted to $I-G(f)$ can be very complex. In the continuous case (i.e. if there exists a continuous function g on I into itself with $g(x)=f(x)$ for all $x \in I-S(f)$) the asymptotic behaviour of f on $I-G(f)$ is analysed in [**16**] by using certain factors of f which essentially 'kill off' $G(f)$.

For $\varphi \in S(f)$ and $n \geq 1$ put

$$I_n(\varphi)=\left(\varphi-\frac{1}{n}, \varphi+\frac{1}{n}\right) \cap I$$

and

$$L_n(\varphi)=\bigcup_{m \geq 0} f^m(I_n(\varphi)-S(f^m)).$$

Then $L_n(\varphi)$ is non-empty, open and f-almost-invariant and we have $L_{n+1}(\varphi) \subseteq L_n(\varphi)$ for all $n \geq 1$. Furthermore, for $\varphi \in S(f)$ put

$$\Delta(\varphi)=\{x \in I: f^m(x)=\varphi \text{ for some } m \geq 0\};$$

note that $\Delta(\varphi)$ and thus also $\overline{\Delta(\varphi)}$ and int $\overline{(\Delta(\varphi))}$ are f-almost-invariant.

The proof of theorem 5.1 will consist in showing that for each $\varphi \in S(f)$ with int $\overline{(\Delta(\varphi))} \neq \varnothing$ there exists $m \geq 1$ such that either $\overline{L_m(\varphi)}$ is a topologically transitive f-cycle or $\{\overline{L_n(\varphi)}\}_{n \geq m}$ is a generator for some f-register-shift R. In order to prove that (for sufficiently large n) $\overline{L_n(\varphi)}$ is an f-cycle we will first study some properties of certain (connected) components of open, f-almost-invariant subsets of I.

Let $U \subseteq I$ be open; a component J of U is called *regular* if there exists a component L of U with $L \cap S(f) \neq \varnothing$ such that $f^k(L) \cap J \neq \varnothing$ for some $k \geq 0$. Let $\varphi \in S(f)$ and $n \geq 1$; note that each component of $L_n(\varphi)$ is regular.

LEMMA 5.2. *Let $U \subseteq I$ be non-empty, open and f-almost-invariant and let J be a regular component of U. Then there exists a component K of $U-S(f)$ and $m \geq 0$ such that $f^m(K) \subseteq J$, $K \cap S(f^m)=\varnothing$ and $K \subseteq L$ for some component L of U with $L \cap S(f) \neq \varnothing$.*

Proof. Let

$$m=\min\{k \geq 0: f^k(L) \cap J \neq \varnothing \text{ for some component } L \text{ of } U \text{ with } L \cap S(f) \neq \varnothing\}.$$

Choose a component L of U with $L \cap S(f) \neq \varnothing$ such that $f^m(L) \cap J \neq \varnothing$. Since U is f-almost-invariant there exists a component K of $L-S(f)$ such that $f^m(K) \subseteq J$. Then K is a component of $U-S(f)$ and by the choice of m we have $K \cap S(f^m)=\varnothing$. □

LEMMA 5.3. *Let $U \subseteq \overline{M(f)}$ be non-empty, open and f-almost-invariant. Then the set of the regular components of U is non-empty and finite.*

Proof. By Lemma 3.5 we have $U \cap S(f) \neq \varnothing$; hence we can find at least one regular component of U. Let $\mathcal{Y}$ be the set of the components K of $U-S(f)$ which are contained in some component L of U with $L \cap S(f) \neq \emptyset$. $\mathcal{Y}$ is finite because we have for all $K \in \mathcal{Y}$ and $\varphi \in S(f)$, $\bar{K} \cap S(f) \neq \emptyset$ and card $(\{K \in \mathcal{Y}: \varphi \in \bar{K}\}) \leq 2$. Since $U \subseteq \overline{M(f)}$ we can find $N \geq 1$ such that for all $K \in \mathcal{Y}$ there exists $0 \leq n \leq N$ with $f^n(K) \cap S(f) \neq \emptyset$. Now let J be a regular component of U. By lemma 5.2 there exist $K \in \mathcal{Y}$ and $m \geq 0$ such that $f^m(K) \subseteq J$, $K \cap S(f^m) = \emptyset$ and $m < N$. Since J is the unique component of U with $f^m(K) \subseteq J$, this shows that there are at most $N \cdot$ card $(\mathcal{Y})$ regular components of U. □

Lemma 5.4. *Let U, U' be open and f-almost-invariant subsets of I with $U' \subseteq U$ and $U \cap S(f) = U' \cap S(f)$. Then each regular component of U contains at least one regular component of U'.*

Proof. Let J be a regular component of U. By lemma 5.2 there exist a component K of $U - S(f)$ and $m \geq 0$ such that $f^m(K) \subseteq J$, $K \cap S(f^m) = \emptyset$ and $K \subseteq L$ for some component L of U with $L \cap S(f) \neq \emptyset$. Since $U \cap S(f) = U' \cap S(f)$ we can find a component L' of U' with $L' \cap S(f) \neq \emptyset$ such that $L' \subseteq L$ and $L' \cap K \neq \emptyset$. Then since U' is f-almost-invariant $f^m(L' \cap K)$ is a non-empty open interval which is contained in $U' \cap J$. Let J' be the unique component of U' with $f^m(L' \cap K) \subseteq J'$. Then $J' \subseteq J$ and J' is regular. □

Next we want to show that if $\varphi \in S(f)$ with int $\overline{(\Delta(\varphi))} \neq \emptyset$ and n is sufficiently large then $\overline{L_n(\varphi)}$ is an f-cycle. For this we need the following lemma.

Lemma 5.5. *Let $\varphi \in S(f)$ with* int $\overline{(\Delta(\varphi))} \neq \emptyset$. *Then*:

(1) *If $U \subseteq \overline{\Delta(\varphi)}$ is non-empty, open and f-almost-invariant, then $L_n(\varphi) \subseteq U$ for some $n \geq 1$.*

(2) *There exists $m \geq 1$ such that $L_n(\varphi) \subseteq \overline{\Delta(\varphi)}$ and*

$$L_n(\varphi) \cap S(f) = L_m(\varphi) \cap S(f) \qquad \textit{for all } n \geq m.$$

Proof. (1) Suppose that $U \subseteq \overline{\Delta(\varphi)}$ is non-empty, open and f-almost-invariant. Since $\Delta(\varphi) \cap U \neq \emptyset$ we have $\varphi \in f^j(U - S(f^j)) \subseteq U$ for some $j \geq 0$. Hence there exists $n \geq 1$ such that $I_n(\varphi) \subseteq U$. Again using that U is f-almost-invariant gives us that $L_n(\varphi) \subseteq U$.

