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PROJECTIVE DESCRIPTIONS OF THE (LF)-SPACES OF TYPE $LB(\lambda_p(A), F)$

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Abstract

Let $1 \le p < +\infty$ or $p = 0$ and let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I. Let F be a Fréchet space. It is proved that if $\lambda_p(A)$ satisfies the density condition of Heinrich and a certain condition (C_t) holds, then the (LF)-space $LB_i(\lambda_p(A), F)$ is a topological subspace of $L_b(\lambda_p(A), F)$. It is also proved that these conditions are necessary provided $F = \lambda_q(A)$ or F contains a complemented copy of l_q with $1 < p \le q < +\infty$.

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The space $LB(E, F)$ of all linear bounded maps between two locally convex spaces E and F has been considered and studied under different points of view by several authors (see [2, 4, 10, 11, 16], [17, 11.6], [22, 23] *et al.*). In particular, taking E and F Fréchet spaces and endowing the space $LB_i(E, F)$ with the canonical structure of countable inductive limit of Fréchet spaces, the problem of finding conditions on E and/or F such that the (LF)-space $LB_i(E, F)$ is a topological subspace of $L_b(E, F)$, or equivalently the canonical inclusion map $LB_i(E, F) \hookrightarrow L_b(E, F)$ is also open, has been investigated (see [11, 4, 10]). This question is said to be the problem of the *projective description* of the (LF)-space $LB_i(E, F)$. It is worth noting that, if F is a Banach space, the question is closely related to the Grothendieck's one if $L_b(E, F)$ is again a (DF)-space.

In connection with the study of this problem, in [1, Theorem 1] the following characterization was given: $LB_i(\lambda_1(A), F)$ is a topological subspace of $L_b(\lambda_1(A), F)$ if and only if $\lambda_1(A)$ satisfies the density condition of Heinrich and a so-called condition (C_t) holds (thus obtaining a complete solution of an open problem posed by Bierstedt and Bonet [4, Section 4]). The aim of this paper is to extend the criterion to the class of the (LF)-spaces of type $LB(\lambda_p(A), F)$ with $1 < p < +\infty$ or $p = 0$. Actually, we prove that '$\lambda_p(A)$ satisfies the density condition' and 'condition (C_l) holds' always imply that $LB_i(\lambda_p(A), F)$ is a topological subspace of $L_b(\lambda_p(A), F)$ (Section 2). We also prove that these conditions become necessary provided $F = \lambda_q(A)$ or F contains a complemented copy of l_q with $1 < p \leq q < +\infty$ (Section 3).

As immediate consequences, we derive the well-known results that $L_b(\lambda_p(A), F)$ and $C(S, k_q(\mathscr{V}))$ $(1/p + 1/q = 1$ if $1 \leq p < +\infty$ or $q = 1$ if $p = 0)$ are complete (LB)-spaces (hence bornological and barrelled (DF)-spaces) whenever $\lambda_p(A)$ satisfies the density condition of Heinrich and F is a Banach space and $\lambda_p(A)$ is a Fréchet Montel space and S is a compact Hausdorff space, respectively.

1. Notation

We begin with the necessary notation. For other notation and definitions we refer the reader to [15, 17].

Let E and F be Fréchet spaces. Let $L(E, F)$ be the space of all linear and continuous maps from E into F and $L_b(E, F)$ the same space endowed with the topology of uniform convergence on bounded subsets of E.

Suppose that E is the reduced projective limit of the sequence $(E_n, \| \|_n)_n$ of Banach spaces with linking maps $\rho_{nm}\colon E_m \to E_n$, $n < m$, and $\rho_n\colon E \to E_n$, $n \in \mathbb{N}$ (that is, $\rho_{nm} \circ \rho_m = \rho_n$ for $n < m$) such that the sets $U_n := \{x \in E; \|\rho_n x\|_n \leq 1\}$ form a basis of 0-neighbourhoods in E. The duals norms are defined by $\|f\|'_n := \sup\{|f(x)|; x \in U_n\}$, $f \in E'$; hence $\| \|'_n$ is the gauge of $\mathring{U}_n$ in E' and $E'_n := (\operatorname{span} \mathring{U}_n, \| \|'_n)$ is a Banach space. Then for each $n < m$ the map $J_{nm}\colon L_b(E_n, F) \to L_b(E_m, F)$ defined by $J_{nm}T := T \circ \rho_{nm}$ is one-to-one, linear and continuous. Therefore, we can consider the inductive limit of Fréchet spaces $\operatorname{ind}_n(L_b(E_n, F), J_{nm})$, which continuously embeds in $L_b(E, F)$. Indeed, for each $n \in \mathbb{N}$ the map $J_n\colon L_b(E_n, F) \to L_b(E, F)$ defined by $J_nT := T \circ \rho_n$ is one-to-one, linear and continuous; thereby implying that there is a linear, one-to-one and continuous map $J\colon \operatorname{ind}_n L_b(E_n, F) \to L_b(E, F)$, whose image is the space $LB(E, F)$ of linear bounded maps from E into F. We define $LB_i(E, F) := \operatorname{ind}_n L_b(E_n, F)$.

In the sequel, for $n, k \in \mathbb{N}$ let $B_{n,k} := \{T \in L(E_n, F); \sup_{\|x\|_n \leq 1} |Tx|_k \leq 1\}$, where $(| |_k)_k$ denotes a fundamental increasing sequence of seminorms defining the topology of F such that the sets $V_k := \{x \in F;\ |x|_k \leq 1\}$ form a basis of 0-neighbourhoods in F.

Recall that, if F is a Banach space, then $LB(E, F) = L(E, F)$ algebraically and every bounded set of $L_b(E, F)$ is contained and bounded in $L_b(E_n, F)$ for some n; hence the (LB)-space $LB_i(E, F)$ is always regular.

A Fréchet space E is called *distinguished* if its inductive dual $E'_i = \mathrm{ind}_n E'_n$ and its strong dual E'_b are topologically isomorphic. In general, $E'_i = E'$, the inclusion map $E'_i \hookrightarrow E'_b$ is continuous and E'_i is the bornological space associated with E'_b. A Fréchet space E is said to satisfy the *density condition* of Heinrich [14] (see [2, Proposition 2]) if for any sequence $(\lambda_n)_n$ of strictly positive numbers there exists a bounded subset B of E such that

$$\forall n \in \mathbb{N}\ \exists m > n\ \exists \lambda > 0 : \bigcap_{j=1}^{m} U_j \subset \lambda B + U_n.$$

This density condition was introduced by Heinrich [14] and thoroughly studied for Fréchet and Köthe spaces by Bierstedt and Bonet [2]. It was proved in [2, Theorem 1.4] that a Fréchet space E has the density condition if and only if the bounded subsets of its strong dual are metrizable; hence every Fréchet space with the density condition is distinguished (see, [15, Section 29, 3.12 and 4.3]). Moreover, every quasinormable and every Fréchet Montel space has the density condition.

For the notations for Köthe echelon and co-echelon spaces we refer the reader to [6]. Nevertheless, we recall the following.

