Canad. Math. Bull. Vol. **17** (1), 1974

# INCLUSION RELATIONS FOR GENERAL RIESZ TYPICAL MEANS

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Let $\alpha$ be a non-negative real number, $\lambda\equiv\{\lambda_n\}(n\geq 0)$ a strictly increasing unbounded sequence with $\lambda_0\geq 0$ and let $\sum_{m=0}^{\infty} a_m$ be an arbitrary series with partial sums $s\equiv\{s_n\}$. Write

$$A^\alpha(\omega)\equiv A^\alpha(\lambda,\omega)\equiv A^\alpha(\lambda,\sum a_m;\omega)\equiv A^\alpha(\lambda,s,\omega)\equiv\sum_{\lambda_n\leq\omega}(\omega-\lambda_n)^\alpha a_n=\int_0^\omega(\omega-t)^\alpha\,ds(t)$$

where $s(t)=s_n$ for $\lambda_n<t\leq\lambda_{n+1}$, $s(t)=0$ for $0\leq t\leq\lambda_0$. The series $\sum a_n$ or the sequence of partial sums $s=\{s_n\}$ is summable to $\hat{s}$ by the Riesz method $(R,\lambda,\alpha)$ if

$$(R,\lambda,\alpha,\omega)\equiv(R,\lambda,\alpha,\sum a_m,\omega)\equiv(R,\lambda,\alpha,s,\omega)\equiv\omega^{-\alpha}A^\alpha(\omega)\to\hat{s}$$

as $\omega\to\infty$.

For a given non-negative integer $p$ and a strictly increasing unbounded sequence $\lambda\equiv\{\lambda_n\}(n\geq 0)$ with $\lambda_0\geq 0$, denote by $\bar{T}^{(p)}$ and $T^{(p)}$ the $(C,\lambda,p)$ series-to-sequence and sequence-to-sequence matrices, respectively; thus for $p>0$

$$\bar{T}^{(p)}_{n\nu}=(1-\lambda_\nu/\lambda_{n+1})\cdots(1-\lambda_\nu/\lambda_{n+p})\quad(0\leq\nu\leq n),\qquad\bar{T}^{(p)}_{n\nu}=0(\nu>n)$$

$$T^{(p)}_{n\nu}=\Delta_\nu\bar{T}^{(p)}_{n\nu}\equiv\bar{T}^{(p)}_{n\nu}-\bar{T}^{(p)}_{n,\nu+1}$$

and

$$\bar{T}^{(0)}_{n\nu}=1\quad(0\leq\nu\leq n),\qquad\bar{T}^{(0)}_{n\nu}=0\quad(\nu>n)$$
$$T^{(0)}_{n\nu}=0\quad(\nu\neq n),\qquad T^{(0)}_{n\nu}=1\quad(\nu=n).$$

The $(C,\lambda,p)$ mean of a series $\sum a_m$ with partial sums $s$ is

$$t^{(p)}_n\equiv t^{(p)}_n(s)\equiv t^{(p)}_n(\sum a_m)\equiv\sum_{\nu=0}^n\bar{T}^{(p)}_{n\nu}a_\nu=\sum_{\nu=0}^n T^{(p)}_{n\nu}s_\nu\equiv C^{(p)}_n(s)/(\lambda_{n+1}\cdots\lambda_{n+p}).$$

The series $\sum a_m$ or the sequence of partial sums $\{s_m\}$ is summable $(C,\lambda,p)$ to $\hat{s}$ if $t^{(p)}_n\to s$ as $n\to\infty$. The inverse matrices

$$T''^{(p)}\equiv(T''^{(p)}_{nm})\qquad T'^{(p)}\equiv(T'^{(p)}_{nm})\qquad(n,m=0,1,2,\ldots)$$

of $\bar{T}^{(p)}$ and $T^{(p)}$, respectively, are given (see [21] p. 297–298) by

$$(1)\quad\begin{cases}T''^{(p)}_{rk}=(-1)^{p+1}(\lambda_{k+p+1}-\lambda_k)\lambda_{k+1}\cdots\lambda_{k+p}/\beta^{(p)}_{rk} & (0\leq k\leq r\leq k+p+1)\\ T''^{(p)}_{rk}=0\quad\text{otherwise, where }\beta^{(p)}_{rk}=\prod_{j=k}^{k+p+1}{}'(\lambda_r-\lambda_j)\end{cases}$$

$$(2)\qquad T'^{(p)}_{rk}=\sum_{\nu=k}^{r}T''^{(p)}_{\nu k}\quad(0\leq k\leq r\leq k+p),\qquad T'^{(p)}_{rk}=0\text{ otherwise};$$