(2) Since int $\overline{(\Delta(\varphi))}$ is f-almost-invariant and $S(f)$ is finite there exists by (1) $m \geq 1$ such that $L_n(\varphi) \subseteq \overline{\Delta(\varphi)}$ and $L_n(\varphi) \cap S(f) = L_m(\varphi) \cap S(f)$ for all $n \geq m$. □

Proposition 5.6. *Let $\varphi \in S(f)$ with* int $\overline{(\Delta(\varphi))} \neq \emptyset$. *Then for all sufficiently large $n \geq 1$ the number of components of $L_n(\varphi)$ is finite and $\overline{L_n(\varphi)}$ is an f-cycle contained in $\overline{\Delta(\varphi)}$.*

Proof. By lemma 5.5(2) there exists $m \geq 1$ such that $L_n(\varphi) \subseteq \overline{\Delta(\varphi)}$ and $L_n(\varphi) \cap S(f) = L_m(\varphi) \cap S(f)$ for all $n \geq m$. Now let $n \geq m$. Since each component of $L_n(\varphi)$ is regular lemma 5.3 immediately gives us that the number of components of $L_n(\varphi)$ is finite. Hence $\overline{L_n(\varphi)}$ can be written as a disjoint union of non-trivial closed intervals $B_1, \ldots, B_p$ (with $p \geq 1$). Let $U \subseteq \overline{L_n(\varphi)}$ be open and non-empty. In order to prove that $\overline{L_n(\varphi)}$ is an f-cycle it remains to show that for each $1 \leq i \leq p$ we have $B_i \cap f^k(U) \neq \emptyset$ for some $k \geq 1$. Let $1 \leq i \leq p$ and put $V = \bigcup_{k \geq 1} f^k(U - S(f^k))$. Then V is open and (since V and $\overline{L_n(\varphi)}$ are both f-almost-invariant) we have $V \subseteq \overline{L_n(\varphi)} \subseteq \overline{\Delta(\varphi)}$. Hence by lemma 5.5(1) there exists $j \geq n$ with $L_j(\varphi) \subseteq V$. B_i contains at least one component J of $L_n(\varphi)$, which is regular. By lemma 5.4 there exists a component J' of $L_j(\varphi)$ with $J' \subseteq J \subseteq B_i$. Hence $J' \subseteq B_i \cap V$ and therefore $B_i \cap f^k(U) \neq \emptyset$ for some $k \geq 1$. □

PROPOSITION 5.7. *Let* $L_\infty = \bigcap_{n\geq 1} \overline{L_n(\varphi)}$ *for some* $\varphi \in S(f)$ *with* $\operatorname{int}\overline{(\Delta(\varphi))} \neq \emptyset$. *Then*:

(1) *If* $\operatorname{int}(L_\infty) \neq \emptyset$ *then* L_∞ *is a topologically transitive f-cycle and* $L_\infty = \overline{L_m(\varphi)}$ *for some* $m \geq 1$.

(2) *If* $\operatorname{int}(L_\infty) = \emptyset$ *then* L_∞ *is an f-register-shift and for all sufficiently large m,* $\{\overline{L_n(\varphi)}\}_{n\geq m}$ *is a generator for* L_∞ *with* $A(L_\infty, f) = \bigcap_{n\geq m} A(\overline{L_n(\varphi)}, f)$.

Proof. (1) Put $V = \operatorname{int}(L_\infty)$ and suppose that V is non-empty. By lemma 5.5(2) we have $V \subseteq \overline{L_k(\varphi)} \subseteq \overline{\Delta(\varphi)}$ for some $k \geq 1$ and by lemma 5.5(1) we can find $m \geq 1$ with $L_m(\varphi) \subseteq V$. Thus $\overline{L_n(\varphi)} = \overline{L_m(\varphi)} = L_\infty$ for all $n \geq m$, and so by proposition 5.6 $\overline{L_m(\varphi)}$ is an f-cycle contained in $\overline{\Delta(\varphi)}$. Now let $U \subseteq \overline{L_m(\varphi)}$ be non-empty, open and f-almost-invariant. Again by lemma 5.5(1) we have $L_j(\varphi) \subseteq U$ for some $j \geq m$ and hence $\overline{L_j(\varphi)} = \bar{U} = \overline{L_m(\varphi)}$. Therefore by proposition 4.2, $\overline{L_m(\varphi)}$ is a topologically transitive f-cycle.

(2) Suppose that $\operatorname{int}(L_\infty) = \emptyset$. By proposition 5.6 there exists $m \geq 1$ such that for each $n \geq m$ $\overline{L_n(\varphi)}$ is an f-cycle contained in $\overline{\Delta(\varphi)} \subseteq \overline{M(f)}$. Hence L_∞ is an f-register-shift and $\{\overline{L_n(\varphi)}\}_{n\geq m}$ is a generator for L_∞. In order to show that $A(L_\infty, f) = \bigcap_{n\geq m} A(\overline{L_n(\varphi)}, f)$, consider an f-cycle K with $L_\infty \subseteq K$; put $U = \operatorname{int}(K) \cap \operatorname{int}(\overline{\Delta(\varphi)})$. Then U is non-empty because $\varphi \in L_\infty \cap \operatorname{int}(\overline{\Delta(\varphi)}) \subseteq K \cap \operatorname{int}(\overline{\Delta(\varphi)})$. Moreover, U is open and f-almost-invariant; hence by lemma 5.5(1) we have $L_n(\varphi) \subseteq U \subseteq K$ and thus $\operatorname{int}(\overline{L_n(\varphi)}) \subseteq \operatorname{int}(K)$ for some $n \geq m$. Therefore $A(\overline{L_n(\varphi)}, f) \subseteq A(K, f)$ which shows that $A(L_\infty, f) = \bigcap_{n\geq m} A(\overline{L_n(\varphi)}, f)$.

LEMMA 5.8. *Let* $U \subseteq \overline{M(f)}$ *be non-empty, open and f-almost-invariant. Then there exists* $\varphi \in S(f)$ *with* $\operatorname{int}(\overline{\Delta(\varphi)}) \neq \emptyset$ *such that* $L_n(\varphi) \subseteq U$ *for some* $n \geq 1$.

Proof. $\bigcup_{\varphi \in S(f)} \operatorname{int}(\overline{\Delta(\varphi)})$ is dense in $\overline{M(f)}$ since

$$\overline{M(f)} - \bigcup_{\varphi \in S(f)} \operatorname{int}\overline{(\Delta(\varphi))} = \bigcup_{\varphi \in S(f)} \overline{\Delta(\varphi)} - \bigcup_{\varphi \in S(f)} \operatorname{int}(\overline{\Delta(\varphi)}) \subseteq \bigcup_{\varphi \in S(f)} \partial(\overline{\Delta(\varphi)}).$$

Hence there exists $\varphi \in S(f)$ with $\operatorname{int}(\overline{\Delta(\varphi)}) \cap U \neq \emptyset$, and thus by lemma 5.5(1) we have $L_n(\varphi) \subseteq U$ for some $n \geq 1$. □

Proof of Theorem 5.1. By propositions 4.3 and 4.4 there exists $m \geq 1$ such that for all $n \geq m$ the open and f-biinvariant sets $A(C_1, f), \ldots, A(C_r, f)$, $A(K_n^{(1)}, f), \ldots, A(K_n^{(s)}, f)$, $\operatorname{Sink}(f)$, $\operatorname{Homt}(f)$ are disjoint and each of them apart from $\operatorname{Sink}(f)$ and $\operatorname{Homt}(f)$ contains at least one element of $S(f) \cap (a, b)$. Thus $r + s \leq \operatorname{card}(S(f) \cap (a, b))$. For $n \geq m$ put

$$G_n = I - (A(C_1, f) \cup \ldots \cup A(C_r, f) \cup A(K_n^{(1)}, f) \cup \ldots \cup A(K_n^{(s)}, f) \cup \operatorname{Sink} k, (f) \\ \cup \operatorname{Homt}(f)).$$

Then G_n and thus $\operatorname{int}(G_n)$ are f-almost-invariant, and by proposition 3.4 we have $G_n \subseteq \overline{M(f)}$. Lemma 5.8 and proposition 5.7 immediately give us that $\operatorname{int}(G_n) = \emptyset$. □

The next theorem shows how each topologically transitive f-cycle and how for a given f-register-shift R a generator $\{K_n\}_{n\geq 1}$ for R with $A(R,f)=\bigcap_{n\geq 1} A(K_n,f)$ can be constructed; in particular, this implies that $A(R,f)$ is a G_δ-set.