In the sequel, I will always denote a non void index set. Let $A = (a_n)_n$ and $\mathcal{V} = (v_n)_n$ ('weights') be sequences on I, with $0 < a_n(i) \leq a_{n+1}(i)$ and $v_n(i) = 1/a_n(i)$ for all $i \in I$ and $n \in \mathbb{N}$. The *maximal Nachbin family associated with* $\mathcal{V}$ is given by

$$\bar{V} := \left\{ \bar{v} \colon I \mapsto [0, +\infty[;\ \forall n \in \mathbb{N}\ \sup_{i \in I} \frac{\bar{v}(i)}{v_n(i)} = \sup_{i \in I} \bar{v}(i) a_n(i) < +\infty \right\}.$$

Let $1 \leq p < +\infty$ or $p = 0$. The Köthe echelon space of order p and the Köthe co-echelon space of order q, where $1/p + 1/q = 1$ if $1 \leq p < +\infty$ or $q = 1$ if $p = 0$, are defined by

$$\lambda_p(A) := \left\{ x = (x_i)_{i \in I};\ \forall n \in \mathbb{N}\ \|x\|_{n,p} := \left(\sum_{i \in I} a_n^p(i) |x_i|^p \right)^{1/p} < +\infty \right\},$$

if $1 \leq p < +\infty$, ($\lambda_0(A) := \{x = (x_i)_{i \in I};\ \forall n \in \mathbb{N}\ \lim_i a_n(i)|x_i| = 0\}$ and $\|x\|_{n,0} := \sup_{i \in I} a_n(i)|x_i|$, if $p = 0$) and

$$k_q(\mathcal{V}) := \left\{ x = (x_i)_{i \in I};\ \exists n \in \mathbb{N}\ \|x\|'_{n,q} := \left(\sum_{i \in I} v_n^q(i) |x_i|^q \right)^{1/q} < +\infty \right\},$$

if $1 \leq q < +\infty$, ($k_\infty(\mathcal{V}) := \{x = (x_i)_{i \in I};\ \exists n \in \mathbb{N}\ \|x\|'_{n,\infty} := \sup_{i \in I} v_n(i)|x_i| < +\infty\}$ if $q = +\infty$), respectively. Clearly, $k_q(\mathcal{V}) = \mathrm{ind}_n\, l_q(v_n)$. Also, for $1<p<+\infty$ or

$p = 0$, $(\lambda_p(A))'_b = (\lambda_p(A))'_i = k_q(\mathscr{V})$ by [6, Corollary 2.8] and hence $\lambda_p(A)$ is distinguished. By [6, Theorem 2.7], $(\lambda_p(A))'_b$ coincides algebraically and topologically with $K_q(\bar{V}) := \text{proj}_{\bar{v}\in\bar{V}}\, l_q(\bar{v})$. Finally, we recall that the sequence A is said to satisfy condition (D) if there exists an increasing sequence $J = (I_m)_m$ of subsets of I such that:

(N, J) For each $m \in \mathbb{N}$, there is an $n(m) \in \mathbb{N}$, $n(m) \geq m$, with $\inf_{i\in I_m} a_{n(m)}(i)/a_k(i) > 0$, $k = n(m)+1, n(m)+2, \ldots$.

(M, J) For each $n \in \mathbb{N}$ and for each subset $I_0 \subseteq I$ such that $I_0 \cap (I \setminus I_m) \neq \emptyset$ for all $m \in \mathbb{N}$, there exists an $n' = n'(n, I_0) > n$ with $\inf_{i\in I_0} a_n(i)/a_{n'}(i) = 0$.

This condition was introduced in [5, Theorem 2.3]. Moreover, it was proved in [2, Theorem 2.10] that, for $1 \leq p < +\infty$ or $p = 0$, $\lambda_p(A)$ has the density condition if and only if the sequence A satisfies condition (D). In particular, by [2, Theorem 2.4] $\lambda_1(A)$ has the density condition if and only if it is distinguished.

We will denote by $[l_p(I)]_1$ and by $[c_0(I)]_1$ the unit closed ball of $l_p(I)$ for $1 \leq p < +\infty$ and of $c_0(I)$ for $p = 0$, respectively.

REMARK 1.1. All proofs will be carried out for $1 \leq p < +\infty$, the case $p = 0$ being similar.

2. Sufficient conditions for projective descriptions of the (LF)-spaces $LB_i(\lambda_p(A), F)$

The aim of this section is to prove the following:

THEOREM 2.1. *Let $1 \leq p < +\infty$ or $p = 0$ and let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I. Let F be a Fréchet space with a fundamental increasing sequence $(|\ |_k)_k$ of continuous seminorms. If the following conditions hold*

(i) *the sequence A satisfies condition* (D) *and*

(ii) *for each $(\lambda_l)_l \subset \mathbb{R}_+$ and for each $(k(l))_l$ non-decreasing sequence of positive integers, there are $(\gamma_j)_j \subset \mathbb{R}_+$ and $k \in \mathbb{N}$ such that*

$$(C_t) \qquad \forall n \in \mathbb{N} \quad \sum_{j=1}^{n} \gamma_j B_{j,k} \subseteq \bigcup_{m\in\mathbb{N}} \sum_{l=1}^{m} \lambda_l B_{l,k(l)},$$

then $LB_i(\lambda_p(A), F)$ is a topological subspace of $L_b(\lambda_p(A), F)$.

To show the above theorem we need some preliminary results. The first one could be interesting in itself.

LEMMA 2.2. *Let* $1 \le p < +\infty$ *or* $p = 0$. *Let* $A = (a_n)_n$ *be an increasing sequence of strictly positive weights on* I *satisfying condition* (M, J). *Then, for each* $\bar{v} \in \bar{V}$ *and* $k \in \mathbb{N}$,

$$\lim_m \sup_{\mu \in [l_p(I)]_1} \left(\sum_{i \in I \setminus I_m} \bar{v}^p(i) a_k^p(i) |\mu_i|^p \right)^{1/p} = 0, \quad \text{if } 1 \le p < +\infty$$

or

$$\lim_m \sup_{\mu \in [c_0(I)]_1} \sup_{i \in I \setminus I_m} \bar{v}(i) a_k(i) |\mu_i| = 0, \quad \text{if } p = 0.$$

PROOF. Suppose that there exist $\bar{v} \in \bar{V}$ and $k_0 \in \mathbb{N}$ such that the sequence

$$\left\{ \sup_{\mu \in [l_p(I)]_1} \left(\sum_{i \in I \setminus I_m} \bar{v}^p(i) a_{k_0}^p(i) |\mu_i|^p \right)^{1/p} \right\}_m$$

does not converge to 0. Thus, there are $\epsilon_0 > 0$ and an increasing sequence of positive integers $(k(m))_m$ so that, for each $m \in \mathbb{N}$,

$$\sup_{\mu \in [l_p(I)]_1} \left(\sum_{i \in I \setminus I_{k(m)}} \bar{v}^p(i) a_{k_0}^p(i) |\mu_i|^p \right)^{1/p} > \epsilon_0.$$

It follows that, for each $m \in \mathbb{N}$, there is $\mu^m = (\mu_i^m)_i \in [l_p(I)]_1$ such that

$$\left(\sum_{i \in I \setminus I_{k(m)}} \bar{v}^p(i) a_{k_0}^p(i) |\mu_i^m|^p \right)^{1/p} > \epsilon_0$$

and hence

$$\epsilon_0 < \left(\sum_{i \in I \setminus I_{k(m)}} \bar{v}^p(i) a_{k_0}^p(i) |\mu_i^m|^p \right)^{1/p} \le \sup_{i \in I \setminus I_{k(m)}} \bar{v}(i) a_{k_0}(i) \left(\sum_{i \in I \setminus I_{k(m)}} |\mu_i^m|^p \right)^{1/p}$$
$$\le \sup_{i \in I \setminus I_{k(m)}} \bar{v}(i) a_{k_0}(i).$$

