7

Quantum Register Dynamics

7.1 Introduction

In this chapter we move beyond the classical register scenario discussed in the previous chapter, extending the discussion to experiments described by time-dependent quantum registers of varying rank. We apply our previous discussion of the signal basis representation (SBR), the computational basis representation (CBR), signal operators, signal classes, and the CBR of signal operators to the quantum case. Our discussion of dynamics covers persistence, that is, the stability of apparatus, observers, and laboratory time, and the Born probability rule. We state the principles of quantized detector network (QDN) dynamics and show how they apply to the description of quantum experiments. We discuss the signal theorem and path summations.

7.2 Persistence

Our first step is to clarify what QDN assumes about the evolution of apparatus in time, because this affects the modeling. QDN is designed to reflect the behavior of apparatus in the real world and so it is not assumed in general that a given observer's apparatus is constant in time, even during a given run of an experiment.

Although many experiments appear to be carried out with apparatus that persists over any given run of the experiment, and indeed, perhaps over all the runs of that experiment, that is just an incidental factor that reflects no more than an economy in construction. In practice it is invariably easier and more economical to use the same equipment over and over again rather than use it once, throw it away after each run, and build a new version ready for the next run. Lest this be thought of as a trivial point, it is nevertheless an integral and costly feature of many experiments, involving maintenance and upgrading. Indeed, in laboratories such as the Large Hadron Collider, actual run time is a

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small fraction of total project time. A related, significant issue has to do with the concept of *ensemble*, discussed in the Appendix.

Persistence has everything to do with *time scale*, specifically, the relative laboratory time over which any piece of equipment can be meaningfully discussed as such. The degree of persistence of apparatus is as contextual as anything else in physics. If a run is relatively brief, say, over in very small fractions of a second, as in high-energy particle scattering experiments, then in such an experiment, the apparatus will behave as if it persists forever. On the other hand, in some experiments, individual runs can involve enormous intervals of time between state preparation and observation, as always happens in the case of astrophysical observations of stars and galaxies. In such cases, light from a distant star may be received by astronomers long after that star had ceased to exist as a star.

A necessary criterion for an experiment to be describable by QDN is that any detector exists at least during a given stage. Indeed, we can take the definition of a stage to be that interval of laboratory time over which a given detector can be assumed to persist, that is, have an identity that has physical relevance in the context of the experiment concerned.

In this respect, a stage is to be identified not with a moment or point in time but with the *interval* of time over which information could be extracted by the observer. Such an interval is of contextual temporal length in the laboratory, being as short or as long as the observer requires to acquire a bit or "quantum" of information. The temporal divide between two stages will be called a *quantum tick*, or *q-tick*. It is necessarily a heuristic concept, because any attempt to empirically measure a q-tick and assign a numerical value to a *chronon*, or assumed fundamental unit of time, will be self-referential in some way or other. We note that the concept of *Planck time*, commonly regarded as the shortest meaningful interval of time in physics, is a theoretical construct that could never be observed directly and has been criticized on such grounds (Meschini, 2007).

7.3 Quantized Detector Networks

In the projection-valued measure (PVM) formulation of standard quantum mechanics (QM) (von Neumann, 1955; Peres, 1995), it is generally asserted that state vectors of systems under observation (SUOs) evolve in Hilbert spaces of fixed dimension. Any time dependence of the apparatus itself, such as externally imposed time-dependent electromagnetic fields, is encoded into an explicit time dependence of the Hamiltonian or whatever observables are involved. This approach encodes the idea that experiments are done in fixed laboratories and that information is extracted from states of SUOs that evolve unitarily in time.

This temporal architecture is known as the Schrödinger picture and is widely used in QM. A variant but nominally equivalent temporal architecture is that of the *Heisenberg picture*, in which the quantum states appear frozen in time but

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now it is the observables of the theory that have unitary temporal evolution over and above any intrinsic, explicit evolution.

In standard QM, these two related architectures were eventually recognized as too limited. One reason is that in the PVM approach, the number of mutually orthogonal possible outcome states associated with a given observable is the number of its eigenstates, which is fixed and cannot exceed the dimension of the Hilbert space concerned. The positive operator-valued measure (POVM) formalism on the other hand was developed to deal with the possibility that the number of observational outcome channels set up in a laboratory may be different from the dimension of the Hilbert space used (Ludwig, 1983a,b; Kraus, 1983; Peres, 1995) and, indeed, may exceed that number, in the case of finite dimensions.

In contrast, the QDN formalism assumes from the outset that the Hilbert space representing outcome possibilities is always different from one stage to the next, even if the dimensionality remains constant, and even if the detectors involved in the experiment appear to persist over several or all stages of an experiment.

To be specific, let us denote the observer's apparatus at any given stage Σ_n by $\mathcal{A}_n$. Some parts of $\mathcal{A}_n$ will correspond to actual, real detectors, while other parts will be virtual, or potential, detectors. Whatever the case, $\mathcal{A}_n$ will be always be modeled by a countable number r_n of vertices at any stage of the relevant stage diagram for that experiment. In this book, we denote the ith vertex at stage Σ_n by i_n, where the superscript i runs from 1 to r_n and labels all of the distinct vertices in that stage. Note that i_n (a vertex) should not be confused with $\boldsymbol{i}_n$ (bold font), our notation for a CBR element of the preferred basis at stage Σ_n. Note also that detector vertices should not be confused with *modules*, which are nondetecting parts of apparatus that sit in the information void between stages.

In the description of real experiments, r_n will always be finite, in contrast to the situation in QM, where the Hilbert space may be infinite dimensional. The harmonic oscillator in one spatial dimension is an example where QM assumes either that the position observable has a continuous spectrum or, equivalently, that the Hamiltonian has a countable infinity of energy eigenstates. The harmonic oscillator is discussed in Chapter 24. In the world of empirical reality, there are no such harmonic oscillators, just SUOs for which such a description may be a reasonable approximation.

The question as to whether r_n is finite or not is central to many if not all of the technical difficulties encountered in the refinement of standard QM known as quantum field theory. The harmonic oscillator appears to be intimately involved in many of these problems in one way or another. Although the mathematical properties of the quantized harmonic oscillator play an essential role in accounting for the particle concept in free quantum field theory, those same properties generate fundamental problems in interacting field theories. For instance, the ultraviolet divergences encountered in most Feynman loop integrals are linked to the unbounded energy spectrum of the standard QM oscillator, while infrared

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divergences are linked to the assumed continuity of spacetime and the zero-point energy of the quantized oscillator.