THEOREM 5.9. *Let C be a topologically transitive f-cycle and let R be an f-register-shift. Then*:

(1) $C=\bigcap_{n\geq 1}\overline{L_n(\varphi)}=\overline{L_m(\varphi)}$ *for some* $\varphi\in S(f)$ *with* $\operatorname{int}\overline{\Delta(\varphi)}\neq\emptyset$.

(2) $R=\bigcap_{n\geq 1}\overline{L_n(\varphi)}$ *and* $\{\overline{L_n(\varphi)}\}_{n\geq m}$ *is a generator for* R *with* $A(R,f)=\bigcap_{n\geq m} A(\overline{L_n(\varphi)},f)$ *for some* $\varphi\in S(f)$ *with* $\operatorname{int}(\overline{\Delta(\varphi)})\neq\emptyset$ *and all large enough* m.

Proof. (1) By proposition 4.3(2) we have $C\subseteq\overline{M(f)}$. Hence by lemma 5.8 there exists $\varphi\in S(f)$ with $\operatorname{int}\overline{(\Delta(\varphi))}\neq\emptyset$ such that $L_m(\varphi)\subseteq C$ for some $m\geq 1$. Since C is topologically transitive we have $C=\overline{L_m(\varphi)}=\bigcap_{n\geq 1}\overline{L_n(\varphi)}$.

(2) Let $\{K_n\}_{n\geq 1}$ be a generator for R. Since $S(f)$ is finite there exists, by lemma 5.8, $\varphi\in S(f)$ with $\operatorname{int}\overline{(\Delta(\varphi))}\neq\emptyset$ such that for each $n\geq 1$ we have $L_j(\varphi)\subseteq K_n$ for some $j\geq n$. Hence propositions 4.4(3) and 5.7(2) give us that $R=\bigcap_{n\geq 1}\overline{L_n(\varphi)}$ and that for sufficiently large m, $\{\overline{L_n(\varphi)}\}_{n\geq m}$ is a generator for R with $A(R,f)=\bigcap_{n\geq m} A(\overline{L_n(\varphi)},f)$.

6. *More on register-shifts and topologically transitive cycles*

In this section we will first study some more properties of f-register-shifts. In particular, we will show that each f-register-shift R is a Cantor-like set, that $R-Q\subseteq f(R-S(f))\subseteq R$ where Q is a finite set of points in I and that the orbit of each point in $R-M(f)$ is dense in R. In the second part of this section we will prove that each topologically transitive f-cycle C is in fact strongly transitive, i.e. for each open and non-empty subset U of C, $\bigcup_{n=0}^m f^n(U-S(f^n))$ is dense in C for some $m\geq 0$.

For the sequel it will be convenient to consider a new dynamical system associated with f. For $x\in I$ and $\varepsilon>0$ let $B_\varepsilon(x,1)=(x,x+\varepsilon)\cap I$ and $B_\varepsilon(x,-1)=(x-\varepsilon,x)\cap I$. For $A\subseteq I$ put

$$A^*=\{(x,\alpha)\in I\times\{1,-1\}\colon B_\varepsilon(x,\alpha)\cap A\neq\emptyset\quad\text{for each }\varepsilon>0\}.$$

Let K be an f-cycle; note that $(x,\alpha)\in K^*$ if and only if $B_\varepsilon(x,\alpha)\subseteq K$ for some $\varepsilon>0$. Let $(x,\alpha)\in I^*$ and $n\geq 0$; put $f^n(x,\alpha)=\lim_{y\downarrow x} f^n(y)$ if $\alpha=1$ and $f^n(x,\alpha)=\lim_{y\uparrow x} f^n(y)$ if $\alpha=-1$. This is well defined and clearly we have $f^n(x,\alpha)\in I$. Moreover, there exists $\varepsilon>0$ such that f^n is continuous and monotone on $B_\varepsilon(x,\alpha)$; put $\Pi^n(x,\alpha)=\alpha$ if f^n is increasing on $B_\varepsilon(x,\alpha)$ and $\Pi^n(x,\alpha)=-\alpha$ if f^n is decreasing on $B_\varepsilon(x,\alpha)$. In the following we will write $f(x,\alpha)$ (resp. $\Pi(x,\alpha)$) instead of $f^1(x,\alpha)$ (resp. $\Pi^1(x,\alpha)$). Clearly $(f(x,\alpha),\Pi(x,\alpha))\in I^*$ for each $(x,\alpha)\in I^*$.

Finally, let $F: I^*\to I^*$ be the mapping given by $F(x,\alpha)=(f(x,\alpha),\Pi(x,\alpha))$ and define the mapping $F^n: I^*\to I^*$ inductively by $F^0(x,\alpha)=(x,\alpha)$ and $F^n(x,\alpha)=F(F^{n-1}(x,\alpha))$ for all $n\geq 1$. It is not difficult to see that for all $(x,\alpha)\in I^*$ and $n,m\geq 0$ we have:

$$f^{m+n}(x,\alpha)=f^m(f^n(x,\alpha),\Pi^n(x,\alpha)),$$
$$\Pi^{m+n}(x,\alpha)=\Pi^m(f^n(x,\alpha),\Pi^n(x,\alpha))$$

and hence

$$F^n(x, \alpha) = (f^n(x, \alpha), \Pi^n(x, \alpha)).$$

Furthermore, it is easily checked that for each f-cycle K we have $F(K^*) \subseteq K^*$.

In order to show that each f-register-shift is a Cantor-like set we need the next two lemmas which will also be useful later. For $(x, \alpha) \in I^*$ put

$$O_f(x, \alpha) = \{f^n(x, \alpha) \colon n \geq 0\},$$

for $B \subseteq I^*$ let

$$O_f(B) = \{f^n(x, \alpha) \colon (x, \alpha) \in B \text{ and } n \geq 0\}$$

and for an f-cycle K put

$$S(f, K) = \{(x, \alpha) \in K^* \colon x \in S(f)\}.$$

Note that if $(x, \alpha) \in K^*$ for some f-cycle K then $O_f(x, \alpha) \subseteq K$.

LEMMA 6.1. *Let K be an f-cycle and let $c, d \in K$ with $c < d$ and $(c, d) \subseteq K$. Then for each $n \geq 0$,*

$$\partial(f^n((c, d))) \subseteq O_f(S(f, K)) \cup \{f^n(c, 1), f^n(d, -1)\}.$$

Proof. Let $n \geq 0$. If $u \in (c, d) \cap S(f^n)$ then there exists $0 \leq j < n$ with $f^j(u) \in S(f) \cap \operatorname{int}(K)$ and hence $f^n(u, \alpha) \in O_f(S(f, K))$ for all $\alpha \in \{1, -1\}$. Now let $(c, d) \cap S(f^n) = \{u_1, \ldots, u_m\}$ with $u_1 < u_2 < \cdots < u_m$. Put $u_0 = c$ and $u_{m+1} = d$. Then f^n is continuous and monotone on each of the intervals (u_i, u_{i+1}), $i = 0, 1, \ldots, m$. Thus

$$\begin{aligned} \partial(f^n((c, d))) &= \partial(f^n((c, d) - S(f^n))) \\ &\subseteq \bigcup_{1 \leq i \leq m} \partial(f^n((u_i, u_{i+1}))) \\ &\subseteq O_f(S(f, K)) \cup \{f^n(c, 1), f^n(d, -1)\}. \end{aligned}$$ □

LEMMA 6.2. *Let $\{K_n\}_{n \geq 1}$ be a generator for some f-register-shift R and let $(x, \alpha) \in I^*$ with $(x, \alpha) \in (K_n)^*$ for each $n \geq 1$. Then* $\operatorname{card}(O_f(x, \alpha)) = +\infty$.