We can then find another increasing sequence $(m_r)_r$ of positive integers and a sequence $(i_r)_r \subset I$ such that, for each $r \in \mathbb{N}$, $i_r \in I_{k(m_r)} \setminus I_{k(m_{r-1})}$ $(m_0 := 1)$ and $\bar{v}(i_r) a_{k_0}(i_r) > \epsilon_0$. Let $Y := \{i_r; r \in \mathbb{N}\}$. Then $Y \cap (I \setminus I_m) \neq \emptyset$ for all $m \in \mathbb{N}$. By (M, J), for the given Y and k_0, there is $k > k_0$ so that $\inf_{r \in \mathbb{N}} a_{k_0}(i_r)/a_k(i_r) = 0$. But, since $\bar{v} \le \alpha_k v_k$ on I for some $\alpha_k > 0$, we obtain that

$$\epsilon_0 < a_{k_0}(i_r) \bar{v}(i_r) \le \alpha_k a_{k_0}(i_r) v_k(i_r) = \alpha_k a_{k_0}(i_r)/a_k(i_r)$$

for all $r \in \mathbb{N}$; thereby implying that $0 < \epsilon_0/\alpha_k \le \inf_{r \in \mathbb{N}} a_{k_0}(i_r)/a_k(i_r)$ which is a contradiction. □

Next, for a given increasing sequence $J = (I_m)_m$ of subsets of I such that $I = \bigcup_{m\in\mathbb{N}} I_m$, we introduce the following space

$$L(\lambda_p(A), F; J) := \left\{ T \in L(\lambda_p(A), F);\ \exists m \in \mathbb{N}\ \forall\lambda \in \lambda_p(A)\ T\left(\sum_{i\in I}\lambda_i e_i\right) = T\left(\sum_{i\in I_m}\lambda_i e_i\right)\right\},$$

where $(e_i)_{i\in I}$ denotes the usual vector basis of $\lambda_p(A)$, that is, $e_i = (\delta_{ij})_{j\in I}$. We observe that if $T \in L(\lambda_p(A), F; J)$, then there is $m \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there are $c_k > 0$ and $n_k \in \mathbb{N}$ for which

$$(1) \qquad |T(\lambda)|_k \le c_k \left\| \sum_{i\in I_m} \lambda_i e_i \right\|_{n_k,p}$$

for all $\lambda \in \lambda_p(A)$ (this follows from the fact that $T \in L(\lambda_p(A), F)$ and that, for each $m \in \mathbb{N}$, $\lambda_p(A; I_m) := \{\lambda \in \lambda_p(A);\ \lambda_i = 0 \text{ for all } i \notin I_m\}$ is a sectional subspace of $\lambda_p(A)$). Moreover, we have:

LEMMA 2.3. *Let $1 \le p < +\infty$ or $p = 0$. Let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I satisfying condition (M, J). Let F be a Fréchet space with a fundamental increasing sequence $(|\ |_k)_k$ of continuous seminorms. Then $L(\lambda_p(A), F; J)$ is a dense subspace of $L_b(\lambda_p(A), F)$.*

PROOF. Let $T \in L(\lambda_p(A), F)$. Let B be an absolutely convex bounded subset of $\lambda_p(A)$, $\epsilon > 0$ and $k \in \mathbb{N}$. By [6, Proposition 2.5] we can suppose that $B = \bar{v}[l_p(I)]_1 = \{\bar{v}\mu = (\bar{v}(i)\mu_i)_i;\ \mu \in [l_p(I)]_1\}$ for some $\bar{v} \in \bar{V}$.

Consider the absolutely convex 0-neighbourhood of $L_b(\lambda_p(A), F)$ given by

$$W := \left\{R \in L(\lambda_p(A), F);\ R(\bar{v}[l_p(I)]_1) \subseteq V_k\right\}.$$

We claim that there is $S \in L(\lambda_p(A), F; J)$ such that $T - S \in \epsilon W$, or equivalently $\sup_{\lambda\in\bar{v}[l_p(I)]_1} |T(\lambda) - S(\lambda)|_k < \epsilon$. Since T is linear and continuous, for the given k, there are $c_k > 0$ and $n_k \in \mathbb{N}$ such that, for each $\lambda \in \lambda_p(A)$, $|T(\lambda)|_k \le c_k\|\lambda\|_{n_k,p}$; thereby implying that, for each $\lambda = \bar{v}\mu \in \bar{v}[l_p(I)]_1$,

$$(2) \qquad |T(\lambda)|_k \le c_k \left(\sum_{i\in I} \bar{v}^p(i) a^p_{n_k}(i) |\mu_i|^p\right)^{1/p},$$

where by Lemma 2.2 there is $m_0 \in \mathbb{N}$ such that, for each $m \ge m_0$,

$$(3) \qquad \sup_{\mu\in[l_p(I)]_1} \left(\sum_{i\in I\setminus I_m} \bar{v}^p(i) a^p_{n_k}(i)|\mu_i|^p\right)^{1/p} < \epsilon/2c_k.$$

Now, we define a map $S : \lambda_p(A) \to F$ by $S(\lambda) := T\big(\sum_{i\in I_{m_0}} \lambda_i e_i\big) = \sum_{i\in I_{m_0}} \lambda_i T(e_i)$, $\lambda \in \lambda_p(A)$; clearly $S \in L(\lambda_p(A), F; J)$. Moreover, by (2) and (3), for each $\lambda = \bar{v}\mu \in \bar{v}[l_p(I)]_1$,

$$|T(\lambda) - S(\lambda)|_k = \left| T\left(\sum_{i\in I\setminus I_{m_0}} \lambda_i e_i\right)\right|_k \leq c_k \left(\sum_{i\in I\setminus I_{m_0}} \bar{v}^p(i) a_{n_k}^p(i) |\mu_i|^p\right)^{1/p} < \epsilon/2.$$

It follows that $\sup_{\lambda\in\bar{v}[l_p(I)]_1} |T(\lambda) - S(\lambda)|_k \leq \epsilon/2 < \epsilon$ and the proof is then complete. □

REMARK 2.4. From the proof of Lemma 2.3, it is clear that given $T \in L(\lambda_p(A), F)$ and, for each $m \in \mathbb{N}$, $T_m : \lambda_p(A) \to F$ defined by $T_m(\lambda) := T\big(\sum_{i\in I_m} \lambda_i e_i\big)$ for $\lambda \in \lambda_p(A)$, it holds that $(T_m)_m \subset L(\lambda_p(A), F; J)$ and $T_m \overset{m}{\mapsto} T$ in $L_b(\lambda_p(A), F)$.

On the other hand, we have:

LEMMA 2.5. *Let $1 \leq p < +\infty$ or $p = 0$. Let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I satisfying (N, J). Let F be a Fréchet space with a fundamental increasing sequence $(|\ |_k)_k$ of continuous seminorms. Then $L(\lambda_p(A), F; J)$ is a subspace of $LB(\lambda_p(A), F)$.*

PROOF. Let $T \in L(\lambda_p(A), F; J)$. Then, by (1) there is $m \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exist $c_k > 0$ and $n_k \in \mathbb{N}$, $n_k \geq k$, for which

$$|T(\lambda)|_k \leq c_k \left\|\sum_{i\in I_m} \lambda_i e_i\right\|_{n_k,p} = c_k \left(\sum_{i\in I_m} |\lambda_i|^p a_{n_k}^p(i)\right)^{1/p}$$

for all $\lambda \in \lambda_p(A)$.