$\Pi'$ in (1) indicates that the zero factor corresponding to $j=r$ is to be omitted.



https://doi.org/10.4153/CMB-1974-009-0 Published online by Cambridge University Press

For an arbitrary $B=(b_{\rho\nu})$ ($\rho$ may be a continuous or discrete parameter) we denote by $c_B$ and $c_B^0$, respectively, the linear space of all $B$-limitable and $B$-limitable to zero sequences. It was proved by Peyerimhoff [12, §8] that the linear spaces $c^0_{(R,\lambda,\alpha)}$ and $c_{(R,\lambda,\alpha)}$ with the norm $\|x\|=\sup_{\omega\geq 0}|(R,\lambda,\alpha,x,\omega)|$ are $BK$-spaces. Denote these two $BK$-spaces, respectively, by $R_{\lambda\alpha}(c^0)$ and $R_{\lambda\alpha}(c)$ and the norm by $\|\cdot\|_{\lambda\alpha}$. Given two matrices $A$ and $B$, we say that $B$ is stronger than $A$ or includes $A$ if $c_A\subseteq c_B$. Limits of summation are assumed throughout $0,\infty$ unless otherwise specified, and $\Delta x_n=x_n-x_{n+1}$; $\Lambda_n=\lambda_{n+1}/(\lambda_{n+1}-\lambda_n)$. Sums $\sum_{j=m}^{n}$ where $n<m$ are defined as equal to zero.

A number of special results exist for summability methods $B$ which include Riesz summability $(R,\lambda,\alpha)$—see Kuttner [8], Russell [15], Rangachari [13], Meir [11] and Borwein and Russell [3]. The question of necessary and sufficient conditions to be satisfied by an arbitrary method in order that it will include $(R,\lambda,\alpha)$ has received an answer for limited values of $\lambda$ and $\alpha$. A complete solution was given when $0\leq\alpha\leq 1$ by Russell [20], without any restrictions on $\lambda$. Maddox [9] obtained necessary conditions for a series-to-sequence method to include $(R,\lambda,\alpha)$ when $\alpha>0$ and $\lambda$ is suitably restricted. Maddox [9] conjectured that the necessary conditions are also sufficient. This conjecture was proved by Russell [20, Theorem 2] for $0<\alpha\leq 1$, by Jakimovski and Tzimbalario [6] for $1<\alpha\leq 2$ and in Russell [21, page 300] for $\alpha=2,3,4,\ldots$, with a weaker restriction on $\lambda$. Here we give a complete solution for a sequence-to-sequence or series-to-sequence method $B$ to be stronger than $(R,\lambda,\alpha)$ if $\alpha>2$ too. Using this result we prove the conjecture by Maddox for $\alpha>2$ with the weaker restriction on $\lambda$ given by Russell. These results are obtained by showing that certain sequences are a Schauder-basis in $R_{\lambda\alpha}(c^0)$.

The main results to be proved here are as follows:

THEOREM 1. *Let $\alpha>0$ and denote $p<\alpha\leq p+1$, where $p$ is an integer. In order that a sequence-to-sequence or sequence-to-function method $B=(b_{\rho\nu})$ shall include $(R,\lambda,\alpha)$ it is necessary that*

$$\exists \lim_{\rho} b_{\rho\nu}\equiv\beta_\nu \qquad (\nu=0,1,2,\ldots), \tag{3}$$

$$\exists \lim_{\rho}\sum_{\rho} b_{\rho\nu}\equiv\beta, \tag{4}$$

*and that a family of functions $\{g_\rho\}$ exists, defined in $[\lambda_0,\infty)$ such that*

$$\text{(i)}\ b_{\rho\nu}=\Delta_\nu\int_{\lambda_\nu}^{\infty}(\omega-\lambda_\nu)^\alpha\,dg_\rho(\omega), \qquad \text{(ii)}\ \int_{\lambda_0}^{\infty}\omega^\alpha|dg_\rho(\omega)|\equiv M_\rho\leq M<\infty. \tag{5}$$