Proof. Assume that $\operatorname{card}(O_f(x, \alpha)) < +\infty$. Then also $\operatorname{card}(\{F^n(x, \alpha) \colon n \geq 0\}) < +\infty$, and hence there exist $q \geq 0$ and $p \geq 1$ such that $F^p(y, \beta) = (y, \beta)$ where $(y, \beta) = F^q(x, \alpha)$. We have $(x, \alpha) \in (K_n)^*$ and thus $(y, \beta) \in (K_n)^*$ for each $n \geq 1$. Since by proposition 3.4, $K_1 \cap \operatorname{Sink}(f) = \varnothing$ there exists $\varepsilon > 0$ such that f^p is continuous and increasing on $B_{2\varepsilon}(y, \beta)$ and

$$\beta \cdot f^p(z) > \beta \cdot z \qquad \text{for all } z \in B_{2\varepsilon}(y, \beta).$$

Hence for all $\delta > 0$ there exists $m \geq 0$ such that $B_\varepsilon(y, \beta) \subseteq f^m(B_\delta(y, \beta) - S(f^m))$; thus $B_\varepsilon(y, \beta) \subseteq K_n$ for each $n \geq 1$ which is not possible. Therefore, $\operatorname{card}(O_f(x, \alpha)) = +\infty$. □

Let $\mathscr{C} = \{C_n\}_{n \geq 1}$ be a decreasing sequence of f-cycles and $m \geq 1$; we call a component B of C_m *$\mathscr{C}$-splitting* if for some $k > m$ B contains at least two (distinct) components of C_k. $\mathscr{C}$ is said to be *splitting* if for all $n \geq 1$ each component of C_n is $\mathscr{C}$-splitting.

PROPOSITION 6.3. *Let $\{K_n\}_{n \geq 1}$ be a generator for some f-register-shift R. Then $\{K_n\}_{n \geq 1}$ is splitting.*

Proof. Assume that $\{K_n\}_{n \geq 1}$ is not splitting. Then there exists $m \geq 1$ and a component B of K_m such that for all $n \geq m$, B contains exactly one component of K_n (since by lemma 4.1(1) B contains at least one component of K_n). For $n \geq m$ let B_n be the unique component of K_n with $B_n \subseteq B$. Then $B_{n+1} \subseteq B_n$ and card $(\bigcap_{n \geq m} B_n) = 1$. Without loss of generality we can assume that $S(f, K_n) = S(f, K_m)$ for all $n \geq m$. By proposition 4.4(2) we have $S(f, K_m) \neq \varnothing$, and thus there exists $(x, \alpha) \in I^*$ with $(x, \alpha) \in S(f, K_m) \subseteq (K_n)^*$ for all $n \geq 1$. Lemma 6.2 shows that card $(O_f(x, \alpha)) = +\infty$; hence there exist $c, d \in O_f(x, \alpha) \subseteq O_f(S(f, K_m))$ with $c < d$ such that $(c, d) \subseteq K_m$. We can find $p \geq 0$ such that $f^p((c, d)) \cap B \neq \varnothing$. By lemma 6.1 we have

$$\begin{aligned}\partial(f^p((c, d))) &\subseteq O_f(S(f, K_m)) \cup \{f^p(c, 1), f^p(d, -1)\} \\ &\subseteq O_f(S(f, K_m));\end{aligned}$$

thus there exist $u, v \in O_f(S(f, K_m))$ with $(u, v) \subseteq B$. But since $O_f(S(f, K_m)) \subseteq K_n$ for all $n \geq 1$ and B is not $\{K_n\}_{n \geq 1}$-splitting, it follows that $(u, v) \subseteq B_n$ for all $n \geq m$ which is not possible. □

Note that by the above proposition each point of an f-register-shift R is a limit point of R. Since in addition R is non-empty, closed and nowhere dense, R is a Cantor-like set. In particular, for each $x \in R$ there exists $\alpha \in \{1, -1\}$ with $(x, \alpha) \in R^*$. The next proposition shows that f acts minimally on each f-register-shift.

PROPOSITION 6.4. *Let R be an f-register-shift. Then $\overline{O_f(x, \alpha)} = R$ for each $(x, \alpha) \in R^*$.*

Proof. Let $(x, \alpha) \in R^*$, $\{K_n\}_{n \geq 1}$ be a generator for R and for each $n \geq 1$ and $(\varphi, \beta) \in S(f, K_n)$ let $K_n(\varphi, \beta)$ denote the unique component of $\operatorname{int}(K_n) - S(f)$ with $(\varphi, \beta) \in (K_n(\varphi, \beta))^*$. We can find $m \geq 1$ such that for each $(\varphi, \beta) \in S(f, K_m)$ either

$$O_f(x, \alpha) \cap (\overline{K_m(\varphi, \beta)} - \{\varphi\}) = \varnothing$$

or

$$O_f(x, \alpha) \cap K_n(\varphi, \beta) \neq \varnothing$$

for all $n \geq m$. Now let $z \in R$ and for $n \geq m$ let B_n denote the unique component of K_n with $z \in B_n$. Then $\{z\} = \bigcap_{n \geq m} B_n$. Since $(x, \alpha) \in (K_n)^*$ we have $O_f(x, \alpha) \subseteq K_n$ for each $n \geq 1$ and thus $O_f(x, \alpha) \subseteq R$. Hence it is sufficient to show that $O_f(x, \alpha) \cap B_n \neq \varnothing$ for all $n \geq m$. So let $n \geq m$. Since $(x, \alpha) \in R^* \subseteq (K_j)^*$ for all $j \geq 1$, lemma 6.2 shows that there exist $c, d \in O_f(x, \alpha)$ with $c < d$ and $(c, d) \subseteq K_n$. Moreover, we can find $p \geq 0$ such that

$$f^p((c, d) - S(f^p)) \cap B_n \neq \varnothing.$$

Hence $O_f(x, \alpha) \cap B_n \neq \varnothing$ follows if we can show that $\partial(f^p((c, d) - S(f^p))) \subseteq \overline{O_f(x, \alpha)}$. Let $u, v \in \overline{O_f(x, \alpha)}$ with $(u, v) \subseteq K_n$ and let $w \in \partial(f((u, v) - S(f)))$. Then either $w = f(u)$, $w = f(v)$ or $w = f(\varphi, \beta)$ for some $(\varphi, b) \in S(f, K_n)$. If $w = f(u)$ or $w = f(v)$ then clearly $w \in \overline{O_f(x, \alpha)}$. If $w = f(\varphi, \beta)$ for some $(\varphi, \beta) \in S(f, K_n)$ then by the choice of m we have $K_j(\varphi, \beta) \cap O_f(x, \alpha) \neq \varnothing$ for all $j \geq m$; hence $w \in \overline{O_f(x, \alpha)}$ (since $\bigcap_{j \geq m} \overline{K_j(\varphi, \beta)} = \{\varphi\}$). Repeating this argument shows that $\partial(f^k((u, v) - S(f^k))) \subseteq \overline{O_f(x, \alpha)}$ for all $k \geq 0$; in particular, we have

$$\partial(f^p((c, d) - S(f^p))) \subseteq \overline{O_f(x, \alpha)}. \qquad \square$$

Let R be an f-register-shift; the above result gives us that $O_f(x)$ is dense in R for all $x \in R - M(f)$ (note that since $M(f)$ is countable and R is a Cantor-like set,

$R-M(f)$ is uncountable). Next we will show that R is the union of $f(R-S(f))$ and a finite set. For this we need the following lemma which will also be useful later.