The issue of zero-point energy of the quantized oscillator was explicitly addressed in quantum optics by Glauber in his landmark papers on photon correlations (Glauber, 1963a,b,c). He pointed out that the irreversibility of quantum detection processes involves the non-Hermitian positive energy electric field operator $\boldsymbol{E}^{(+)}(t,\boldsymbol{x})$ rather than the Hermitian electric field operator $\boldsymbol{E}(t,\boldsymbol{x}) \equiv \boldsymbol{E}^{(+)}(t,\boldsymbol{x}) + \boldsymbol{E}^{(-)}(t,\boldsymbol{x})$. It is the vacuum expectation values (VEVs) involving the $\boldsymbol{E}$ field operators that are affected by zero-point energy, whereas VEVs involving the $\boldsymbol{E}^{(+)}$ alone are not. Glauber's conclusion was the following:

> *The electric field in the vacuum undergoes zero-point oscillations which, in the correctly formulated theory, have nothing to do with the detection of photons.* (Glauber, 1963c)

According to Glauber, then, the standard quantum field theory approach to quantum detection, if it involves VEVs of Hermitian operators, will be inadequate to model irreversible, local detection processes as they actually occur in real laboratories. The reason the standard Lehmann–Symanzik–Zimmerman (LSZ) (Lehmann et al., 1955) scattering formalism works is that the *in* and *out* Hilbert spaces representing prepared and outcome states, respectively, are applied at infinitely remote times in the past and future, respectively, in a causally irreversible way. In between those two remote times, a quantum process is technically in what we have called in earlier chapters the information void, a laboratory regime during which no signal detection takes place. During such a regime, unitary evolution can be assumed to take place without violating any of the principles of QM. That is the essential reason why the standard model Lagrangian, which respects CPT[1] inversion symmetry and is used to work out unitary evolution, is a good basis for particle scattering calculations. However, finite time processes, which are generally not considered in high-energy physics, would require the same review that Glauber carried out for photonic processes.

Given the apparatus $\mathcal{A}_n$ at stage Σ_n, its associated detectors, both real and virtual, are represented by a set of signal qubits $\{Q_n^i : i = 1, 2, \ldots, r_n\}$, with qubit Q_n^i being identified with vertex i_n. Here as elsewhere, upper indices label individual qubits and detectors, while lower indices denote stages. The tensor product $Q_n^1 Q_n^2 \ldots Q_n^{r_n}$, plus the information held by the observer about the physical significance of those qubits, constitute a quantum register of rank r_n. This is a Hilbert space of dimension 2^{r_n}.

A fundamental property of any quantum register of rank greater than one is that it contains entangled states as well as separable states. Separability and entanglement are discussed in some detail in Chapter 22.

[1] Charge, space (parity), and time.

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Example 7.1 Given a rank-three quantum register $Q^1Q^2Q^3$, the state $\mathbf{\Psi} \equiv \mathbf{1}^1\mathbf{0}^2\mathbf{1}^3 - \mathbf{0}^1\mathbf{0}^2\mathbf{0}^3$ is partially separable relative to the chosen basis, because we can write $\mathbf{\Psi} = \{\mathbf{1}^1\mathbf{1}^3 - \mathbf{0}^1\mathbf{0}^3\}\mathbf{0}^2$. On the other hand, $\mathbf{\Phi} \equiv \mathbf{1}^1\mathbf{0}^2\mathbf{1}^3 - \mathbf{0}^1\mathbf{1}^2\mathbf{0}^3$ cannot be written as a product, so is entangled relative to the chosen basis.

In QDN, entanglement is regarded as contextual on the observer's information about their apparatus, and not as an intrinsic property of SUOs. QDN tries to avoid terms such as "entangled photons," but we reserve the right to use such terminology occasionally, provided it does not mislead. The concept of an entangled labstate is not only perfectly acceptable in QDN but actually essential for the correct calculation of outcome signal detection probabilities in many experiments.

7.4 Persistence and Ensembles

A conventional assumption in QM is that pure states of a system under observation may be represented by time-dependent elements of a fixed Hilbert space. The chosen Hilbert space is usually assumed fixed for two reasons. First, there is the conditioned belief that an SUO "exists" in time as a separate entity long enough for the observer to study it. Another contributory factor is the *persistence of the apparatus*, or the tendency of actual apparatus to exist in its original form and functionality in a laboratory before and after its useful role has ended.

Most physics experiments deal with persistent apparatus. That is generally arranged by the observer as a matter of economy: experimentalists generally do not have the resources to scrap their apparatus at the end of each run and then rebuild it in time for the next run.

There are situations, however, where persistence cannot be assumed. For example, astronomers can catch light from a supernova shock wave only during an extremely limited time, and that particular observation cannot be repeated because the source of the signal has long gone. What helps the observers is the vast numbers of photon signals that they manage to detect during that limited window of opportunity.

A similar issue arises in quantum cosmology. The Universe is believed to be expanding. On that account, any approach to quantum cosmology should take the attendant irreversibility into consideration, and not treat the evolution of the Universe in traditional QM terms as a typical SUO. The Universe is in an ensemble of one, and not only contains the observer and the apparatus, but will outlast both of them.

In QDN, individual detectors are never persistent. Each detector is assigned a particular stage at which it operates as an detector, and outside that time, has no role in the formalism. This is the QDN analogue of the concept of an *event* in relativity. Some applications of QDN will for convenience assume persistence

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of apparatus (Jaroszkiewicz, 2004), the only effect being to increase the number of qubits used in the formalism.

7.5 Observers and Time

Observers generally come equipped with their own sense of time, and quantum experiments are carried out relative to that time. Relativity teaches that there are two observer-related time concepts with different properties; *coordinate time* and *proper time*. In both special relativity (SR) and general relativity (GR), the former time concept is used to label events in spacetime and is generally locally integrable. This means that spacetime can be discussed in terms of coordinate patches (Schutz, 1980). Within a given coordinate patch, events can be labeled by spacetime coordinates in a path-independent way. When a particular coordinate patch is related to clocks and rods in a specific laboratory, we shall refer to that coordinate time as *labtime*. In GR, the description of labtime requires some assumption about time-like foliations of spacetime and frame fields. On the other hand, proper time is nonintegrable, which is to say that the proper time between two events depends on the particular path taken between those events. In other words, proper time is contextual.

In QDN, the time parameter associated with an experiment can normally be identified with the proper time of an idealized inertial observer moving along a time-like worldline, and for whom their laboratory appears to be at rest at all times. In some situations, we may have to apply QDN to what we call *interframe* experiments. These involve state preparation in one inertial frame and signal detection in another. The Doppler effect in observational cosmology is an example of interframe physics.

What are important in such situations in SR and GR are *space-like hypersurfaces*: these are the analogues of the concept of stages in QDN. On a space-like hypersurface, no two events can be causally connected; that is, no event can be the cause of any other event on the same hypersurface of simultaneity. In QDN, the analogous statement is that no two detectors on a given stage can affect each other in any way. This could be because the detectors are really relatively space-like separated, but the possibility exists, frequently in real experiments, that the detectors are relatively time-like separated but shielded from each other.

In the real world, observers have finite existence: they come and go. Observers and their apparatus are created at certain times and disappear at later times, as seen by other observers in the wider universe. QDN as formulated here allows for a discussion of different observers, each with their individual time parameters and lifetimes. The use of quantum registers also raises the possibility of accounting for the origin of various temporally related concepts such as light cones, time dilation, and other metric-based phenomena in terms of quantum register dynamics. A useful way to discuss what is going on is in terms of *causal sets*, the structures of which arise naturally within quantum register dynamics (Eakins and Jaroszkiewicz, 2005). Causal sets are discussed in Chapter 23.

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During their operational lifetimes, observers quantify their time in terms of real numbers, usually read from clocks. Most clocks give only a crude estimate of the passage of time, and as a result, the ordinary human perception of time as a one-dimensional continuum is just a convenient approximation. The classical view of time is that it is a continuum at all scales and for all phenomena. Certainly, things appear consistent with that view in the ordinary world.

In QM, however, the situation is quite different. What matters in a quantum experiment is information acquisition from the observer's apparatus and this can only ever be done in a discrete way, regardless of any theoretical assumption to the contrary (Misra and Sudarshan, 1977). While an observer's effective sense of time can be modeled accurately as continuous, it is certainly the case that an observer can look at a detector and determine its status in a discrete way only. There are no truly continuous-in-time observations. It is important here to distinguish between what experimentalists actually do in experiments and what theorists imagine they do.