Since A satisfies condition (N, J), there exists $n(m) \geq m$ such that, for each $h > n(m)$, $\inf_{i\in I_m} a_{n(m)}(i)/a_h(i) = \alpha_h > 0$, or equivalently $a_{n(m)} \geq \alpha_h a_h$ on I_m. Thus, for each $k > n(m)$ (and hence $n_k \geq k > n(m)$) and $\lambda \in \lambda_p(A)$,

$$\begin{aligned}|T(\lambda)|_k &\leq c_k \left(\sum_{i\in I_m} |\lambda_i|^p a_{n_k}^p(i)\right)^{1/p} \\ &\leq c_k \alpha_{n_k}^{-1} \left(\sum_{i\in I_m} |\lambda_i|^p a_{n(m)}^p(i)\right)^{1/p} = c_k \alpha_{n_k}^{-1} \left\|\sum_{i\in I_m} \lambda_i e_i\right\|_{n(m),p}.\end{aligned}$$

This means that $T \in L(l_p(a_{n(m)}), F)$ and hence $T \in LB(\lambda_p(A), F)$. □

At this point we are able to state and prove the following basic result towards Theorem 2.1.

LEMMA 2.6. *Let* $1 \le p < +\infty$ *or* $p = 0$ *and let* $A = (a_n)_n$ *be an increasing sequence of strictly positive weights on* I. *Let* F *be a Fréchet space with a fundamental increasing sequence* $(|\ |_k)_k$ *of continuous seminorms. If the sequence* A *satisfies condition* (N, J) *and condition* (C_l) *holds, then* $LB_i(\lambda_p(A), F)$ *and* $L_b(\lambda_p(A), F)$ *induce the same topology on* $L(\lambda_p(A), F; J)$.

PROOF. By Lemma 2.5, $L(\lambda_p(A), F; J) \subset LB(\lambda_p(A), F)$. We now remark that replacing the increasing sequence $(a_n)_n$ by $(a_{n(m)})_m$ if necessary, we can assume that $n(m) = m$ in condition (N, J) so that, for each $m \in \mathbb{N}$ and each $k > m$, $\inf_{i \in I_m} a_m(i)/a_k(i) > 0$. For each $m \in \mathbb{N}$, let $\delta_m := \inf_{i \in I_m} a_m(i)/a_{m+1}(i) > 0$.

For a fixed absolutely convex 0-neighbourhood U of $LB_i(\lambda_p(A), F)$, for each $l \in \mathbb{N}$, there is $k(l) > k(l-1)$ $(k(0) := 0)$ so that $B_{l,k(l)} \subseteq U$. Since U is an absolutely convex set, we have

$$\bigcup_{m \in \mathbb{N}} \sum_{l=1}^{m} 2^{-l} B_{l,k(l)} \subseteq U.$$

By (C_l), for the given sequences $(2^{-l})_l$ and $(k(l))_l$, there are $(\gamma_j)_j \subset \mathbb{R}_+$ and $k \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} \gamma_j B_{j,k} \subseteq \bigcup_{m \in \mathbb{N}} \sum_{l=1}^{m} 2^{-l} B_{l,k(l)}. \tag{4}$$

Inductively, we may now choose an increasing sequence $(\alpha_j)_j$ of positive numbers with, for each $j \in \mathbb{N}$, $\alpha_j \ge \gamma_j^{-1}$ and $\alpha_{j+1} \ge \alpha_j/\delta_j$ so that, defining $\bar{v} := \inf_{j \in \mathbb{N}} \alpha_j v_j \in \bar{V}$, we conclude (as in [5, Proof of Lemma 3.12]) that, for each $m \in \mathbb{N}$, $\bar{v}_{|I_m} = \min_{j \le m} \alpha_j v_j$. Let $B := \bar{v}[l_p(I)]_1$. By [6, Proposition 2.5], B is an absolutely convex bounded subset of $\lambda_p(A)$ and hence the set $V := \{T \in L(\lambda_p(A), F);\ T(B) \subseteq V_k\}$ is an absolutely convex 0-neighbourhood of $L_b(\lambda_p(A), F)$. We claim that $V \cap L(\lambda_p(A), F; J) \subseteq U$. Indeed, if $T \in V \cap L(\lambda_p(A), F; J)$, then

$$\sup_{\lambda \in B} |T(\lambda)|_k \le 1$$

and there is $K \in \mathbb{N}$ such that $T(\lambda) = T\left(\sum_{i \in I_K} \lambda_i e_i\right)$ for all $\lambda \in \lambda_p(A)$. For each $i \in I_K$ let $h(i) := \min\{j \in \{1, 2, \dots, K\};\ \alpha_j v_j(i) = \bar{v}(i)\}$ and for each $n = 1, \dots, K$, let $J_n := \{i \in I_K;\ h(i) = n\}$. Obviously, $I_K = \bigcup_{n=1}^{K} J_n$ and $J_n \cap J_m = \emptyset$ if $n \neq m$. Next, for each $n = 1, \dots, K$ we define a linear and continuous map $T^n : \lambda_p(A) \to F$ by

$$T^n(\lambda) := T\left(\sum_{i \in J_n} \lambda_i e_i\right) = \sum_{i \in J_n} \lambda_i T(e_i), \quad \lambda \in \lambda_p(A).$$

Clearly, $T = \sum_{n=1}^{K} T^n$. We show that $T^n \in \gamma_n B_{n,k}$ for all $n = 1, \dots, K$.

Since $T \in L(\lambda_p(A), F; J) \subset LB(\lambda_p(A), F)$, $T \in L(l_p(a_l), F)$ for some $l \in \mathbb{N}$ and hence, for each $h \in \mathbb{N}$ there is $c_h > 0$ such that

$$|T(\lambda)|_h \leq c_h \left(\sum_{i \in I_K} |\lambda_i|^p a_l^p(i) \right)^{1/p} \tag{5}$$

for all $\lambda \in l_p(a_l)$. On the other hand, by definition of $\bar{v}$, we have that, for each $i \in J_n$,

$$\frac{1}{a_n(i)} = v_n(i) = \alpha_n^{-1}\alpha_n v_n(i) = \alpha_n^{-1}\bar{v}(i) \leq \alpha_n^{-1}\alpha_l v_l(i) = \alpha_n^{-1}\alpha_l \frac{1}{a_l(i)}.$$

By (5) it follows that, for each $h \in \mathbb{N}$ and $\lambda \in l_p(a_n)$,

$$\begin{aligned} |T^n(\lambda)|_h = \left| T\left(\sum_{i \in J_n} \lambda_i e_i \right) \right|_h &\leq c_h \left(\sum_{i \in J_n} |\lambda_i|^p a_l^p(i) \right)^{1/p} \\ &\leq c_h \alpha_n^{-1}\alpha_l \left(\sum_{i \in J_n} |\lambda_i|^p a_n^p(i) \right)^{1/p}, \end{aligned}$$

which yields $T \in L(l_p(a_n), F)$. Moreover, if $\lambda \in l_p(a_n)$ with $\|\lambda\|_{n,p} \leq 1$, then $\lambda_{|J_n} = \bar{v}(\lambda_{|J_n})/\bar{v} \in 1/\alpha_n \bar{v}[l_p(I)]_1 = 1/\alpha_n B$ because

$$\left\| \frac{\lambda_{|J_n}}{\bar{v}} \right\|_{n,p} = \frac{1}{\alpha_n} \left(\sum_{i \in J_n} |\lambda_i|^p a_n^p(i) \right)^{1/p} \leq \frac{1}{\alpha_n} \|\lambda\|_{n,p} \leq \frac{1}{\alpha_n};$$

therefore $|T^n(\lambda)|_k = |T(\lambda_{|J_n})|_k \leq 1/\alpha_n$ and hence $|T^n(\lambda)|_k \leq \gamma_n$ because $1/\alpha_n \leq \gamma_n$.