*If $0\leq\alpha\leq 1$ (without restrictions on $\lambda$) then (3), (4) and (5) are also sufficient. If $\alpha>1$ it is also necessary that*

$$\lim_{n\to\infty}\sum_{r=1}^{p}\left(\sum_{k=1}^{r}T'^{(p)}_{n+r,n+k}t^{(p)}_{n+k}(x)\right)b_{\rho,n+r}=0 \tag{6}$$

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*for each* $\rho$ *and each* $x \in c^0_{(R,\lambda,\alpha)}$, *and (without restrictions on* $\lambda$*)* (3), (4), (5) *and* (6) *are also sufficient. If the method B is row-finite i.e.* $b_{\rho\nu}=0$ *for* $\nu \geq \nu(\rho)$ *for each* $\rho(\nu(\rho)<+\infty)$, *then* (3), (4) *and* (5) *are necessary and sufficient for B to include* $(R, \lambda, \alpha)$ *for* $\alpha>0$.

THEOREM 2. *Let* $\alpha>1$. *A sequence-to-sequence or sequence-to-function method* $B\equiv(b_{\rho\nu})$ *which satisfies*

$$(7) \qquad |b_{\rho\nu}| \leq H_\rho \Lambda_\nu^{-\alpha} \text{ for each } \rho \text{ and each } \nu = 0, 1, 2, \ldots$$

*includes* $(R, \lambda, \alpha)$ *if, and only if*, (3) (4) *and* (5) *are satisfied.*

THEOREM 3. *Let* $\alpha>1$ *and assume* $\Lambda_{n-1}=0(\Lambda_n)$. *A sequence-to-sequence or sequence-to-function method* $B\equiv(b_{\rho\nu})$ *includes* $(R, \lambda, \alpha)$ *if, and only if,* (3), (4), (5) *and* (7) *are satisfied.*

THEOREM 4. *Let* $\alpha>0$, *assume* $\Lambda_n\neq 0(1)$, *and when* $\alpha>1$ *assume* $\Lambda_{n-1}=0(\Lambda_n)$. *In order that a series-to-sequence or series-to-function method* $\bar{B}=(\bar{b}_{\rho\nu})$ *shall include* $(R, \lambda, \alpha)$ *it is necessary and sufficient that*

$$(8) \qquad \exists \lim_{\rho} \bar{b}_{\rho\nu} \equiv \bar{\beta}_\nu \quad \text{for} \quad \nu = 0, 1, 2, \ldots,$$

$$(9) \qquad |\bar{b}_{\rho\nu}| \leq H_\rho \Lambda_\nu^{-\alpha},$$

*and that a family of functions* $\{g_\rho\}$ *exists, defined in* $[\lambda_0, \infty)$, *such that:*

$$(10) \qquad \bar{b}_{\rho\nu} = \int_{\lambda_\nu}^{\infty} (\omega-\lambda_\nu)^\alpha \, dg_\rho(\omega), \qquad \int_{\lambda_0}^{\infty} \omega^\alpha |dg_\rho(\omega)| \equiv M_\rho \leq M < \infty.$$

*If the method B is row-finite, it is not necessary to assume that* $\Lambda_{n-1}=0(\Lambda_n)$ *when* $\alpha>1$ *and* (8) *and* (10) *are necessary and sufficient for* $\bar{B}$ *to include* $(R, \lambda, \alpha)$ *when* $\alpha>0$.

No real generality is lost by the assumption $\Lambda_n\neq 0(1)$, since otherwise $(R, \lambda, \alpha)$ will be equivalent to convergence for all $\alpha>0$ (Hardy and Riesz [5, Theorem 21])

THEOREM 5. *For each* $\alpha>0$, $\alpha=p+\delta$ *where p is an integer and* $0<\delta\leq 1$, *the sequence* $\{\delta^{(p,j)}\}_{j\geq 0}$ *defined by* $\delta^{(p,j)}=T'^{(p)}e^j$, $e^j=(0, 0, \ldots, 0, 1, 0, \ldots)$ *where* 1 *is the j-th coordinate, is a Schauder-basis in* $R_{\lambda\alpha}(c^0)$ *and we have* $x=\sum_{j=0}^{\infty} t_j^{(p)}(x)\delta^{(p,j)}$ *for each* $x \in R_{\lambda\alpha}(c^0)$, *where convergence is in the norm of* $R_{\lambda\alpha}(c^0)$.