The discreteness of the information extraction process forms the basis of the time concept in QDN. In general, a given observer will represent the state of their apparatus (the labstate) at a finite sequence of their own (observer) times, denoted by the integer n. These times will be referred to as *stages.* In QDN, a pure labstate at stage Σ_n will be denoted by $\mathbf{\Psi}_n$.

In QDN, stage Σ_{n+1} is always regarded as definitely *later* than stage Σ_n. There is no scope in QDN for the concept of closed time-like curve (CTC) found in some GR spacetimes, such as the Gödel model (Gödel, 1949). Invariably, discussions that do involve CTCs cannot accommodate quantum processes properly, at least not those that involve probabilities.[2]

A final point on this topic: there is no need to assume that the laboratory time interval between stages has a definite value or that a succession of stages involves equal intervals of labtime.

7.6 The Born Probability Rule

One of the most significant attributes of quantum processes is the randomness of quantum outcomes. Given identical state preparation, different runs of a given experiment generally demonstrate *controlled* unpredictability. Controlled, because the observer may know all about the range of possible outcomes and the probabilities of those outcomes before one is actually observed, but unpredictable because the observer cannot in general say beforehand which particular one of those outcomes will occur in any particular run.

[2] This is for the good reason that the probability concept requires an observer accumulating information with an irreversible sense of time, something that a genuine CTC is incompatible with.

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Remark 7.2 This subject is a minefield of issues that lurk unseen until we stumble across them. Reference was made to "identical state preparation" in the previous paragraph. That is clearly a vacuous concept. By definition, different runs of a given experiment cannot have absolutely identical state preparation: the Universe will have aged for sure during any two runs, and there will be vast changes in the local environment of any apparatus on atomic scales. Yet clearly, most of those changes will *not* influence the outcome probabilities. As pointed out by Kraus, what appears significant are *equivalence classes* of state preparation processes (Kraus, 1983). Exactly where the line between influence and noninfluence of external factors in state preparation should be drawn is one of those deep questions that no one knows anything much about. The *Heisenberg cut* is a hypothetical line demarcating the classical world of the observer and the quantum world of the SUO. We propose the term *Kraus cut* to denote the hypothetical line between those factors that have no influence on state preparation or outcome and those that do. All we can do is to observe that some things are critical in state preparation, and everything else seems unimportant. Much the same point is emphasized in Peres (1995).

In practice the QM approach to probability works brilliantly and we use it in QDN. The Born probability rule (Born, 1926) in QM states that if a final state $|\Psi\rangle$ is represented by a superposition of the form

$$|\Psi\rangle = \sum_{i=1}^{d} \Psi^i |i\rangle, \tag{7.1}$$

where the possible outcomes are represented by orthonormal vectors $|i\rangle$, $i = 1, 2, \ldots, d$, in some Hilbert space, then the conditional probability P_i of outcome $|i\rangle$ is given by $P_i = |\langle i|\Psi\rangle|^2$, if the final state is normalized to unity.

This rule is used in much the same way in QDN, as follows. Consider a normalized pure labstate $\boldsymbol{\Psi}_n$ at stage Σ_n. This can always be expanded in terms of the computational basis representation (CBR) of the preferred register basis B_n at that stage in the form

$$\boldsymbol{\Psi}_n = \sum_{i=0}^{2^{r_n}-1} \Psi_n^i \boldsymbol{i}_n, \tag{7.2}$$

where the coefficients Ψ_n^i are complex and

$$\sum_{i=0}^{2^{r_n}-1} |\Psi_n^i|^2 = 1. \tag{7.3}$$

In QDN, labstates are usually normalized to unity for convenience. Because the preferred basis states form a complete orthonormal basis set, we may immediately

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read off from (7.2) the various CBR conditional probabilities $\Pr(\boldsymbol{i}_n|\boldsymbol{\Psi}_n)$, which are given by the rule

$$\Pr(\boldsymbol{i}_n|\boldsymbol{\Psi}_n) \equiv |\overline{\boldsymbol{i}_n}\boldsymbol{\Psi}_n|^2 = |\Psi_n^i|^2, \quad 0 \leqslant i < 2^{r_n}. \tag{7.4}$$

$\Pr(\boldsymbol{i}_n|\boldsymbol{\Psi}_n)$ is the conditional (Bayesian) probability for the observer to find the apparatus in signal state $\boldsymbol{i}_n$ at stage Σ_n, *if the observer looked at their apparatus at that time*. These probabilities are conditional on the observer being sure, just before they looked, that the labstate at stage Σ_n was $\boldsymbol{\Psi}_n$.

There is no natural restriction in QDN to labstates that are eigenstates of signality; i.e., superpositions of basis states from different signality classes are permitted in principle. QDN is analogous in this respect to the extension of Schrödinger wave mechanics to Fock space and to quantum field theory.

7.7 Principles of QDN Dynamics

We are now in a position to discuss the principles of labstate dynamics from the perspective of a single observer. At stage Σ_n, this observer will hold in their memory current information about their apparatus $\mathcal{A}_n$, the associated quantum register $\mathcal{Q}_n$, and the labstate $\boldsymbol{\Psi}_n$ (assumed pure here). An analogous statement will hold for each stage in a finite sequence of stages running from some initial stage Σ_0 to some final stage Σ_N, where $N > 0$. QDN does not assume observers exist over unbounded intervals of time, so the formalism is applied only over a finite number of stages.

We restrict attention in this chapter to pure labstates, that is, labstates fully specified by single elements in a quantum register. A mixed-state, density matrix approach to QDN dynamics is discussed in Chapter 9.

For the most basic sort of experiment, labstate preparation will be assumed to have taken place by initial stage Σ_0 and outcome detection is to take place at final stage Σ_N. For each integer n such that $0 \leqslant n \leqslant N$, the observer associates with their apparatus $\mathcal{A}_n$ at that time a quantum register $\mathcal{Q}_n$. This register is the tensor product $\mathcal{Q}_n \equiv Q_n^1 Q_n^2 \ldots Q_n^{r_n}$ of a finite number r_n of qubits, Q_n^1, $Q_n^2, \ldots, Q_n^{r_n}$, each qubit Q_n^i representing the ith real or virtual detector i_n in $\mathcal{A}_n$. The quantum register $\mathcal{Q}_n$ is a Hilbert space with preferred basis B_n consisting of the 2^{r_n} CBR signal states.

There is no requirement in QDN or implication in our notation for the detector represented by Q_{n+1}^i to be related in any obvious way to the detector represented by Q_n^i; that is, we do not assume persistence. In other words, successive quantum registers are understood as different Hilbert spaces, even if they have the same rank, that is, if $r_{n+1} = r_n$. This is one of the factors which makes QDN more general in its scope than standard QM, although all the principles of QM are incorporated in QDN.

At stage Σ_n, the observer describes the quantum state of their apparatus at that time by a labstate $\boldsymbol{\Psi}_n$, which is some normalized vector in $\mathcal{Q}_n$. Using

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the CBR, this state can be written in the form (7.2) and normalized according to (7.3).

A given run of an experiment will be described by the observer in terms of a sequence $\{\mathbf{\Psi}_n : 0 \leqslant n \leqslant N\}$ of normalized labstates, each element of which is associated with a particular quantum register $\mathcal{Q}_n$, ending with state outcome observation at the final stage Σ_N.[3] The question now is how successive labstates relate to each other between times M and N.