LEMMA 6.5. *Let $A\subseteq I$ be closed and let $(y,\beta)\in(\overline{f(A-S(f))})^*$. Then there exists $(x,\alpha)\in A^*$ with $F(x,\alpha)=(y,\beta)$.*

Proof. For each $n\geq 1$ there exists $y_n\in f(A-S(f))\cap B_{1/n}(y,\beta)$. For $n\geq 1$ let $x_n\in A$ with $f(x_n)=y_n$ and let x be a point of accumulation of $\{x_n\}_{n\geq 1}$. Then there exists $\alpha\in\{1,-1\}$ such that $B_\varepsilon(x,\alpha)\cap\{x_n\colon n\geq 1\}\neq\varnothing$ for each $\varepsilon>0$, and so $(x,\alpha)\in A^*$ and $F(x,\alpha)=(y,\beta)$. □

PROPOSITION 6.6. *Let R be an f-register-shift. Then*:

(1) $\overline{f(R-S(f))}=R$.

(2) $f(R-S(f))\cup\{f(x,\alpha)\colon(x,\alpha)\in R^*$ *and* $x\in S(f)\}=R$.

(3) $F(R^*)=R^*$.

Proof. (1) Let $\{K_n\}_{n\geq 1}$ be a generator for R. Then

$$f(R-S(f))\subseteq f(K_n-S(f))\subseteq K_n \qquad \text{for each } n\geq 1;$$

hence $f(R-S(f))\subseteq R$. Let $x\in R-M(f)$; then $O_f(f(x))\subseteq f(R-S(f))\subseteq R$ and thus by proposition 6.4 we have $\overline{f(R-S(f))}=R$.

(2) Clearly by (1) we have

$$f(R-S(f))\cup\{f(x,\alpha)\colon(x,\alpha)\in R^* \text{ and } x\in S(f)\}\subseteq\overline{f(R-S(f)}=R.$$

Now let $y\in R-f(R-S(f))$; then $(y,\beta)\in R^*=(\overline{f(R-S(f))})^*$ for some $\beta\in\{1,-1\}$, and so by lemma 6.5 there exists $(x,\alpha)\in R^*$ with $f(x,\alpha)=y$. Since $y\notin f(R-S(f))$ we must have $x\in S(f)$.

(3) Let $(x,\alpha)\in R^*$; then $f(x,\alpha)\in R$ and for all $\varepsilon>0$ we have $B_\varepsilon(f(x,\alpha),\Pi(x,\alpha))\cap R\neq\varnothing$. Hence $F(x,\alpha)\in R^*$. On the other hand, let $(y,\beta)\in R^*$. Then by (1) $(y,\beta)\in(\overline{f(R-S(f))})^*$ and so by lemma 6.5 there exists $(x,\alpha)\in R^*$ with $F(x,\alpha)=(y,\beta)$. □

Let $A\subseteq I$ be non-empty; for $x\in I$ put $d(x,A)=\inf\{|x-y|\colon y\in A\}$.

Let R be an f-register-shift; the next result shows that each element of $A(R,f)$ is attracted to R.

PROPOSITION 6.7. *Let R be an f-register-shift and $\{K_n\}_{n\geq 1}$ be a generator for R. Then*:

(1) *For each $\varepsilon>0$ there exists $m\geq 1$ such that if $x\in A(K_m,f)$ and $\alpha\in\{1,-1\}$ then* $\limsup_{n\to\infty} d(f^n(x,\alpha),R)\leq\varepsilon$.

(2) *If $x\in A(R,f)$ and $\alpha\in\{1,-1\}$ then* $\lim_{n\to\infty} d(f^n(x,\alpha),R)=0$.

Proof. (1) Let $\varepsilon>0$; there exists $m\geq 1$ such that the length of each component of K_m is smaller than ε and

$$\operatorname{int}(K_m)\cap S(f)=\operatorname{int}(K_n)\cap S(f) \qquad \text{for all } n\geq m.$$

Let $x\in A(K_m,f)$ and $\alpha\in\{1,-1\}$. If $x\in A(K_m,f)\cap M(f)$ then there exists $p\geq 0$ such that $f^p(x)\in\operatorname{int}(K_m)\cap S(f)\subseteq R$; thus $f^n(x,\alpha)\in R$ for all $n\geq p$.

If $x\in A(K_m,f)-M(f)$ then there exists $p\geq 0$ such that $f^n(x)\in K_m$ for all $n\geq p$. Hence in any case $\limsup_{n\to\infty} d(f^n(x,\alpha),R)\leq\varepsilon$.

(2) This follows immediately from (1). □

Let R be an f-register-shift; one can show that if $x \in I - M(f)$ with $\lim_{n\to\infty} d(f^n(x), R) = 0$ then it does not necessarily follow that $x \in A(R, f)$.

Let C be an f-cycle; we call C *strongly transitive* if for each non-empty open subset U of C we have $\bigcup_{n=0}^{m} f^n(U - S(f^n)) = C$ for some $m \geq 0$.

In order to prove that each topologically transitive f-cycle is strongly transitive we need the following result.

PROPOSITION 6.8. *Let $\varphi \in S(f)$ and suppose that $\overline{L_m(\varphi)}$ is a topologically transitive f-cycle for some $m \geq 1$. Then there exists $n \geq 0$* such that

$$L_m(\varphi) = \bigcup_{k=1}^{n} f^k(I_m(\varphi) - S(f^k)).$$

Proof. By proposition 4.3 we have $L_m(\varphi) \subseteq \overline{M(f)}$; hence by lemma 5.3, $L_m(\varphi) = (c_1, d_1) \cup \cdots \cup (c_q, d_q)$ can be written as a disjoint and finite union of non-empty open intervals. Put $B(1) = \{c_1, \ldots, c_q\}$ and $B(-1) = \{d_1, \ldots, d_q\}$. For $n \geq 0$ let

$$E_n = \bigcup_{0 \leq k \leq n} f^k(I_m(\varphi) - S(f^k));$$

then $L_m(\varphi) = \bigcup_{n\geq 0} E_n$, $E_n \subseteq E_{n+1}$ and $f^j(E_n - S(f^j)) \subseteq E_{n+j}$ for all $n, j \geq 0$. For $\alpha \in \{1, -1\}$ put

$$R(\alpha) = \{z \in B(\alpha): (z, \alpha) \notin (\overline{E_n})^* \text{ for all } n \geq 0\}.$$

Now let us assume that $L_m(\varphi) \neq E_n$ for all $n \geq 0$. Then $R(1) \cup R(-1) \neq \varnothing$. Let $\varepsilon > 0$; for $\alpha \in \{1, -1\}$ and $u \in B(\alpha)$ put $u(\varepsilon) = u + \alpha \cdot \varepsilon$ if $u \in R(\alpha)$ and $u(\varepsilon) = u$ if $u \notin R(\alpha)$. Furthermore, let $D(\varepsilon) = \bigcup_{1\leq i\leq q} (c_i(\varepsilon), d_i(\varepsilon))$. By compactness there exists $k \geq 0$ such that $\bigcup_{1\leq i\leq q} (c_i + \varepsilon, d_i - \varepsilon) \subseteq E_k$. Hence there exists $j \geq 0$ such that $D(\varepsilon) \subseteq E_j$.