In the proof of these theorems we use the following lemmas.

LEMMA 1. *Suppose p is a non-negative integer and* $0<\delta\leq 1$. *If* $\sum a_n$ *is summable* $(R, \lambda, p+\delta)$ *to zero, then for* $k=0, 1, \ldots, p$ $(R, \lambda, k, \sum a_m, \omega)=o(\Lambda_n^{\alpha-k})$ *for* $\lambda_n\leq\omega\leq\lambda_{n+1}$.

**Proof.** This is a limitation theorem for Riesz means in a form given by Borwein [1, Lemma 2 in o-form].

LEMMA 2. *If* $\alpha>0$ *and* $\beta>0$, *then*

$$A^{\alpha+\beta}(\omega)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^\omega(\omega-u)^{\beta-1}A^\alpha(u)\,du, \tag{11}$$

**Proof.** For this lemma see Hardy and Riesz [5, Lemma 6, p. 27].

LEMMA 3. *Let* $\alpha>0$; *if* $\alpha>1$ *assume* $\Lambda_{n-1}=0(\Lambda_n)$. *Then in order that* $\sum b_na_n$ ($\sum b_ns_n$) *should converge whenever* $\sum a_n(s)$ *is summable* $(R,\lambda,\alpha)$, *it is necessary that* $b_n=0(\Lambda_n^{-\alpha})$.

**Proof.** For this lemma see Russell [16, Theorem 2].

Given a function $f$, defined in an interval $[a, b]$, and distinct points $x_j$ in this interval, we define the divided differences by $f[x]=f(x)$ and

$$f[x_0,\ldots,x_n]=\{f[x_0,\ldots,x_{n-1}]-f[x_1,\ldots,x_n]\}/(x_0-x_n)\qquad(n=1,2,\ldots).$$

LEMMA 4. *Let* $p$ *be a given positive integer. For each* $n\geq 1$, *there exist real numbers* $c_j^{(n,p)}$, $\omega_j^{(n,p)}$ $(j=0,1,\ldots,p)$ *satisfying* $\sum_{j=0}^p c_j^{(n,p)}=1$, $|c_j^{(n,p)}|\leq H^{(p)}$ *for* $j=0,1,\ldots,p$, *where* $H^{(p)}$ *depends on* $p$ *but not on* $n$, $\lambda_{m(n)}\leq w_j^{(n,p)}\leq\lambda_{m(n)+1}$ *for* $j=0,1,\ldots,p$, *where* $m=m(n)$ *is defined by*

$$\lambda_{m+1}-\lambda_m=\max_{n\leq j<n+p}(\lambda_{j+1}-\lambda_j)\quad\text{if}\quad\lambda_{n+p}/\lambda_n\leq(p+1)^p,$$

*and* $m=n+r$ *where* $0\leq r<p$, $\lambda_{n+r+1}/\lambda_{n+r}>p+1$ *and*

$$\lambda_{n+j+1}/\lambda_{n+j}\leq p+1\quad\text{for}\quad 0\leq j<r\quad\text{if}\quad\lambda_{n+p}/\lambda_n>(p+1)^p;$$

*and* $t_n^p(x)=\sum_{j=0}^p c_j^{(n,p)}\,(R,\lambda,p,x,\omega_j^{(n,p)})$ *for any sequence* $x$. *In particular*

$$\sum_{j=0}^p|c_j^{(n,p)}|\leq(p+1)H^{(p)}$$

*for* $n\geq 1$.

**Proof.** For this lemma see A. Meir [11, The Lemma and its proof], D. Borwein and D. C. Russell [3, The Lemma and its proof], D. Borwein [2, Proof of the Theorem], and D. C. Russell [17, Lemma 2′].