Provided each run is prepared in the same way, and provided the apparatus during eah run is controlled in the same way, we can discuss a typical labstate $\mathbf{\Psi}_n$ as a representative for an ensemble of runs. Indeed, that is the only interpretation that makes sense if an objectivized or hidden-variables interpretation of the wave function is to be avoided. QDN follows the traditional QM view as emphasized by Peres in this respect (Peres, 1995).

The dynamical transition from labstate $\mathbf{\Psi}_n$ to labstate $\mathbf{\Psi}_{n+1}$ involves a mapping from quantum register $\mathcal{Q}_n$ to quantum register $\mathcal{Q}_{n+1}$ that satisfies two criteria fundamental to quantum mechanics: *linearity* and *norm preservation*. This leads us to give a number of definitions and theorems that have proved central to QDN.

7.8 Born Maps and Semi-unitarity

In the following, we restrict the discussion to Hilbert spaces of finite dimension. This is in line with our general philosophy in QDN that there are no empirically observable infinities in physics.

Definition 7.3 A *Born map* is a norm-preserving map from one Hilbert space $\mathcal{H}$ to some other Hilbert space $\mathcal{H}'$; if Ψ in $\mathcal{H}$ is mapped into $\Psi' \equiv \mathfrak{B}(\Psi)$ in $\mathcal{H}'$ by a Born map $\mathfrak{B}$, then $(\Psi', \Psi')' = (\Psi, \Psi)$.

Born maps are used in QDN in order to preserve total probabilities (hence the terminology), but unfortunately, their properties are insufficient to model all quantum processes. Born maps are not necessarily linear, as can be seen from the elementary example $\mathfrak{B}(\Psi) = |\Psi|\Phi'$ for all Ψ in $\mathcal{H}$, where Φ' is a fixed element of $\mathcal{H}'$ normalized to unity and $|\Psi|$ is the norm of Ψ in $\mathcal{H}$. To go further, it is necessary to impose linearity.

Henceforth, we adopt the rule that if U is a linear map, then we may drop the parentheses and write $U\Psi$ instead of $U(\Psi)$.

Definition 7.4 A *semi-unitary operator* is a linear Born map. If U is such a map, then for any elements ψ, ϕ in $\mathcal{H}$ and complex α, β, we may write $|\alpha\psi + \beta\phi| \ = |\alpha U\psi + \beta U\phi|'$.

[3] QDN allows partial observations to be made at intermediate stages.

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The proofs of the following theorems are left to the reader.

Theorem 7.5 *A semi-unitary operator from Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{H}'$ exists if and only if* $\dim \mathcal{H} \leqslant \dim \mathcal{H}'$.

Theorem 7.6 *A semi-unitary operator from $\mathcal{H}$ to $\mathcal{H}'$ is an injection; that is, $\{U\psi = U\phi\}$ if and only if $\psi = \phi$.*

Theorem 7.7 *If U is a semi-unitary operator from $\mathcal{H}$ to $\mathcal{H}'$, then its retraction $\overline{U}$ exists; that is, we have $\overline{U}U = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ is the identity operator over $\mathcal{H}$.*

Corollary 7.8 A semi-unitary operator preserves inner products and not just norms. This means that an orthonormal basis set for Hilbert space $\mathcal{H}$ is mapped by a semi-unitary operator into a mutually orthonormal set of vectors in $\mathcal{H}'$ with the same cardinality.

Theorem 7.9 *If U is a semi-unitary operator from $\mathcal{H}$ to $\mathcal{H}'$ and* $\dim \mathcal{H} = \dim \mathcal{H}'$, *then the retraction $\overline{U}$ of U is also a semi-unitary operator from $\mathcal{H}'$ to $\mathcal{H}$. For such an operator, $\overline{U}U = I_{\mathcal{H}}$ and $U\overline{U} = I_{\mathcal{H}'}$.*

Definition 7.10 An operator U satisfying the conditions of Theorem 7.9 will be called a *unitary*.

Remark 7.11 If U is a semi-unitary operator from $\mathcal{H}$ to $\mathcal{H}'$ and $\dim \mathcal{H} < \dim \mathcal{H}'$, then $U\overline{U} \neq I_{\mathcal{H}'}$, simply because the retraction $\overline{U}$ is defined not over $\mathcal{H}'$ but over the proper subset $U(\mathcal{H})$ of $\mathcal{H}'$.

7.9 Application to Dynamics

It is normally assumed in QDN that a labstate $\mathbf{\Psi}_n$ in $\mathcal{Q}_n$ at stage Σ_n is mapped into a labstate $\mathbf{\Psi}_{n+1}$ in $\mathcal{Q}_{n+1}$ by some Born map $\mathcal{B}_{n+1,n}$. Because $\overline{\mathbf{\Psi}_{n+1}}\mathbf{\Psi}_{n+1} = \overline{\Psi_n}\Psi_n$ under such a map, the Born rule used in conjunction with the signal bases B_n and B_{n+1} means that total probability is conserved. This is not the same thing as conservation of signality, charge, particle number, or any other quantum variable.

The following three scenarios are possible.

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$\mathfrak{B}_{n+1,n}$ Is Nonlinear

By Theorem 7.5, nonlinearity is unavoidable if the rank r_n of $\mathcal{Q}_n$ is greater than the rank r_{n+1} of $\mathcal{Q}_{n+1}$, but can arise even if this is not the case. Nonlinearity here is interpreted as a marker of classical intervention by the observer. For example, switching off any apparatus at stage Σ_{n+1} would be modeled by the Born map $\mathcal{B}_{n+1,n}(\boldsymbol{\Psi}_n) = \mathbf{0}_{n+1}$ for any state $\boldsymbol{\Psi}_n$ in $\mathcal{Q}_n$, where $\mathbf{0}_{n+1}$ is the signal ground state of the apparatus at stage Σ_{n+1}. Another example is *state reduction* due to observation; that is, if at stage Σ_{n+1} the observer actually looks at the apparatus and determines its signal status, then this would be modeled by the nonlinear Born map $\mathfrak{B}_{n+1,n}(\boldsymbol{\Psi}_n) = \boldsymbol{k}_{n+1}$, where now $\boldsymbol{k}_{n+1}$ is some element of the CBR of the preferred basis B_{n+1}, chosen randomly with a probability weighting given by the Born rule. In this particular case, however, there are actually *two* labstates associated with stage Σ_{n+1}: $\boldsymbol{\Psi}_{n+1}$ representing the state of the apparatus immediately prior to state reduction and $\boldsymbol{k}$ representing the actual observed outcome immediately after. None of this represents anything more than mathematical modeling of the observer's actual or potential knowledge about the signal status of their apparatus. QDN makes no comment on whether anything deeper than changes in apparatus signal status has occurred. In particular, any speculation of superluminal information flow concerns *correlations*, which is a phenomenon that occurs in all forms of mechanics and has everything to do with what the observer did in the past.