We have thus shown that $T = \sum_{n=1}^{K} T^n \in \sum_{n=1}^{K} \gamma_n B_{n,k}$. By (4) it follows that $T \in \sum_{l=1}^{m} 2^{-l} B_{l,k(l)}$ for some $m \in \mathbb{N}$ and then $T \in U$. □

Finally, we can give

PROOF OF THEOREM 2.1. Since the sequence A satisfies condition (D) and (C_t) holds, by Lemma 2.6, $LB_i(\lambda_p(A), F)$ and $L_b(\lambda_p(A), F)$ induce the same topology on $L(\lambda_p(A), F; J)$ (where $L(\lambda_p(A), F; J) \subset LB(\lambda_p(A), F)$ by Lemma 2.5) and by Lemma 2.3 $L(\lambda_p(A), F; J)$ is a dense subspace of $L_b(\lambda_p(A), F)$. Consequently, $L(\lambda_p(A), F; J)$ is also a dense subspace of $LB_i(\lambda_p(A), F)$. Thus Lemma 1.2 of [7] implies that $LB_i(\lambda_p(A), F)$ is a topological subspace of $L_b(\lambda_p(A), F)$. □

REMARK 2.7. If F is a Banach space, condition (C_t) clearly holds because, denoting the norm of F by $|\ |$, $|\ |_k = |\ |$ for all $k \in \mathbb{N}$. Therefore, for $1 \leq p < +\infty$ or $p = 0$, $LB_i(\lambda_p(A), F) = L_b(\lambda_p(A), F)$ holds topologically whenever the sequence A

satisfies condition (D), that is, if $\lambda_p(A)$ has the density condition of Heinrich; in this case, it follows that $LB_i(\lambda_p(A), F) \simeq L_b(\lambda_p(A), F)$ is a complete (LB)-space. Unfortunately, this result remains true only in the setting of echelon spaces satisfying the density condition. Indeed, there are examples of Fréchet-Montel spaces E (hence with the density condition) such that $L_b(E, l_2)$ is not even a (DF)-space (see [21]) and of Fréchet-Schwartz spaces E and Banach spaces X such that $L_b(E, X)$ is not a (DF)-space (see [18]).

Then, by Remark 2.7 we immediately obtain well-known results of Bonet and Dìaz ([8, Theorem 13] for $1 < p < +\infty$) and of Bierstedt and Bonet ([2, Proposition 2.3], [3, Proposition 2.3] for $p = 1$), that is,

COROLLARY 2.8. *Let $1 \leq p < +\infty$ or $p = 0$. Let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I and let F be a Banach space. If the sequence A satisfies condition* (D), *then $L_b(\lambda_p(A), F)$ is a bornological (DF)-space.*

Moreover, by Remark 2.7 and by the fact that $L_b(C(S), E)$ is topologically isomorphic to $C(S, E'_b)$ for every Fréchet-Montel space E and compact Hausdorff space S, we also obtain the following result of Domański [13, Theorem 3.1]

COROLLARY 2.9. *Let S be a compact Hausdorff space and let $1 \leq p < +\infty$ or $p = 0$. Let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I and $\mathcal{V} = (1/a_n)_n$. If $\lambda_p(A)$ is a Fréchet-Montel space, then $C(S, k_q(\mathcal{V}))$, with $1/q + 1/p = 1$ if $1 \leq p < +\infty$ or $q = 1$ if $p = 0$, is a complete (LB)-space and hence bornological.*

Recall that the problem if $C(S, E'_b)$ is an (LB)-space for every Fréchet-Montel space E and every compact Hausdorff space S was posed by Bierstedt and Schmets [19, page 103]. Only other some partial solutions are known: $C(\beta\mathbb{N}, E'_b)$ and $c_0(E'_b)$ are (LB)-spaces for every Fréchet-Montel space E [12, Corollary 6.3] (the case $c_0(E'_b)$ has been solved by Dierolf); if E is a Fréchet-Schwartz space, $C(S, E'_b)$ is an (LB)-space for every compact Hausdorff space S.

3. Sufficient and necessary conditions for projective descriptions of the (LF)-spaces $LB_i(\lambda_p(A), F)$

It was proved in [1, Theorem 1] that conditions (i) and (ii) of Theorem 2.1 are also necessary for $LB_i(\lambda_1(A), F)$ to be a topological subspace of $L_b(\lambda_1(A), F)$. But this is no longer true for $1 < p < +\infty$ or $p = 0$ as the following result shows.

THEOREM 3.1. *Let E be a Fréchet space with a fundamental increasing sequence $(\| \|_n)_n$ of continuous seminorms. Then the following conditions are equivalent:*

(i) *$LB_i(E,\omega)$ is a topological subspace of $L_b(E,\omega)$.*
(ii) *E is distinguished and condition (C_t) holds.*

The above theorem should be compared with [4, Proposition 3.12]. Also, we point out that $\lambda_1(A)$ is distinguished if and only if it has the density condition, that is, the sequence A satisfies condition (D); while for $1 < p < +\infty$ or $p = 0$, $\lambda_p(A)$ is always distinguished and has the density condition if and only if the sequence A satisfies (D). Thus, for $E = \lambda_1(A)$, Theorem 3.1 is a direct consequence of [1, Theorem 1] and for $E = \lambda_p(A)$, $1 < p < +\infty$ or $p = 0$, gives a complete characterization.

PROOF. We start by observing that $LB_i(E,\omega) = \mathrm{ind}_n(E'_n)^{\mathbb{N}}$ and $L_b(E,\omega) = (E'_b)^{\mathbb{N}}$ hold topologically; hence condition (C_t) turns out as: for each $(\lambda_l)_l \subset \mathbb{R}_+$ and for each $(k(l))_l$ non-decreasing sequence of positive integers, there are $(\gamma_j)_j \subset \mathbb{R}_+$ and $k \in \mathbb{N}$ such that

$$(C_t)' \qquad \forall n \in \mathbb{N} \quad \sum_{j=1}^{n} \gamma_j (\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}} \subseteq \bigcup_{m\in\mathbb{N}} \sum_{l=1}^{m} \lambda_l (\mathring{U}_l)^{k(l)} \times (E'_l)^{\mathbb{N}}.$$

(i) implies (ii): Let define a map $J : E'_i \to \mathrm{ind}_n(E'_n)^{\mathbb{N}}$ by $J(u) := (u, 0, 0, \dots)$ for $u \in E'$ and a map $P : (E'_b)^{\mathbb{N}} \to E'_b$ by $P(u_n)_n := u_1$ for $(u_n)_n \in (E'_b)^{\mathbb{N}}$. Clearly, J is a topological isomorphism into and P is a topological surjection. Denoting by I the canonical inclusion of $\mathrm{ind}_n(E'_n)^{\mathbb{N}}$ into $(E'_b)^{\mathbb{N}}$, we have that $P \circ I \circ J = \mathrm{Id}_{E'}$. Since I is a topological isomorphism into, it follows that the map $E'_i \hookrightarrow E'_b$ is also open and hence E is distinguished.