LEMMA 5. *Let* $p$ *be a positive integer. Then for any infinite sequence* $x$, *we have:*

$$A^p(\lambda,x,\omega)=\sum_{\nu=n-p}^n\beta_\nu^p(\omega)t_\nu^{(p)}(x)\qquad(\lambda_n<\omega\leq\lambda_{n+1})$$

*where*

$$\beta_\nu^p(\omega)=(-1)^{p+1}c_\omega[\lambda_\nu,\ldots,\lambda_{\nu+p+1}](\lambda_{\nu+p+1}-\lambda_\nu)\lambda_{\nu+1}\cdots\lambda_{\nu+p},$$

*and*

$$c_\omega[t]=c_\omega^{(p)}(t)=\begin{cases}(\omega-t)^p & \text{if}\quad 0\leq t<\omega\\ 0 & \text{if}\quad t\geq\omega.\end{cases}$$

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*We have also for* $\lambda_n < \omega \le \lambda_{n+1}$ *and* $n-p \le \nu \le n$ $\beta_\nu^{(p)}(\omega) \ge 0$ *and*

$$\lim_{\omega\to\infty} \frac{1}{\omega^p} \sum_{\nu=n-p}^{n} \beta_\nu^p(\omega) = 1.$$

**Proof.** For this lemma see D. C. Russell [17, (33) and pp. 426–7; and 18].

LEMMA 6. *Let* $\alpha > 1$. *Suppose* $x \in c^0_{(R,\lambda,\alpha)}$ *and* $\alpha = p+\mu$ *where* $p$ *is a positive integer and* $0 < \mu \le 1$. *Then*

$$t_n^{(k)}(x) = o\left(\min_{n \le r < n+k} \Lambda_r^{\alpha-k}\right) \quad as \quad n \to \infty \quad for \quad k = 1, 2, \ldots, p.$$

**Proof.** Suppose $k=p$. By Lemma 4, we get

$$\begin{aligned} |t_n^{(p)}(x)| &= \left| \sum_{j=0}^{p} c_j^{(n,p)}(R, \lambda, p, x, \omega_j^{(n,p)}) \right| \\ &\le \left\{ \sum_{j=0}^{p} |c_j^{(n,p)}| \right\} \sup_{\lambda_{m(n)} \le \omega \le \lambda_{m(n)+1}} |(R, \lambda, p, x, \omega)| \\ &\le (p+1) H^{(p)} \sup_{\lambda_{m(n)} \le \omega \le \lambda_{m(n)+1}} |(R, \lambda, p, x, \omega)| \end{aligned}$$

(and by Lemma 1)

$$= o(\Lambda^\mu_{m(n)}) \qquad (n \to \infty).$$

Now for each $q$, $0 \le q < p$, we have

$$\Lambda_{m(n)}/\Lambda_{n+q} = \begin{cases} \dfrac{\lambda_{n+q+1} - \lambda_{n+q}}{\lambda_{m(n)+1} - \lambda_{m(n)}} \cdot \dfrac{\lambda_{m(n)+1}}{\lambda_{n+q+1}} \\[2ex] \dfrac{\lambda_{n+q+1} - \lambda_{n+q}}{\lambda_{n+q+1}} \cdot \dfrac{1}{1 - \lambda_{m(n)}/\lambda_{m(n)+1}} \end{cases}$$

(and by Lemma 4)

$$\begin{aligned} &\le \begin{cases} \lambda_{m(n)+1}/\lambda_{n+q+1} & \text{if} \quad \lambda_{n+p}/\lambda_n \le (p+1)^p \\ (1 - \lambda_{m(n)}/\lambda_{m(n)+1})^{-1} & \text{if} \quad \lambda_{n+p}/\lambda_n > (p+1)^p \end{cases} \\ &\le \begin{cases} \lambda_{n+p}/\lambda_n \le (p+1)^p & \text{if} \quad \lambda_{n+p}/\lambda_n \le (p+1)^p \\ \left(1 - \dfrac{1}{p+1}\right)^{-1} = \dfrac{(p+1)}{p} & \text{if} \quad \lambda_{n+p}/\lambda_n > (p+1)^p \end{cases} \\ &= 0(1), \qquad n \to \infty. \end{aligned}$$

Hence, for $\Lambda_{n+q} = \min\limits_{n \le r < n+p} \Lambda_r$

$$\begin{aligned} |t_n^{(p)}(x)| &= o(\Lambda^\mu_{m(n)}) = o((\Lambda_{m(n)}/\Lambda_{n+q})^\mu \Lambda^\mu_{n+q}) \\ &= o(\Lambda^\mu_{n+q}) = o(\min_{n \le r < n+p} \Lambda_r^\mu). \end{aligned}$$