$\mathfrak{B}_{n+1,n}$ Is Linear and $r_n = r_{n+1}$

This scenario corresponds to unitary evolution in standard QM, and to reflect this, we use the notation $\mathcal{B}_n(\boldsymbol{\Psi}_n) \equiv \mathbb{U}_{n+1,n}\boldsymbol{\Psi}_n = \boldsymbol{\Psi}_{n+1}$. From Theorem 7.9, $\mathbb{U}_{n+1,n}$ in this case satisfies the rules

$$\overline{\mathbb{U}}_{n+1,n}\mathbb{U}_{n+1,n} = \mathbb{I}_n, \qquad \mathbb{U}_{n+1,n}\overline{\mathbb{U}}_{n+1,n} = \mathbb{I}_{n+1}, \tag{7.5}$$

where $\overline{\mathbb{U}}_{n+1,n}$ is the retraction of $\mathbb{U}_{n+1,n}$. In such a case it is reasonable to call $\mathbb{U}_{n+1,n}$ *unitary*, being the formal analogue of a unitary operator in QM.

$\mathfrak{B}_{n+1,n}$ Is Linear and $r_n < r_{n+1}$

In this case we use the same notation as in the second case above, i.e., $\mathcal{B}_{n+1,n}(\boldsymbol{\Psi}_n) \equiv \mathbb{U}_{n+1,n}\boldsymbol{\Psi}_n = \boldsymbol{\Psi}_{n+1}$, but now $\mathbb{U}_{n+1,n}$ is properly semi-unitary and only the first relation $\overline{\mathbb{U}}_{n+1,n}\mathbb{U}_{n+1,n} = \mathbb{I}_n$ in (7.5) carries over. Such a scenario arises in particle decay experiments, for example. These are discussed in Chapter 15.

We cannot in general expect the rank r_n of the quantum register $\mathcal{Q}_n$ to be constant with n, so if we wish to preserve probability and restrict the dynamical evolution to be linear in the labstate, then we have to assume

$$r_M \leqslant r_{M+1} \leqslant \cdots \leqslant r_n \leqslant \cdots \leqslant r_N, \tag{7.6}$$

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where Σ_M is the initial stage and Σ_N is the final stage. From this, we can appreciate that unless experimentalists are careful, their quantum registers will grow irreversibly in rank. On the other hand, the particle decay experiments discussed in Chapter 15 specifically require the rank to increase at each time step.

The use of Born maps means total probability is always conserved, even if linearity is absent. In principle, therefore, QDN allows for a discussion of nonlinear QM, still based on most of the familiar Hilbert space concepts used in QM. As we have mentioned in the case of nonlinear Born maps, necessarily nonlinear processes such as state preparation, state reduction, the switching on and off of apparatus, and so on, which are outside the scope of unitary (Schrödinger) evolution in standard QM, can all be discussed in QDN in terms of nonlinear Born maps.

Our interest will generally be in experiments based on linear quantum processes, so (7.6) will be assumed. For such an experiment running from stage Σ_M to stage Σ_N, for $N > M$, and knowing $r_n \leqslant r_{n+1}$, then the labstate $\boldsymbol{\Psi}_n$ will change according to the rule

$$\boldsymbol{\Psi}_n \to \boldsymbol{\Psi}_{n+1} \equiv \mathbb{U}_{n+1,n}\boldsymbol{\Psi}_n, \quad M \leqslant n < N, \tag{7.7}$$

where $\mathbb{U}_{n+1,n}$ is a semi-unitary operator (this terminology will be used from now on even in the case where $r_n = r_{n+1}$).

The CBR at stages Σ_n and Σ_{n+1} can be used to represent $\mathbb{U}_{n+1,n}$. Specifically, we find

$$\mathbb{U}_{n+1,n} = \sum_{j=0}^{2^{r_{n+1}}-1} \sum_{i=0}^{2^{r_n}-1} \boldsymbol{j}_{n+1} U^{j,i}_{n+1,n} \overline{\boldsymbol{i}_n}. \tag{7.8}$$

This representation can be used to define a retraction operator $\overline{\mathbb{U}}_{n+1,n}$, given by

$$\overline{\mathbb{U}}_{n+1,n} = \sum_{j=0}^{2^{r_{n+1}}-1} \sum_{i=0}^{2^{r_n}-1} \boldsymbol{i}_n U^{j,i*}_{n+1,n} \overline{\boldsymbol{j}_{n+1}}, \tag{7.9}$$

where $U^{j,i*}_{n+1,n}$ is the complex conjugate of $U^{j,i}_{n+1,n}$, if the semi-unitarity condition holds, that is, if

$$\sum_{j=0}^{2^{r_{n+1}}-1} U^{j,i}_{n+1,n} U^{j,k*}_{n+1,n} = \delta^{ik}. \tag{7.10}$$

A useful way of thinking about the semi-unitarity condition (7.10) and the rules of semi-unitary operators is in terms of complex vectors as follows. Let V be a finite-dimensional complex vector space with inner product $\boldsymbol{a}^\dagger \cdot \boldsymbol{b}$. An *orthonormal d-subset* of V is any set of elements $\{\boldsymbol{a}^i : i = 1, 2, \ldots, d\}$ of V that are normalized to unity and mutually orthogonal; that is, we have the rule $\boldsymbol{a}^{i\dagger} \cdot \boldsymbol{a}^j = \delta^{ij}$.

Now let V' be another finite-dimensional complex vector space with inner product $\boldsymbol{a}'^\dagger \cdot \boldsymbol{b}'$. The question of semi-unitarity reduces to the possibility of finding an injection from any given orthonormal d-subset of V into at least one

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orthonormal d-subset of V'. The above semi-unitarity theorems tell us that this cannot be done if $d > \dim V'$.

7.10 The Signal Theorem

The mathematical properties of semi-unitary operators and their relationship to signal bases have an important bearing on the permitted physics of QDN dynamics. Consider an experiment at stages Σ_n and Σ_{n+1} and assume semi-unitarity. At stage Σ_n the labstate $\boldsymbol{\Psi}_n$ is given by a superposition of signal states from signal basis $B_n \equiv \{\boldsymbol{i}_n : 0 \leqslant i < 2^{r_n}\}$, while the labstate $\boldsymbol{\Psi}_{n+1}$ is given as a superposition of signal states from signal basis $B_{n+1} \equiv \{\boldsymbol{i}_{n+1} : 0 \leqslant i < 2^{r_{n+1}}\}$. Because of linearity, the crucial question as far as the dynamics is concerned is how individual signal states evolve. Semi-unitarity imposes the following constraint, which we call the *signal theorem*:

Theorem 7.12 *Two different preferred basis states $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ in a signal basis B_n cannot evolve by semi-unitary dynamics into labstates that have only one preferred signal basis state in common.*

Proof Take $0 \leqslant i < j < 2^{r_n}$. Suppose $\boldsymbol{i}_n$ evolves by semi-unitarity dynamics into a labstate according to the rule

$$\boldsymbol{i}_n \to \mathbb{U}_{n+1,n}\boldsymbol{i}_n = \alpha\boldsymbol{k}_{n+1} + \boldsymbol{\phi}_{n+1}, \tag{7.11}$$

while $\boldsymbol{j}_n$ evolves according to the rule

$$\boldsymbol{j}_n \to \mathbb{U}_{n+1,n}\boldsymbol{j}_n = \beta\boldsymbol{k}_{n+1} + \boldsymbol{\psi}_{n+1}. \tag{7.12}$$

Here k is some integer in the semi-open interval $[0, 2^{r_{n+1}})$, α and β are non-zero complex numbers, and $\boldsymbol{\phi}_{n+1}$ and $\boldsymbol{\psi}_{n+1}$ are elements in $\mathcal{Q}_{n+1}$ sharing no signal states in common either with each other or with $\boldsymbol{k}_{n+1}$ in their CBR basis expansions, which means

$$\overline{\boldsymbol{k}_{n+1}}\boldsymbol{\phi}_{n+1} = \overline{\boldsymbol{k}_{n+1}}\boldsymbol{\psi}_{n+1} = \overline{\boldsymbol{\phi}_{n+1}}\boldsymbol{\psi}_{n+1} = 0. \tag{7.13}$$