Next, for fixed $(\lambda_l)_l \subset \mathbb{R}_+$ and $(k(l))_l \subset \mathbb{N}$, consider the set

$$U := \bigcup_{m\in\mathbb{N}} \sum_{l=1}^{m} 2^{-l} \lambda_l (\mathring{U}_l)^{k(l)} \times (E'_l)^{\mathbb{N}}, \tag{6}$$

which is an absolutely convex 0-neighbourhood of $\mathrm{ind}_n(E'_n)^{\mathbb{N}}$. Then, by assumption there are $k \in \mathbb{N}$ and a closed absolutely convex bounded subset B of E such that $V := (\mathring{B})^k \times (E')^{\mathbb{N}}$ is a closed absolutely convex 0-neighbourhood of $(E'_b)^{\mathbb{N}}$ and $V \cap \mathrm{ind}_n(E'_n)^{\mathbb{N}} \subset U$. Since B is a bounded set of E, there is $(\alpha_j)_j \subset \mathbb{R}_+$ so that, for each $j \in \mathbb{N}$, $\sup_{x\in B} \|x\|_j \le \alpha_j$. Put $\gamma_j := 2^{-j-1}\alpha_j^{-1} > 0$, we claim that, for each $n \in \mathbb{N}$, $\sum_{j=1}^{n} \gamma_j (\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}} \subset U$. Fix $n \in \mathbb{N}$ and $u \in \sum_{j=1}^{n} \gamma_j (\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}}$. Then $u = \sum_{j=1}^{n} u_j$, with $u_j = (u_{ij})_i \in \gamma_j (\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}}$ for all $j = 1, \dots, n$. Now, $u_j \in (E'_j)^{\mathbb{N}}$ and $u_{ij} \in \gamma_j \mathring{U}_j$ for all $j = 1, \dots, n$ and $i = 1, \dots, k$; hence for $j = 1, \dots, n$ and $i = 1, \dots, k$,

$$\sup_{x\in B} |u_{ij}(x)| \le \sup_{x\in B} \|u_{ij}\|'_j \|x\|_j \le \gamma_j \sup_{x\in B} \|x\|_j \le \gamma_j \alpha_j = 2^{-j-1}.$$

It follows that, for each $j = 1, \dots, n$, $u_j \in 2^{-j-1}V$; therefore

$$u = \sum_{j=1}^{n} u_j \in \sum_{j=1}^{n} 2^{-j-1} V \subset (1/2)V$$

and so $u \in U$. By $(C_t)'$ and by (6) the proof is then complete.

(ii) implies (i): Let W be an absolutely convex 0-neighbourhood of $\text{ind}_n(E'_n)^{\mathbb{N}}$. Then, there are $(\lambda_l)_l \subset \mathbb{R}_+$ and a non-decreasing sequence $(k(l))_l$ of positive integers such that, for each $l \in \mathbb{N}$, $\lambda_l(\mathring{U}_l)^{k(l)} \times (E'_l)^{\mathbb{N}} \subset W$ and hence

$$\bigcup_{m\in\mathbb{N}} \sum_{l=1}^{m} 2^{-l}\lambda_l(\mathring{U}_l)^{k(l)} \times (E'_l)^{\mathbb{N}} \subseteq W.$$

By (C_t), taking $(\lambda'_l)_l = (2^{-l}\lambda_l)_l$, there exist $(\gamma_j)_j \subset \mathbb{R}_+$ and $k \in \mathbb{N}$ for which

$$\sum_{j=1}^{n} \gamma_j(\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}} \subseteq \bigcup_{m\in\mathbb{N}} \sum_{l=1}^{m} 2^{-l}\lambda_l(\mathring{U}_l)^{k(l)} \times (E'_l)^{\mathbb{N}}.$$

Put $V := \bigcup_{n\in\mathbb{N}} \sum_{j=1}^{n} 2^{-j}\gamma_j \mathring{U}_j$. Then V is an absolutely convex 0-neighbourhood of E'_i. Since E is distinguished, there is a closed absolutely convex bounded set B of E such that $\mathring{B} \subset V$.

Consider the set $U := (\mathring{B})^k \times (E')^{\mathbb{N}}$. This is a 0-neighbourhood in $(E'_b)^{\mathbb{N}}$. We claim that $U \cap \text{ind}_n(E'_n)^{\mathbb{N}} \subset W$. Let $u \in U \cap \text{ind}_n(E'_n)^{\mathbb{N}}$. Then $u = (u_i)_i$ with $u_i \in \mathring{B}$ for all $i = 1, \dots, k$ and $u_i \in E'_{n_0}$ for all $i \in \mathbb{N}$ and some $n_0 \in \mathbb{N}$. Since $\mathring{B} \subset V$, there is $n'_o \in \mathbb{N}$ such that, for each $i = 1, \dots, k$,

$$u_i = \sum_{j=1}^{n'_0} 2^{-j}\gamma_j u_{ij}, \tag{7}$$

with $u_{ij} \in \mathring{U}_j$ for all $j = 1, \dots, n'_0$.

To conclude the proof we have to consider the following two cases.

$(n'_0 < n_0)$: Then, by (7),

$$\begin{aligned} u &= ((u_i)_{i\le k}, (0)_{i>k}) + ((0)_{i\le k}, (u_i)_{i>k}) \\ &= \sum_{j=1}^{n'_0} 2^{-j}\gamma_j((u_{ij})_{i\le k}, (0)_{i>k}) + 2^{-n_0}\gamma_{n_0}((0)_{i\le k}, (2^{n_0}\gamma_{n_0}^{-1}u_i)_{i>k}) \\ &\in \sum_{j=1}^{n'_0} 2^{-j}\gamma_j(\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}} + 2^{-n_0}\gamma_{n_0}(\mathring{U}_{n_0})^k \times (E'_{n_0})^{\mathbb{N}} \\ &\subseteq \sum_{j=1}^{n_0} 2^{-j}\gamma_j(\mathring{U}_j)^k \times (E'_j)^{\mathbb{N}} \subseteq W. \end{aligned}$$

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$(n_0 \leq n_0')$: Put $w_j := ((u_{ij})_{i\leq k}, (0)_{i>k})$ for $j \in \{1, \ldots, n_0'\} \setminus \{n_0\}$ and $w_{n_0} := ((u_{in_0})_{i\leq k}, (2^{n_0}\gamma_{n_0}^{-1}u_i)_{i>k})$. Then, $w_j \in (\mathring{U}_j)^k \times (E_j')^{\mathbb{N}}$ for $j \in \{1, \ldots, n_0'\} \setminus \{n_0\}$ and $w_{n_0} \in (\mathring{U}_{n_0})^k \times (E_{n_0}')^{\mathbb{N}}$ and hence

$$u = \sum_{j=1}^{n_0'} 2^{-j}\gamma_j w_j \in \sum_{j=1}^{n_0'} 2^{-j}\gamma_j(\mathring{U}_j)^k \times (E_j')^{\mathbb{N}} \subseteq W.$$

The proof is now complete. □

Under additional assumptions on F we are able to prove that conditions (i) and (ii) of Theorem 2.1 turn out to be also necessary for $LB_i(\lambda_p(A), F)$ to be a topological subspace of $L_b(\lambda_p(A), F)$ also for $1 < p < +\infty$. Indeed, we have

THEOREM 3.2. *Let $1 < p, q < +\infty$ and let $A = (a_n)_n$ be an increasing sequence of strictly positive weights on I. Let $F = \lambda_q(A)$ or F be a Fréchet space which contains a complemented copy of l_q. If $p \leq q$, then the following conditions are equivalent*:

(i) *$LB_i(\lambda_p(A), F)$ is a topological subspace of $L_b(\lambda_p(A), F)$.*

(ii) *The sequence A satisfies condition* (D) *and* (C_t) *holds.*

PROOF. (ii) implies (i): This follows from Theorem 2.1.