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Suppose now the lemma is true for some $k$, $1<k\le p$. By Russell [17, (28)] we have

$$t_n^{(k-1)}(x)=\{\lambda_{n+k}t_n^{(k)}(x)-\lambda_n t_{n-1}^{(k)}(x)\}/(\lambda_{n+k}-\lambda_n).$$

Since $\lambda_{n+k}/(\lambda_{n+k}-\lambda_n)$ and $\lambda_n/(\lambda_{n+k}-\lambda_n)$ are not larger than $\min_{n\le r<n+k-1}\Lambda_r$, we get

$$\begin{aligned}|t_n^{(k-1)}(x)| &\le \Big(\min_{n\le r<n+k-1}\Lambda_r\Big)(|t_n^{(k)}(x)|+|t_{n-1}^{(k)}(x)|)\\ &= o\Big(\min_{n\le r<n+k-1}\Lambda_r^{\alpha-(k-1)}\Big) \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence Lemma 6 is true for $k-1$ too; and by induction Lemma 6 is true for $1\le k\le p$.

LEMMA 7. *Let p be a positive integer if* $p>1$ *suppose* $\Lambda_{n-1}=0(\Lambda_n)$. *Then we have for* $1\le k\le r\le p$

$$|T'^{(p)}_{n+r,n+k}|\le G_p\Lambda_{n+r}^p.$$

**Proof.** This lemma is (29) in Russell [22].

**Proof of Theorem 5.** For any sequence $\{y_j\}_{j\ge 0}$ we have, since $\delta^{(p,j)}=T'^{(p)}e^j$

$$\sum_{j=0}^n y_j\,\delta^{(p,j)}=\sum_{j=0}^n y_jT'^{(p)}e^j=T'^{(p)}\sum_{j=0}^n y_je^j\equiv\sum_{j=0}^{n+p}\xi_j^{(n)}e^j$$

where $\xi_j^{(n)}=(T'^{(p)}y)_j$ for $0\le j\le n$, since $T'^{(p)}$ is a normal matrix and only the elements $T'^{(p)}_{nk}$ $(n-p\le k\le n,\ n=0,1,2,\dots)$ may be different from zero. Since in the space $R_{\lambda\alpha}(c^0)$ the coordinates are continuous (see Peyerimhoff [12, §8]) $x=\sum_{j=0}^{\infty}y_j\delta^{(p,j)}$ implies $(T'^{(p)}y)_j=x_j$ for $j\ge 0$, or $T'^{(p)}y=x$. Hence $y=T^{(p)}x$ or $y_j=t_j^{(p)}(x)$ for $j\ge 0$. We get in particular for any sequence $x$

$$\sum_{j=0}^n t_j^{(p)}(x)\,\delta^{(p,j)}=\sum_{j=0}^{n+p}x_je^j-\sum_{r=1}^{p}\Big(\sum_{k=1}^{r}T'^{(p)}_{n+1,n+k}t_{n+k}^{(p)}(x)\Big)e^{n+r}, \tag{12}$$

since $T'^{(p)}$ is a normal matrix and only the elements $T'^{(p)}_{nk}$ with $n-p\le k\le n$, $n=0,1,2,\dots$ may be different from zero. To complete the proof we have to show that for each $x\in R_{\lambda\alpha}(c^0)$ the norm of the sequence ${}^nx$ defined by

$${}^nx=x-\sum_{j=0}^n t_n^{(p)}(x)\,\delta^{(p,j)}$$

tends to zero as $n\to\infty$. We have

$$\begin{aligned}{}^nx &= x-\sum_{j=0}^n t_j^{(p)}(x)\,\delta^{(p,j)}\\ &= T'^{(p)}(T^{(p)}x)-\sum_{j=0}^n t_j^{(p)}(x)\,\delta^{(p,j)}\\ &= T'^{(p)}\Big(T^{(p)}x-\sum_{j=0}^n t_j^{(p)}(x)e^j\Big)\\ &\equiv T'^{(p)}\xi^{(n)}(x)\end{aligned} \tag{13}$$

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where $\xi_k^{(n)}(x)=0$ for $0\le k\le n$ and $\xi_k^{(n)}(x)=t_k^{(p)}(x)$ for $k>n$. By Lemma 5 we get now

$$(14)\qquad A^p(\lambda, {}^n x, \omega) = \begin{cases} 0 & \text{for } 0\le\omega\le\lambda_{n+1} \\ \sum_{\nu=n+1}^{n+r} \beta_\nu^p(\omega) t_\nu^{(p)}(x) & \text{for } \lambda_{n+r}<\omega\le\lambda_{n+r+1} \\ & \quad 1\le r\le p \\ A^p(\lambda, x, \omega) & \text{for } \omega>\lambda_{n+p+1}. \end{cases}$$