From Corollary 7.8, semi-unitarity preserves inner products and not just norms, so we must have

$$0 = \overline{\boldsymbol{i}_n}\boldsymbol{j}_n = \overline{\boldsymbol{i}_n}\,\overline{\mathbb{U}}_{n+1,n}\mathbb{U}_{n+1,n}\boldsymbol{j}_n = \overline{\alpha\boldsymbol{k}_{n+1} + \boldsymbol{\phi}_{n+1}}(\beta\boldsymbol{k}_{n+1} + \boldsymbol{\psi}_{n+1}) \tag{7.14}$$

$$= (\alpha^*\overline{\boldsymbol{k}_{n+1}} + \overline{\boldsymbol{\phi}_{n+1}})(\beta\boldsymbol{k}_{n+1} + \boldsymbol{\psi}_{n+1}) = \alpha^*\beta, \tag{7.15}$$

using (7.12). This establishes the theorem. □

The signal theorem leads to the following important result for conventional physics. Suppose an observer constructs an apparatus that, if prepared at stage Σ_n to be in its signal ground state $\mathbf{0}_n$, would evolve into $\mathbf{0}_{n+1}$. If the dynamics is semi-unitary, then we may write

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$$\mathbf{0}_n \to \mathbb{U}_{n+1,n}\mathbf{0}_n = \mathbf{0}_{n+1}. \tag{7.16}$$

This condition models an important physical property expected of most laboratory apparatus; we would not expect equipment that was in its signal ground state to spontaneously generate outcome signals subsequently, unless it was interfered with by some external agency. Any apparatus that satisfies (7.16) will be called *calibrated* between stages Σ_n and Σ_{n+1} on that account. The analogue of such a situation in Schwinger's source theoretic approach to quantum field theory (Schwinger, 1969) would be one where the external sources were switched off during some interval of time, so that the vacuum (empty space) remained unchanged during that time.

Suppose now that, given such a calibrated apparatus, the observer had instead prepared at stage Σ_n some labstate $\mathbf{\Psi}_n$ of the CBR form

$$\mathbf{\Psi}_n = \sum_{i=1}^{2^{r_n}-1} \Psi^i \boldsymbol{i}_n, \tag{7.17}$$

that is, a labstate with no signal ground component (note that the summation (7.17) in runs from 1, not 0). Then for calibrated apparatus with semi-unitary labstate evolution, the signal theorem tells us that there can be no signal ground component in the labstate $\mathbf{\Psi}_{n+1}$ at time $n+1$, and so we may write

$$\mathbf{\Psi}_n \to \mathbb{U}_{n+1,n}\mathbf{\Psi}_n = \sum_{j=1}^{2^{r_{n+1}}-1} \Phi^j \boldsymbol{j}_{n+1}, \tag{7.18}$$

where

$$\Phi^j = \sum_{i=1}^{2^{r_n}-1} U^{ji}_{n+1,n}\Psi^i. \tag{7.19}$$

This is an important result, because it tells us that under normal circumstances, calibrated apparatus does not normally fall into its signal ground state during an experiment, unless forced to do so by an external agency, such as the observer switching it off.

Example 7.13 Consider a calibrated rank-one apparatus evolving into a rank-one apparatus between stages Σ_n and Σ_{n+1} under semi-unitary evolution. Then by a suitable choice of phase of basis elements, we may always write

$$\begin{aligned} \mathbf{0}_n &\to \mathbb{U}_{n+1,n}\mathbf{0}_n = \mathbf{0}_{n+1}, \\ \mathbf{1}_n &\to \mathbb{U}_{n+1,n}\mathbf{1}_n = \mathbf{1}_{n+1}, \end{aligned} \tag{7.20}$$

from which we conclude the dynamics is essentially trivial.

The following example is important, as it models what happens in various quantum optics modules such as beam splitters and Wollaston prisms.

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Example 7.14 Consider a calibrated rank-two apparatus evolving into a rank-two apparatus between stages Σ_n and Σ_{n+1} under semi-unitary evolution. Then calibration means that we must have $\mathbb{U}_{n+1,n}\mathbf{0}_n = \mathbf{0}_{n+1}$. Suppose further that it is known that any signality-one labstate always evolves into a signality-one labstate. Then we may write

$$\begin{aligned}\widehat{\mathbb{A}}^1_n\mathbf{0}_n \equiv \mathbf{1}_n \rightarrow \mathbb{U}_{n+1,n}\mathbf{1}_n = \alpha\mathbf{1}_{n+1} + \beta\mathbf{2}_{n+1},\\ \widehat{\mathbb{A}}^2_n\mathbf{0}_n \equiv \mathbf{2}_n \rightarrow \mathbb{U}_{n+1,n}\mathbf{2}_n = \gamma\mathbf{1}_{n+1} + \delta\mathbf{2}_{n+1},\end{aligned} \tag{7.21}$$

where the complex coefficients α, β, γ, and δ satisfy the unitarity constraints

$$|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1, \quad \alpha^*\gamma + \beta^*\delta = 0. \tag{7.22}$$

Now consider the signality-two state $\widehat{\mathbb{A}}^1_n\widehat{\mathbb{A}}^2_n\mathbf{0}_n \equiv \mathbf{3}_n$. Since the apparatus is calibrated, this state must necessarily evolve into a state that has no signal ground component. Therefore, we may write

$$\mathbb{U}_{n+1,n}\mathbf{3}_n = a\mathbf{1}_{n+1} + b\mathbf{2}_{n+1} + c\mathbf{3}_{n+1}, \tag{7.23}$$

where $|a|^2 + |b|^2 + |c|^2 = 1$. Moreover, since the evolution is given as semi-unitary, then inner products are preserved. Hence we deduce

$$\alpha^*a + \beta^*b = \gamma^*a + \delta^*b = 0. \tag{7.24}$$

Writing these relations in matrix form, we find

$$\begin{bmatrix}\alpha^* & \beta^*\\ \gamma^* & \delta^*\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{7.25}$$

The 2×2 matrix on the left-hand side of this expression is necessarily invertible, leading to the conclusion that $a = b = 0$ and therefore that

$$\mathbf{3}_n \rightarrow \mathbb{U}_{n+1,n}\mathbf{3}_n = \mathbf{3}_{n+1}, \tag{7.26}$$

modulo some arbitrary phase.

The following application of the signal theorem is surprising and somewhat counterintuitive, because what appears to be a trivial mathematical result rules out an entire class of physics experiment.