(i) implies (ii): First we show that (C_t) holds. Its proof is very similar to the one of Theorem 3.1 showing that (i) implies (ii).

Fix $(\lambda_l)_l \subset \mathbb{R}_+$ and a non-decreasing sequence $(k(l))_l$ of positive integers and consider the set $U := \bigcup_{m\in\mathbb{N}} \sum_{l=1}^{m} 2^{-l}\lambda_l B_{l,k(l)}$, which is an absolutely convex 0-neighbourhood in $LB_i(\lambda_p(A), F)$. By assumption, there are $k \in \mathbb{N}$ and an absolutely convex bounded subset B of E such that $V := \{T \in L(\lambda_p(A), F) : T(B) \subseteq V_k\}$ is a 0-neighbourhood of $L_b(\lambda_p(A), F)$ and $V \cap LB(\lambda_p(A), F) \subset U$.

Since B is a bounded set of $\lambda_p(A)$, for each $j \in \mathbb{N}$, $\alpha_j := \sup_{\lambda\in B} \|\lambda\|_{j,p} < +\infty$. Put $\gamma_j := 2^{-j-1}\alpha_j^{-1} > 0$ for all $j \in \mathbb{N}$. We prove that $\sum_{j=1}^{n} \gamma_j B_{j,k} \subset U$ holds for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $T \in \sum_{j=1}^{n} \gamma_j B_{j,k}$. Then $T = \sum_{j=1}^{n} T_j$, with $T_j \in \gamma_j B_{j,k}$ for all $j = 1, \ldots, n$. Consequently, for $j = 1, \ldots, n$, $T_j \in L(l_p(a_j), F)$ and

$$\sup_{\lambda\in B} |T_j(\lambda)|_k \leq \sup_{\lambda\in B} \gamma_j \|\lambda\|_{j,p} \leq \gamma_j\alpha_j = 2^{-j-1}.$$

This means that, for each $j = 1, \ldots, n$, $T_j \in 2^{-j-1}V$ and hence $T = \sum_{j=1}^{n} T_j \in \sum_{j=1}^{n} 2^{-j-1}V \subset V$. On the other hand, it is clear that $T \in LB(\lambda_p(A), F)$. Thus $T \in V \cap LB(\lambda_p(A), F) \subset U$.

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It remains to show that the sequence A satisfies condition (D). Proof of this is inspired by the one of [11, 4.8]. For this we have to consider two cases: (a) $F = \lambda_q(A)$, $p \leq q$; (b) F contains a complemented copy of l_q, $p \leq q$.

(a) Suppose that the sequence A does not satisfy condition (D). Then, by [3], $\lambda_p(A)$ has a sectional subspace isomorphic to $\lambda_p(B)$, where $B = (b_n)_n$ is a sequence on $\mathbb{N} \times \mathbb{N}$ given by

$$(8) \qquad \begin{cases} b_n(k,j) = b_1(k,j) = 1 & \text{for all } k \geq n; \\ b_{n+1}(n,j) \overset{j}{\mapsto} +\infty & \text{for all } n \in \mathbb{N}. \end{cases}$$

Since $\lambda_p(B)$ is a sectional subspace of $\lambda_p(A)$ and hence $\lambda_q(B)$ is also a sectional subspace of $\lambda_q(A)$, by assumption it clearly follows that $LB_i(\lambda_p(B), \lambda_q(B))$ is a topological subspace of $L_b(\lambda_p(B), \lambda_q(B))$.

For each $n \in \mathbb{N}$, put

$$U_n := \left\{ \lambda = (\lambda_{kj})_{kj} \in \lambda_p(B); \|\lambda\|_n := \left(\sum_{j,k=1}^{\infty} |\lambda_{kj}|^p b_n^p(k,j) \right)^{1/p} \leq 1 \right\}$$

and

$$V_n := \left\{ \mu = (\mu_{kj})_{kj} \in \lambda_q(B); |\mu|_n := \left(\sum_{j,k=1}^{\infty} |\mu_{kj}|^q b_n^q(k,j) \right)^{1/q} \leq 1 \right\}.$$

Then $(U_n)_n$ and $(V_n)_n$ form a basis of closed absolutely convex 0-neighbourhoods of $\lambda_p(B)$ and of $\lambda_q(B)$, respectively. Also, put

$$B_n := \{T \in L(l_p(b_n), \lambda_q(B));\ T(U_n) \subseteq V_1\} \quad \text{for all } n \in \mathbb{N},$$

we have that $B_n \subseteq B_{n+1}$ for every $n \in \mathbb{N}$ and hence $W := \overline{\bigcup_{n\in\mathbb{N}} B_n}$ (the closure is taken in $LB_i(\lambda_p(B), \lambda_q(B))$) is a closed absolutely convex 0-neighbourhood in $LB_i(\lambda_p(B), \lambda_q(B))$. Since $LB_i(\lambda_p(B), \lambda_q(B))$ is a topological subspace of $L_b(\lambda_p(B), \lambda_q(B))$, there is a closed absolutely convex bounded subset C of $\lambda_p(B)$ and $k_0 \in \mathbb{N}$ such that

$$U := \{T \in LB(\lambda_p(B), \lambda_q(B)):\ T(C) \subseteq V_{k_0}\} \subseteq (1/2)\, W.$$

Clearly, $C \subseteq \bigcap_{n=1}^{\infty} \sigma_n U_n$, with $(\sigma_n)_n \subset \mathbb{R}_+$ an increasing sequence such that $\sigma_n \geq 1$ for every $n \in \mathbb{N}$.

Now, we can find inductively an increasing sequence $(m(n))_n$ of positive integers such that, for each $n \in \mathbb{N}$,

$$(9) \quad m(n) > \max\left\{m(n-1), 2^{n+1}\sigma_{n+1}\right\} \quad \text{and} \quad b_n(n-1, m(n)+1) \geq m(n-1)+1,$$

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with $m(0) := 2\sigma_1$ and $m(1) := 2^2\sigma_2$. Indeed, suppose that we have determined $(m(n))_{n=1}^k$ such that (9) holds for $n = 1, \dots, k$. Since $b_{k+1}(k, j) \overset{j}{\mapsto} +\infty$ by (8), there is $m(k+1) > \max\{m(k), 2^{k+2}\sigma_{k+2}\}$ such that $b_{k+1}(k, m(k+1)+1) \geq m(k)+1$.