We will first show that for each $u \in R(\alpha)$ there exists $j \geq 1$ such that $F^j(u, \alpha) = (u, \alpha)$. Let $u \in R(\alpha)$ for some $\alpha \in \{1, -1\}$. Since $\overline{L_m(\varphi)}$ is a topologically transitive f-cycle we have

$$\overline{f(\overline{L_m(\varphi)} - S(f))} = \overline{L_m(\varphi)};$$

hence by lemma 6.5 there exists $(u_1, \alpha_1) \in (\overline{L_m(\varphi)})^*$ with $F(u_1, \alpha_1) = (u, \alpha)$. Then $(u_1, \alpha_1) \notin (\overline{E_n})^*$ for all $n \geq 0$ and thus $u_1 \in R(\alpha_1)$. Therefore, we can inductively find a sequence $\{(u_n, \alpha_n)\}_{n\geq 0}$ of elements of $(\overline{L_m(\varphi)})^*$ such that $(u_0, \alpha_0) = (u, \alpha)$, $u_n \in R(\alpha_n)$ and $F(u_{n+1}, \alpha_{n+1}) = (u_n, \alpha_n)$ for each $n \geq 0$. Since card $(R(1) \cup R(-1)) < +\infty$ there exist $n \geq 0$ and $j \geq 1$ such that $(u_{j+n}, \alpha_{j+n}) = (u_n, \alpha_n)$. Hence $(u_j, \alpha_j) = (u_0, \alpha_0)$ and thus $F^{lj}(u, \alpha) = (u, \alpha)$ for each $l \geq 0$. Since by proposition 4.3(1) int $(\overline{L_m(\varphi)}) \cap \operatorname{Sink}(f) = \varnothing$ there exist $N \geq 1$ and $\varepsilon > 0$ such that

(i) for each $v \in R(\beta), \beta \in \{1, -1\}$, f^N is continuous and increasing on $B_\varepsilon(v, \beta)$, $F^N(v, \beta) = (v, \beta)$ and $\beta \cdot f^N(z) > \beta \cdot z$ for all $z \in B_\varepsilon(v, \beta)$

(ii) $I_m(\varphi) \cup (L_m(\varphi) \cap S(f^N)) \subseteq D(\varepsilon)$.

Furthermore, there exist $p, k \geq 1$ and $0 < \delta < \varepsilon$ such that

$$D(\varepsilon) \subseteq E_k \subseteq E_{k+N} \subseteq D(\delta) \subseteq E_p.$$

If $f^N(D(\delta) - S(f^N)) \subseteq D(\delta)$ then (since $I_m(\varphi) \subseteq D(\delta)$) we would have $L_m(\varphi) \subseteq \bigcup_{i=0}^{N-1} f^i(D(\delta) - S(f^i)) \subseteq E_{p+N-1}$ which contradicts our assumption. Hence there exists $x \in D(\delta)$ with $f^N(x) \in L_m(\varphi) - D(\delta)$. Without loss of generality let us assume that f^N is increasing in x and that $f^N(x) \in (c, c+\delta]$ for some $c \in R(1)$. Put

$$y = \max\{z \leq x : z \in S(f^N) \cup B(1)\};$$

so $y < x$ and $f^N((y, x)) \subseteq (c, c+\delta)$. If $y \in S(f^N)$ or $y \in B(1) - R(1)$ then we would have $(y, x) \cap E_k \neq \varnothing$ and thus $f^N((y, x)) \cap E_{k+N} \neq \varnothing$; but this is not possible since $E_{k+N} \cap (c, c+\delta) = \varnothing$. Hence $y \in R(1)$. Since $x \in D(\delta)$ we have $B_\delta(y, 1) \subseteq (y, x)$ and so $f^N(B_\delta(y, 1)) \subseteq (c, c+\delta)$. But again this is not possible since by (i) the length of the interval $f^N(B_\delta(y, 1))$ is strictly larger than δ. Therefore, $L_m(\varphi) = E_n$ for some $n \geq 0$. □

PROPOSITION 6.9. *C is a topologically transitive f-cycle if and only if C is a strongly transitive f-cycle.*

Proof. If C is a strongly transitive f-cycle then clearly, by proposition 4.2, C is topologically transitive. Conversely, suppose that C is a topologically transitive f-cycle. By theorem 5.9 there exist $\varphi \in S(f)$ with $\operatorname{int}(\overline{\Delta(\varphi)}) \neq \varnothing$ and $m \geq 1$ such that $C = \overline{L_n(\varphi)}$ for all $n \geq m$. Now let $U \subseteq C$ be non-empty and open. Since $U \subseteq C \subseteq \overline{\Delta(\varphi)}$ we have $I_j(\varphi) \subseteq f^q(U - S(f^q))$ for some $q \geq 0$ and $j \geq m$. Hence by proposition 6.8 there exists $p \geq q$ such that

$$L_j(\varphi) \subseteq \bigcup_{k=0}^{p} f^k(U - S(f^k)).$$

Since $\overline{L_j(\varphi)} = C$ this shows that C is strongly transitive. □

Remark. We call $(x, \alpha) \in I^*$ periodic if $F^n(x, \alpha) = (x, \alpha)$ for some $n \geq 1$. We say that $g \in \mathcal{N}(I)$ is uniformly piecewise linear with slope $\eta > 0$ if on each component U of $I - S(f)$ g is linear with slope η or $-\eta$. Suppose that I is a strongly transitive f-cycle, and assume further that either $f(U) \cap f(V) \neq \varnothing$ for some components U and V of $I - S(f)$ with $U \neq V$ or that I^* contains no periodic element. As in [**13**, Theorem 5] it can be shown that f is then conjugate to a uniformly piecewise linear mapping $g \in \mathcal{N}(I)$ (with slope $\eta \geq 1$), i.e. there exists a homeomorphism ψ of I' such that $\psi f = g\psi$.

Let C be an f-cycle and assume that there exists a continuous function $g: C \to C$ such that $f(x) = g(x)$ for all $x \in C - S(f)$. Then we can label the components $B_1, \ldots, B_m$ of C in such a way that $g(B_i) \subseteq B_{i+1}$ for $1 \leq i \leq m-1$ and $g(B_m) \subseteq B_1$. For $n \geq 0$ define inductively $g^0(x) = x$ and $g^{n+1}(x) = g(g^n(x))$ for all $x \in C$. Let $1 \leq i \leq m$; then $g^m(B_i) \subseteq B_i$, and if in addition C is strongly transitive then for each non-empty open subset $U \subseteq B_i$ there exists $p \geq 0$ such that $\bigcup_{k=0}^{p} g^{mk}(U) = B_i$. The next result shows that in this situation proposition 6.9 can be improved.