Example 7.15 Suppose an experimentalist prepares a rank-two, signality-one labstate of the form

$$\mathbf{\Psi}_n = (\alpha\widehat{\mathbb{A}}^1_n + \beta\widehat{\mathbb{A}}^2_n)\mathbf{0}_n, \tag{7.27}$$

where $|\alpha|^2 + |\beta|^2 = 1$. Suppose further that the dynamics is semi-unitary and that the apparatus at stage Σ_{n+1} is of rank three. Then by the signal theorem, semi-unitary evolution such that

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$$\widehat{\mathbb{A}}^1_n \mathbf{0}_n \to \mathbb{U}_{n+1,n}\widehat{\mathbb{A}}^1_n \mathbf{0}_n = (a\widehat{\mathbb{A}}^1_{n+1} + b\widehat{\mathbb{A}}^2_{n+1})\mathbf{0}_{n+1}, \quad |a|^2 + |b|^2 = 1,$$
$$\widehat{\mathbb{A}}^2_n \mathbf{0}_n \to \mathbb{U}_{n+1,n}\widehat{\mathbb{A}}^2_n \mathbf{0}_n = (c\widehat{\mathbb{A}}^2_{n+1} + d\widehat{\mathbb{A}}^3_{n+1})\mathbf{0}_{n+1}, \quad |c|^2 + |d|^2 = 1, \tag{7.28}$$

is not possible.

This result tells us that a double-slit type of experiment where each slit has only one quantum outcome site in common with the other cannot be physically constructed. Experiments where two or more quantum outcome sites are in common are possible, and then inevitable quantum interference terms will occur in final state amplitudes. For example, in a standard double-slit experiment, every site on the detector screen is affected by the presence of each of the two slits.

This result reinforces an important rule in QM: we cannot simply add pieces of apparatus together and expect the result to conform to an addition of classical expectations. A double-slit experiment where both slits are open is not equivalent to two single-slit experiments run coincidentally and simultaneously.

7.11 Null Evolution

There is an interesting class of quantum process described by evolution operators called *null evolution operators*, associated with the concept of a *null test*. Recall that a null test is one that occurs between two or more stages but no information is extracted. The phenomenon of *persistence* is associated with null evolution operators

Persistence

To understand the action of a null evolution operator, consider the idealized scenario of an initial labstate $\mathbf{\Psi}_n$ in a rank r quantum register $\mathcal{Q}_n$ evolving into labstate $\mathbf{\Psi}_{n+1}$ in a quantum register $\mathcal{Q}_{n+1}$ of the same rank r. In the following, we shall use the CBR at all stages.

Suppose we are given that

$$\mathbf{\Psi}_n = \sum_{i=0}^{2^r-1} \Psi^i_n \boldsymbol{i}_n. \tag{7.29}$$

Now consider a particular evolution operator $\mathbb{N}_{n+1,n}$ defined by

$$\mathbb{N}_{n+1,n} \equiv \sum_{j=0}^{2^r-1} \boldsymbol{j}_{n+1}\overline{\boldsymbol{j}_n}. \tag{7.30}$$

Then under this evolution operator,

$$\mathbf{\Psi}_n \to \mathbb{N}_{n+1,n}\mathbf{\Psi}_n = \sum_{j=0}^{2^r-1} \boldsymbol{j}_{n+1}\overline{\boldsymbol{j}_n} \sum_{i=0}^{2^r-1} \Psi^i_n \boldsymbol{i}_n = \sum_{i,j=0}^{2^r-1} \Psi^i_n \boldsymbol{j}_{n+1}\underbrace{\overline{\boldsymbol{j}_n}\boldsymbol{i}_n}_{\delta^{ij}} = \sum_{i=0}^{2^r-1} \Psi^i_n \boldsymbol{i}_{n+1}. \tag{7.31}$$

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This is a labstate at stage Σ_{n+1} with exactly the same coefficient profile, that is, set of coefficients $\{\Psi_n^i\}$, as the initial labstate. It is reasonable in this context to refer to this phenomenon as an example of *persistence* and refer to $\mathbf{\Psi}_{n+1}$ as a *persistent image* of $\mathbf{\Psi}_n$.

Such an evolution operator will be referred to as a *null evolution operator* (NEO). An NEO $\mathbb{N}_{n+1,n}$ makes sense only under particular circumstances: the rank of the quantum register at stage Σ_{n+1} must be the same as that at stage Σ_n *and* there has to be a one-to-one identification of the elements of the preferred bases.

This form of evolution demonstrates the two faces of time: on the one hand, labtime (the time of the observer) goes on as normal, being collated with relative external context such as the expansion of the Universe. The labtime of a given observer cannot be reversed relative to other observers, according to all known current physics, although it can be slowed relative to the labtime of other observers. On the other hand, some objects such as labstate profiles of persistent labstates may appear to be indifferent to labtime.

Dynamical Null Tests

In Chapter 11 we discuss Newton's famous experiment that showed how a light beam incident on one prism would split into a spectrum of subbeams that could subsequently be refocused onto a second prism and recombined once more into a single beam. This is an example of a nontrivial null test. Because the action of the first prism is nontrivial, and therefore requires nontrivial "undoing" by the second prism, we refer to the overall process as an example of a *dynamical null test*. The generic QDN description of such tests is based on the following.

Consider an labstate $\mathbf{\Psi}_n$ in initial quantum register $\mathcal{Q}_n$ of rank $r = r_n$, given by (7.29). Now apply semi-unitary evolution from stage Σ_n to stage Σ_{n+1} given by evolution operator

$$\mathbb{U}_{n+1,n} = \sum_{i=0}^{2^{r_{n+1}}-1} \sum_{j=0}^{2^{r_n}-1} \boldsymbol{i}_{n+1} U_{n+1,n}^{i,j} \overline{\boldsymbol{j}_n}, \tag{7.32}$$

where $\widehat{d}_n \equiv \dim \mathcal{Q}_n - 1$, $\widehat{d}_{n+1} \equiv \dim \mathcal{Q}_{n+1} - 1$, and the coefficients $\left\{U_{n+1,n}^{i,j}\right\}$ satisfy the semi-unitary condition (7.10). According to our theorems on semi-unitary operators, we require r_{n+1}, the rank of $\mathcal{Q}_{n+1}$, to satisfy $r_{n+1} \geqslant r_n$.

The labstate $\mathbf{\Psi}_{n+1}$ at stage Σ_{n+1} is given by

$$\mathbf{\Psi}_{n+1} = \mathbb{U}_{n+1,n}\mathbf{\Psi}_n = \sum_{i=0}^{2^{r_{n+1}}-1} \sum_{j=0}^{2^{r_n}-1} \boldsymbol{i}_{n+1} U_{n+1,n}^{i,j} \Psi_n^j. \tag{7.33}$$

Clearly, $\mathbf{\Psi}_{n+1}$ will not be a persistent image of $\mathbf{\Psi}_n$ in general. Now consider stage Σ_{n+2}, and suppose that the rank r_{n+2} of $\mathcal{Q}_{n+2}$ is given by $r_{n+2} = r_n + p$, where r_n is the rank of $\mathcal{Q}_n$ and $p \geqslant 0$. Then in terms of the qubits making up the quantum register, we can write

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$$\mathcal{Q}_{n+2} \equiv \underbrace{Q^1_{n+2} Q^2_{n+2} \dots Q^{r_n}_{n+2}}_{\widehat{\mathcal{Q}}_{n+2}} Q^{r_n+1}_{n+2} \dots Q^{r_n+p}_{n+2}. \tag{7.34}$$

Here $\widehat{\mathcal{Q}}_{n+2}$ is a subspace of $\mathcal{Q}_{n+2}$ of dimension equal to that of $\mathcal{Q}_n$. Note that, by construction, the first 2^{r_n} elements of the CBR for $\mathcal{Q}_{n+1}$ involve signal excitations only of the detectors associated with $\widehat{\mathcal{Q}}_{n+2}$.