For each $n \in \mathbb{N}$, let

$$u_n := \left(\delta_{kn}\delta_{j\,m(n+1)+1}\frac{m(n)+1}{m(n)}\right)_{kj}.$$

Then, for each $n \in \mathbb{N}$, $u_n \in 2\mathring{U}_1 \cap 1/m(n)\mathring{U}_{n+1} \setminus \mathring{U}_n$ because of (8) and (9); indeed, for each $n \in \mathbb{N}$, we have

$$\|u_n\|'_n = \frac{1}{b_1(n, m(n+1)+1)}\frac{m(n)+1}{m(n)} = \frac{m(n)+1}{m(n)} \leq 2,$$

$$\|u_n\|'_{n+1} = \frac{1}{b_{n+1}(n, m(n+1)+1)}\frac{m(n)+1}{m(n)} \leq \frac{1}{m(n)+1}\frac{m(n)+1}{m(n)} = \frac{1}{m(n)},$$

$$\|u_n\|'_n = \frac{1}{b_n(n, m(n+1)+1)}\frac{m(n)+1}{m(n)} = \frac{m(n)+1}{m(n)} > 1.$$

Let $T\colon \lambda_p(B) \to \lambda_q(B)$ be a map defined by

$$T(\lambda) := \sum_{n=k_0}^{\infty} u_n(\lambda)e_{n,m(n+1)+1} = \sum_{n=k_0}^{\infty}\lambda_{n,m(n+1)+1}\frac{m(n)+1}{m(n)}e_{n,m(n+1)+1}$$

for $\lambda = (\lambda_{kj})_{kj} \in \lambda_p(B)$ ($(e_{kj})_{kj}$ denotes here the usual vector basis of $\lambda_p(B)$). Since $q \geq p$, it follows from (8) that, for each $h \in \mathbb{N}$ and $\lambda \in \lambda_p(B)$,

$$\begin{aligned}|T(\lambda)|_h &= \left(\sum_{n=k_0}^{\infty}|\lambda_{n,m(n+1)+1}|^q\left(\frac{m(n)+1}{m(n)}\right)^q b_h^q(n, m(n+1)+1)\right)^{1/q}\\ &\leq 2\left(\sum_{n=k_0}^{h-1}|\lambda_{n,m(n+1)+1}|^q b_h^q(n, m(n+1)+1) + \sum_{n\geq h}|\lambda_{n,m(n+1)+1}|^q\right)^{1/q}\\ &\leq 2c_h\left(\sum_{n=k_0}^{\infty}|\lambda_{n,m(n+1)+1}|^q\right)^{1/q} \leq 2c_h\|\lambda\|_1,\end{aligned}$$

where $c_h := \max_{k_0\leq n<h-1} b_h(n, m(n+1)+1)$; thereby implying that

$$|T(\lambda)|_1 \leq 2\|\lambda\|_1$$

holds for all $\lambda \in \lambda_p(B)$. Therefore, $T \in LB(\lambda_p(B), \lambda_q(B))$ and $T \in 2B_1$. Moreover, for each $\lambda \in C \subseteq \bigcap_{n=1}^{\infty}\sigma_n U_n$,

$$|T(\lambda)|_{k_0} = \left(\sum_{n=k_0}^{\infty}|u_n(\lambda)|^q\right)^{1/q} \leq \left(\sum_{n=k_0}^{\infty}\frac{1}{2^{q(n+1)}}\right)^{1/q} < 1$$

because $\lambda \in \sigma_{n+1} U_{n+1}$, $u_n \in 1/m(n)\mathring{U}_{n+1}$ and by (9) $m(n) > 2^{n+1}\sigma_{n+1}$ for every $n \in \mathbb{N}$. Thus $T \in U \subseteq 1/2W$.

Now, $u_n \notin \mathring{U}_n$ for each $n \geq k_0$ and hence there is a closed absolutely convex 0-neighbourhood V_n' in $k_{p'}(\mathcal{V}')$ $(1/p' + 1/p = 1, \mathcal{V}' = (1/b_n)_n)$ so that $u_n \notin \mathring{U}_n + V_n'$. Put $V := \bigcap_{n \geq k_0}(1/2\mathring{U}_n + V_n')$. Since it is bornivorous in $k_{p'}(\mathcal{V}')$, it is also a 0-neighbourhood in $k_{p'}(\mathcal{V}')$ and hence $\mathring{V}$ is a closed absolutely convex bounded subset of $\lambda_p(B)$ and

$$W_1 := \{S \in LB(\lambda_p(B), \lambda_q(B));\ S(\mathring{V}) \subseteq V_1\}$$

is an absolutely convex 0-neighbourhood in $LB_i(\lambda_p(B), \lambda_q(B))$.

Then $T \in 1/2W \subseteq 1/2\bigcup_{n \geq k_0} B_n + W_1$. Consequently, there are $n_0 \geq k_0$ and $T_1 \in 1/2B_{n_0}$, $T_2 \in W_1$ such that $T = T_1 + T_2$; we observe that $T_1(U_{n_0}) \subseteq 1/2V_1$ and $T_2(\mathring{V}) \subseteq V_1$, or equivalently $T_1'(\mathring{V}_1) \subseteq 1/2\mathring{U}_{n_0}$ and $T_2'(\mathring{V}_1) \subseteq V$. It follows that $T' = T_1' + T_2'$ and $(T' - T_1')(\mathring{V}_1) \subseteq V \subseteq 1/2\mathring{U}_{n_0} + V_{n_0}'$ so that $T'(\mathring{V}_1) \subseteq \mathring{U}_{n_0} + V_{n_0}'$. Since $e'_{n_0, m(n_0+1)+1} \in \mathring{V}_1$, by (8) ($(e'_{kj})_{kj}$ denotes the dual basis of $k_{p'}(\mathcal{V}')$), we obtain that $u_{n_0} = T'(e'_{n_0, m(n_0+1)+1}) \in \mathring{U}_{n_0} + V_{n_0}$, which is a contradiction.

(b) Arguing by contradiction as in the case (a), we find a sectional subspace of $\lambda_p(A)$ isomorphic to $\lambda_p(B)$, with B a sequence of weights on $\mathbb{N} \times \mathbb{N}$ satisfying conditions (8).

Since $\lambda_p(B)$ is a sectional subspace of $\lambda_p(A)$ and F contains a complemented copy of l_q, by assumption it clearly follows that $LB_i(\lambda_p(B), l_q)$ is a topological subspace of $L_b(\lambda_p(B), l_q)$. At this point to complete the proof it suffices to proceed in the same way as in the case (a) with the only change to consider a map $T\colon \lambda_p(B) \to l_q$ defined by

$$T(\lambda) := \sum_{n=k_0}^{\infty} u_n(\lambda) e_n$$

for $\lambda_p(B)$, where $(e_n)_n$ denotes the usual vector basis of l_q. □

Now Remark 2.7, Theorem 3.2 and [8, Theorem 13] imply again a well-known result of Bonet, Dìaz and Taskinen [9, Theorem 15], that is,

COROLLARY 3.3. *Let $1 < p \leq q < +\infty$ and let A be an increasing sequence of strictly positive weights on $\mathbb{N}$. Then $L_b(\lambda_p(A), l_q)$ is quasibarrelled if and only if it is bornological if and only if the sequence A satisfies condition* (D).

REMARK 3.4. Let $1 < p \leq q < +\infty$. Let A be an increasing sequence of strictly positive weights on I such that A does not satisfy condition (D). Let $F = \lambda_q(A)$ or let F be a Fréchet space which contains a complemented copy of l_q (as $(l_q)^{\mathbb{N}}$, $(l_q)^{\mathbb{N}} \cap l_r(l_r)$, $r > q$, $L^q_{\text{loc}}(\Omega)$ with Ω an open set of $\mathbb{R}^n$, etc.). Then $LB_i(\lambda_p(A), F)$ is not a topological subspace of $L_b(\lambda_p(A), F)$.

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