PROPOSITION 6.10. *Let C be a topologically transitive f-cycle with m components, let B be a component of C and suppose that there exists a continuous function $g: C \to C$ such that $g(x) = f(x)$ for all $x \in C - S(f)$. Then exactly one of the following statements holds*:

(1) *For each non-trivial interval $J \subseteq B$ there exists $k \geq 0$ with $g^{km}(J) = B$.*

(2) *There exists a closed interval D with $D \cup g^m(D) = B$, $\operatorname{int}(D) \cap \operatorname{int}(g^m(D)) = \varnothing$ and $g^{2m}(D) = D$ such that for each non-trivial interval $J \subseteq D$ there exists $k \geq 0$ with $g^{km}(J) = D$.*

Proof. Put $B=[c, d]$ and $h=g^m$. By Proposition 6.9, C is strongly transitive; in particular, we have $h(B)=B$ and $h(u)=u$ for some $u\in(c, d)$. Suppose that there exists a non-trivial interval $J\subseteq B$ such that $h^n(J)\neq B$ for all $n\geq 0$. Since C is strongly transitive there exists $j\geq 0$ such that $u\in h^n(J)$ for all $n\geq j$; hence $[c, u]\subseteq h^p(J)$ for some $p\geq j$. Since $h^n(J)\neq B$ for all $n\geq 0$ we have $c\notin h([c, u])$, and thus $c\in h([u, d])$. The same argument shows that $d\in h([c, u])$. Therefore, for all $n\geq 0$ we have

$$[c, u]\subseteq h^{2n}([c, u])\subseteq h^{p+2n}(J)\subseteq[c, d)$$

and

$$[u, d]\subseteq h^{2n+1}([c, u])\subseteq h^{p+2n+1}(J)\subseteq(c, d].$$

For $n\geq 0$ put

$$D_n=h^{p+2n}(J)\cap h^{p+2n+1}(J).$$

Then D_n is an interval with $u\in D_n$ and $h(D_n)\subseteq D_{n+1}\subseteq(c, d)$ for each $n\geq 0$. Hence $D_n=\{u\}$ for all $n\geq 0$ because C is strongly transitive. Therefore, putting $D=[c, u]$ we have $h(D)=[u, d]$ and $h^2(D)=D$. Now let $J'\subseteq D$ be a non-trivial interval; as above we have $[c, u]\subseteq h^k(J')$ for some $k\geq 0$, and thus $[c, u]\subseteq h^k(J')\subseteq h^k(D)=[c, u]$. Finally, it is clear that (1) and (2) cannot both hold.

7. *Some examples*

In this section we want to apply our results to some examples. We will make use of the following fact.

PROPOSITION 7.1. *Suppose that* Sink $(f)=$ Homt $(f)=\varnothing$ *and that I is the only f-cycle K with* int $(K)\cap S(f)\cap(a, b)\neq\varnothing$. *Then I is strongly transitive.*

Proof. By proposition 4.4(2) there are no f-register-shifts. Thus by theorem 5.1 there exists a topologically transitive f-cycle K. We have $K=I$ since by proposition 4.3(3) int $(K)\cap S(f)\cap(a, b)\neq\varnothing$, and by proposition 6.9 I is strongly transitive. □

Suppose that $g\in\mathcal{N}([0, 1])$ is given by $g(x)=\beta x \bmod 1$ for all $x\in(0, 1)$ with $\beta x\notin\mathbb{N}$ and some $\beta>1$ (such transformations were discussed in [17]). Then I is a strongly transitive g-cycle; this follows immediately from the next corollary.

For an interval $J\subseteq I$ let $|J|$ denote the length of J.

COROLLARY 7.2. *Suppose that $f(a, 1)=a$, $f(\varphi, 1)=a$ and $f(\varphi, -1)=b$ for all $\varphi\in S(f)\cap(a, b)$ and that $|f(J)|>|J|$ for each interval $J\subseteq I$ with $J\cap S(f)=\varnothing$. Then I is a strongly transitive f-cycle.*

Proof. Since $|f(J)|>|J|$ for each interval $J\subseteq I$ with $J\cap S(f)=\varnothing$ we clearly have Sink $(f)=$ Homt $(f)=\varnothing$. Let K be an f-cycle with int $(K)\cap S(f)\cap(a, b)\neq\varnothing$. Then $f(\varphi, 1)=a$ for each $\varphi\in S(f)\cap(a, b)$ implies that $a\in K$. Since $f(a, 1)=a, f(\varphi, -1)=b$ for each $\varphi\in S(f)\cap(a, b)$ and Sink, $(f)=\varnothing$ it follows that $K=I$. Therefore, by proposition 7.1, I is strongly transitive. □

The next class of function we consider contains the Poincaré map of the geometric Lorentz attractor already mentioned in the Introduction.

COROLLARY 7.3. *Suppose that f is continuous and strictly increasing on (a, φ) and (φ, b) for some $a < \varphi < b$, that $f(\varphi, 1) = a$ and $f(\varphi, -1) = b$ and that $|f(J)| > \sqrt{2}|J|$ for each interval $J \subseteq I$ with $J \cap S(f) = \emptyset$. Then I is a strongly transitive f-cycle.*

Proof. Since $|f(J)| > \sqrt{2}|J|$ for each interval $J \subseteq I$ with $J \cap S(f) = \emptyset$ we have Sink $(f) =$ Homt $(f) = \emptyset$. Let K be an f-cycle with $\varphi \in \operatorname{int}(K) \cap S(f) \cap (a, b)$ and let $B = [c, d]$ be the component of K with $\varphi \in B$. Put $U_1 = (c, \varphi)$ and $U_2 = (\varphi, d)$. Without loss of generality we can assume that $|U_1| \geq |U_2|$ (otherwise 'turn f upside down'). Put $n = \min\{k \geq 1 : f^k(U_1) \cap B \neq \emptyset\}$. Then $f^n(U_1) \subseteq B$, and we have

$$|B| > (\sqrt{2})^n \cdot |U_1| \geq \tfrac{1}{2}(\sqrt{2})^n \cdot |B|;$$

hence $n = 1$ and $d = b$ (because $f(\varphi, -1) = b$). Moreover, we have $\sqrt{2}(\varphi - a) < b - a$; thus

$$|U_2| > b - (\tfrac{1}{2}\sqrt{2}(b-a) + a) = (1 - \tfrac{1}{2}\sqrt{2})(b - a)$$

and

$$|f(U_2)| > (\sqrt{2} - 1)(b - a).$$

Therefore, $|B| + |f(U_2)| > b - a$ which gives us that $f(U_2) \cap B \neq \emptyset$ and thus $f(U_2) \subseteq B$. Hence B is the only component of K. Since $f(\varphi, 1) = a$ we have $c = a$ and therefore $K = I$. Thus, by proposition 7.1, I is strongly transitive. □

Finally we will apply our results to interval exchange transformations (see for instance [**4**] or [**8**]).

Let $S(f) = \{d_0, , , , . d_{m+1}\}$ with $a = d_0 < d_1 < \cdots < d_{m+1} = b$. f is said to be an *interval exchange transformation* (on I) if f 'exchanges' the open intervals (d_k, d_{k+1}), $k = 0, \ldots, m$ according to a permutation of $\{0, 1, \ldots, m\}$, i.e. if

(7.1) f is linear with slope 1 on each of the open intervals (d_k, d_{k+1}), $k = 0, \ldots, m$; and

(7.2) $f((d_k, d_{k+1})) \cap f((d_j, d_{j+1})) = \emptyset$ if $0 \leq k \leq j \leq m$.