Now define the operator

$$\mathbb{V}_{n+2,n+1} \equiv \sum_{i=0}^{2^{r_n}-1} \sum_{j=0}^{2^{r_{n+1}}-1} \boldsymbol{i}_{n+2} U^{j,i*}_{n+1,n} \overline{\boldsymbol{j}_{n+1}}. \tag{7.35}$$

Note that the upper limit on the summation over the index i is $2^{r_n} - 1$, not $2^{r_{n+1}} - 1$. This operator effectively maps states in $\mathbb{U}_{n+1,n}\mathcal{Q}_n$ into $\widehat{\mathcal{Q}}_{n+2}$. Under evolution generated by $\mathbb{V}_{n+2,n+1}$ we find

$$\boldsymbol{\Psi}_{n+2} \equiv \mathbb{V}_{n+2,n+1} \boldsymbol{\Psi}_{n+1} = \sum_{i=0}^{2^{r_n}-1} \Psi^i_n \boldsymbol{i}_{n+2}, \tag{7.36}$$

which is a persistent image, in subspace $\widehat{\mathcal{Q}}_{n+2}$, of the original labstate $\boldsymbol{\Psi}_n$.

This process demonstrates the principle known as *microscopic reversibility*: operator $\mathbb{V}_{n+2,n+1}$ has effectively "undone" the action of $\mathbb{U}_{n+1,n}$ on $\boldsymbol{\Psi}_n$. It is important to understand that as far as QDN is concerned, nothing has remained unchanged: the observer changes from stage to stage and all labstates change with those jumps. Microscopic reversibility is an illusion in a sense, but one with significant empirical content.

7.12 Path Summations

The QDN formulation of dynamics has some of the hallmarks of the Feynman path integral formulation of quantum mechanics (Feynman and Hibbs, 1965), with some significant differences: in QDN, time is not continuous, the Hilbert space changes at each intermediate time step and is assumed finite dimensional, and there is no need to introduce a Lagrangian or Hamiltonian.

A typical run or repetition of a basic experiment will be assumed to start at stage Σ_0 and finish at a later stage Σ_N, for $N > 0$. Given labstate preparation at stage Σ_0, there will be semi-unitary evolution through a sequence of apparatus stages $\{\Sigma_n : 0 < n < N\}$. At these intermediate stages, the observer does not look at their detectors, which are therefore to be regarded as virtual. Outcome detection takes place only at the final stage Σ_N.

At the final stage Σ_N, the observer looks at all of their detectors and works out from rule (5.14) which element of the CBR of the preferred basis B_N corresponds to the observed set of signals, for that given run. The objective in practice is to compare the statistical distribution of observed outcomes with the theoretically derived conditional probability $Pr(\boldsymbol{k}_N|\boldsymbol{\Psi}_0)$ for each of the possible final state signal basis outcomes $\boldsymbol{k}_N,\ 0 \leqslant k < 2^{r_N}$.

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Semi-unitary evolution will be assumed to hold between stages Σ_0 and Σ_N, i.e., condition (7.6) is valid. Given an initial labstate $\boldsymbol{\Psi}_0 \equiv \sum_{i=0}^{2^{r_0}-1} \Psi_0^i \boldsymbol{i}_0$, the next labstate is given by $\boldsymbol{\Psi}_1 = \mathbb{U}_{1,0}\boldsymbol{\Psi}_0$, where $\mathbb{U}_{1,0}$ is semi-unitary, and so on, until finally we may write

$$\boldsymbol{\Psi}_N = \mathbb{U}_{N,N-1}\mathbb{U}_{N-1,N-2}\ldots\mathbb{U}_{1,0}\boldsymbol{\Psi}_0, \qquad N > 0. \tag{7.37}$$

Inserting a resolution of each evolution operator of the form (7.8), the final state can be expressed in the form

$$\boldsymbol{\Psi}_n = \sum_{j_N=0}^{2^{r_N}-1} \sum_{j_{N-1}=0}^{2^{r_{N-1}}-1} \ldots \sum_{j_0=0}^{2^{r_0}-1} \boldsymbol{j}_N U_{N,N-1}^{j_N,j_{N-1}} U_{N-1,N-2}^{j_{N-1},j_{N-2}} \ldots U_{1,0}^{j_1,j_0} \Psi_0^{j_0}. \tag{7.38}$$

We may immediately read off from this expression the coefficient of the signal basis vector $\boldsymbol{i}_N$. This gives the QDN analogue of the quantum mechanics Feynman amplitude $\langle \Phi^i_{\text{final}} | \Psi_{\text{initial}} \rangle$ for the initial state $|\Psi_{\text{initial}}\rangle$ to go to a particular final outcome state $|\Phi^i_{\text{final}}\rangle$. In our case, what we are actually reading off is $\mathcal{A}(\boldsymbol{i}_N|\boldsymbol{\Psi}_0)$, the amplitude for the labstate to propagate from its initial state $\boldsymbol{\Psi}_0$ and then be found in signal basis state $\boldsymbol{i}$ at stage Σ_N. We find

$$\mathcal{A}(\boldsymbol{i}_N|\boldsymbol{\Psi}_0) = \sum_{j_{N-1}=0}^{2^{r_{N-1}}-1} \sum_{j_{N-1}=0}^{2^{r_{N-2}}-1} \ldots \sum_{j_0=0}^{2^{r_0}-1} U_{N,N-1}^{i,j_{N-1}} U_{N-1,N-2}^{j_{N-1},j_{N-2}} \ldots U_{1,0}^{j_1,j_0} \Psi_0^{j_0}. \tag{7.39}$$

The required conditional probabilities are obtained from the Born rule as discussed above, and so we conclude

$$Pr(\boldsymbol{i}_N|\boldsymbol{\Psi}_0) = \left| \sum_{j_{N-1}=0}^{2^{r_{N-1}}-1} \sum_{j_{N-2}=0}^{2^{r_{N-2}}-1} \ldots \sum_{j_0=0}^{2^{r_0}-1} U_{N,N-1}^{i,j_{N-1}} U_{N-1,N-2}^{j_{N-1},j_{N-2}} \ldots U_{1,0}^{j_1,j_0} \Psi_0^{j_0} \right|^2. \tag{7.40}$$

By writing the amplitude (7.39) in the form

$$\mathcal{A}(\boldsymbol{i}_N|\boldsymbol{\Psi}_0) = \sum_{j_{N-1}=0}^{2^{r_{N-1}}-1} U_{N,N-1}^{i,j} \mathcal{A}(\boldsymbol{j}_{N-1}|\boldsymbol{\Psi}_0), \tag{7.41}$$

it is straightforward to use induction and the semi-unitary matrix conditions (7.10) to prove that

$$\sum_{i=0}^{2^{r_N}-1} Pr(\boldsymbol{i}_N|\boldsymbol{\Psi}_0) = 1, \tag{7.42}$$

which means total probability is conserved, as expected.

Feynman derived his path integral for continuous time quantum mechanics by discretizing time and then taking the limit of the discrete time interval going to zero. Technical problems occur in the taking of this limit, and because of these, the path integral in its original formulation (Feynman and Hibbs, 1965) is generally regarded as ill-defined. However, it is an invaluable heuristic tool that

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provides the best way to discuss the quantization of certain classical theories for which other approaches prove inadequate. In QDN, time is discrete and in that sense we follow Feynman's lead while avoiding the pitfalls associated with the continuum limit, which we do not take in QDN.

This completes our introduction to the QDN formalism. An obvious extension is to include *mixed* labstates. These are discussed in Chapter 9.

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