Now suppose that f is an interval exchange transformation. Then $F^n(x, 1) = (f^n(x, 1), 1)$ for all $n \geq 1$, $x \in [a, b)$, $F : I^* \to I^*$ is bijective and

$$f(I - S(f)) \cup \{(\varphi, 1) : \varphi \in S(f) - \{b\}\} = [a, b).$$

Moreover, f^n is also an interval exchange transformation for each $n \geq 1$, and it is not difficult to see that for each f-cycle K we have $f^{-1}(K) \subseteq K$; in particular, this gives us that $A(K, f) \subseteq K$.

PROPOSITION 7.4. *Suppose that f is an interval exchange transformation. Then*:

(1) *If $x \in I$ with $f^n(x) = x$ for some $n \geq 1$ then x is contained in some sink of f.*

(2) Sink $(f) = \{x \in I$: *there exists $n \geq 1$ and $u, v \in S(f^n)$ with $u < x < v$ such that $f^n(z) = z$ for all $z \in (u, v)\}$.*

(3) Homt $(f) = \emptyset$.

Proof. (1) Suppose that $x \in I$ with $f^n(x) = x$ for some $n \geq 1$. Then there exist $u, v \in S(f^n)$ with $u < x < v$ and $(u, v) \cap S(f^n) = \emptyset$. Since f^n is linear on (u, v) with slope 1 we have $f^n(z) = z$ for all $z \in (u, v)$. Hence (u, v) is a sink of f with $x \in (u, v)$.

(2) Clearly, if $u, v \in S(f^n)$ with $n \geq 1$, $u < v$ and $f^n(z) = z$ for all $z \in (u, v)$ then (u, v) is a sink of f and thus $(u, v) \subseteq \text{Sink}(f)$. Conversely, let J be a sink of f and let $n \geq 1$ with $f^n(J) \subseteq J$. Then there exist $u, v \in S(f^n)$ with $J \subseteq (u, v)$ and $(u, v) \cap S(f^n) = \varnothing$. Since f^n is linear on (u, v) with slope 1 we have $f^n(z) = z$ for all $z \in (u, v)$. Moreover, since $f^n(J) = J$ it follows that

$$\{x \in I: f^n(x) \in J \text{ for some } n \geq 0\} = \textstyle\bigcup_{k=0}^{n-1} f^k(J),$$

and thus each element of Sink (f) is already contained in a sink of f.

(3) This follows immediately from Proposition 3.3(3) and from the fact that if $J \subseteq I$ is an interval with $J \cap S(f) = \varnothing$ then $|J| = |f(J)|$. □

COROLLARY 7.5. *Suppose that f is an interval exchange transformation and let $C_1, \ldots, C_r$ be the topologically transitive f-cycles. Then*:

(1) *There are no f-register-shifts.*

(2) $\text{Sink}(f) \cup C_1 \cup \cdots \cup C_r$ *is dense in I.*

Proof. (1) Let R be an f-register-shift. By theorem 5.9 and proposition 5.6 there exists $\varphi \in S(f)$ with int $\overline{\Delta(\varphi)} \neq \varnothing$ and $m \geq 1$ such that $R = \bigcap_{n \geq 1} \overline{L_n(\varphi)}$ and $\overline{L_n(\varphi)} \subseteq \overline{\Delta(\varphi)}$ for all $n \geq m$. But since $f^{-1}\ (\overline{L_n(\varphi)}) \subseteq \overline{L_n(\varphi)}$ we have $\Delta(\varphi) \subseteq \overline{L_n(\varphi)}$ and thus $\overline{L_n(\varphi)} = \overline{\Delta(\varphi)}$ for each $n \geq m$. This is not possible since int $(R) = \varnothing$.

(2) Since $A(K, f) \subseteq K$ for each f-cycle K and since by proposition 7.4(3), Homt $(f) = \varnothing$ this follows immediately from theorem 5.1. □

Finally, we show that if the orbits $O_f(\varphi, 1)$ of the singular points φ in (a, b) are infinite and disjoint then I is a strongly transitive f-cycle.

COROLLARY 7.6 (cf. [8].) *Suppose that f is an interval exchange transformation with $S(f) \cap (a, b) \neq \varnothing$ and assume that* card $(O_f(\varphi, 1)) = +\infty$ *and $f^n(\varphi, 1) \notin S(f) \cap (a, b)$ for all $\varphi \in S(f) \cap (a, b)$ and $n \geq 1$. Then I is a topologically transitive f-cycle. Moreover, we have $O_f(x, 1)$ is dense in I for each $x \in [a, b)$.*

Proof. We will first show that Sink $(f) = \varnothing$. Assume that Sink $(f) \neq \varnothing$. By proposition 7.4(2) there exist $n \geq 1$ and $u, v \in S(f^n)$ with $u < v$ such that $f^n(z) = z$ for all $z \in (u, v)$; then $f^k(u) \in S(f)$ for some $0 \leq k < n$, Put $\varphi = f^k(u)$; so $\varphi < b$ and $f^n(\varphi, 1) = \varphi$. Thus by assumption we have $\varphi = a$. Since $a = f(\eta, 1)$ for some unique $\eta \in S(f) \cap [a, b)$ it follows that $f^n(\eta, 1) = \eta$. But again this is only possible if $\eta = a$; hence $f(a, 1) = a$, and thus $f(z) = z$ for all $z \in (a, \xi)$ where $\xi = \min\ (S(f) \cap (a, b))$. But then there exists $\psi \in S(f) \cap [\xi, b)$ with $f(\psi, 1) = \xi$, which contradicts our assumption. Therefore, Sink $(f) = \varnothing$.

Let $C_1, \ldots, C_r$ be the topologically transitive f-cycles. Then by corollary 7.5(2) $I = C_1 \cup \cdots \cup C_r$. Assume that $r \geq 2$. Then $C_1 \cap C_j \neq \varnothing$ for some $1 \leq i < j \leq r$. Let $z \in C_i \cap C_j$. Suppose first that $z \notin M(f)$. Then $f^n(z) \in C_i \cap C_j$ for all $n \geq 0$, and thus there would exist $p \geq 1$ and $m \geq 0$ such that $f^p(f^m(z)) = f^m(z)$; by proposition 7.4(1) this is not possible. Suppose next that $z \in M(f)$; then $f^m(z) \in S(f) \cap (a, b)$ for some $m \geq 0$ and there exists $(x, 1) \in I^*$ with $f(x, 1) = z$. Since $f^n(\varphi, 1) \notin S(f) \cap (a, b)$ for all $\varphi \in S(f) \cap [a, b)$ and $n \geq 1$ ($f^n(a, 1) \notin S(f) \cap (a, b)$ for all $n \geq 1$ because $a = f(\xi, 1)$ for some $\xi \in S(f) \cap (a, b)$) it follows that $x \notin S(f)$. Thus $x \in f^{-1}(C_i \cap C_j) \subseteq C_i \cap C_j$.

Repeating this argument shows that there exists $p \geq 1$ with $f^p(x) = x$ which by proposition 7.4(1) is not possible. Hence $r = 1$.

Let $x \in [a, b)$ and let $J \subseteq I$ be a non-trivial closed interval. By proposition 6.9 I is strongly transitive; thus we have $I = \bigcup_{n=0}^{m} \overline{f^n(J - S(f^n))}$ for some $m \geq 0$. Let $z = f^m(x, 1)$. It follows from lemma 6.5 that there exist $y \in J$ and $0 \leq k \leq m$ such that $f^k(y, 1) = z$. But this immediately gives us that $f^{m-k}(x, 1) = y \in J$. Therefore, $O_f(x, 1)$ is dense in I. □

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