By (11) and (14) we get for $x\in R_{\lambda\alpha}(c^0)$

$$(15)\qquad \frac{p!\,\Gamma(\alpha-p)}{r(\alpha+1)}(R, \lambda, \alpha, {}^n x, \omega) = \omega^{-\alpha}\int_0^\omega A^p(\lambda, {}^n x, u)(\omega-u)^{\alpha-p-1}\,du =$$

$$(16)\qquad = \begin{cases} 0 & \text{for } 0\le\omega\le\lambda_{n+1} \\ \omega^{-\alpha}\left\{\sum_{r=1}^{q-1}\int_{\lambda_{n+r}}^{\lambda_{n+r+1}}\sum_{\nu=n+1}^{n+r}\beta_\nu^p(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\right. & \\ \qquad\left.+\int_{\lambda_{n+q}}^{\omega}\sum_{\nu=n+1}^{n+q}\beta_\nu^{(p)}(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\right\} \equiv J_{n,q}(x,\omega) & \\ & \text{for } \lambda_{n+q}<\omega\le\lambda_{n+q+1} \\ & \quad 1\le q\le p \\ \omega^{-\alpha}\left\{\sum_{r=1}^{p}\int_{\lambda_{n+1}}^{\lambda_{n+r+1}}\sum_{\nu=n+1}^{n+r}\beta_\nu^p(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\right\} & \\ \qquad+\omega^{-\alpha}\int_{\lambda_{n+p+1}}^{\omega}A^p(\lambda, x, u)(\omega-u)^{\alpha-p-1}\,du & \text{for } \omega>\lambda_{n+p+1} \end{cases}$$

For $\lambda_{n+q}<\omega\le\lambda_{n+q+1}$, $1\le q\le p$ and $n+1\le\nu\le n+q$, we have

$$\left|\omega^{-\alpha}\int_{\lambda_{n+q}}^{\omega}\beta_\nu^p(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\right|$$

$$\le |t_\nu^{(p)}(x)|\left\{\sup_{\lambda_{n+q}\le u\le\omega}|\beta_\nu^p(u)u^{-p}|\right\}\omega^{-(\alpha-p)}\int_{\lambda_{n+q}}^{\omega}(\omega-u)^{\alpha-p-1}\,du$$

(and by Lemma 5, since $\beta_\nu^{(p)}(u)\ge 0$)

$$\le |t_\nu^{(p)}(x)|\left\{\sup_{\lambda_{n+q}\le u\le\omega}(\beta_\nu^p(u)u^{-p})\right\}\frac{1}{\alpha-p}\left(\frac{\omega-\lambda_{n+q}}{\omega}\right)^{\alpha-p}$$

$$\le |t_\nu^{(p)}(x)|\left\{\sup_{\lambda_{n+q}\le u\le\lambda_{n+q+1}}u^{-p}\sum_{\nu=n+q-p}^{n+q}\beta_\nu^p(u)\right\}\frac{1}{\alpha-p}\left(\frac{\omega-\lambda_{n+q}}{\omega}\right)^{\alpha-p}$$

(and by Lemma 5, since $\lim_{u\to\infty}u^{-p}\sum_{\nu=n+q-p}^{n+q}\beta_\nu^p(u)=1, \qquad \lambda_{n+q}<u\le\lambda_{n+q+1}$)

$$\le K\,|\Lambda_{n+q}^{-(\alpha-p)}t_\nu^{(p)}(x)|$$

(and by Lemma 6)

$$\to 0$$

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as $n\to\infty$ uniformly in $\lambda_{n+q}<\omega\le\lambda_{n+q+1}$ and in $1\le q\le p$ and $n+1\le\nu\le n+q$. Hence

$$(17)\qquad \omega^{-\alpha}\int_{\lambda_{n+q}}^{\omega}\sum_{r=n+1}^{n+q}\beta_\nu^{(p)}(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\to 0$$

as $n\to\infty$ uniformly in $\lambda_{n+q}<\omega\le\lambda_{n+q+1}$ and $1\le q\le p$. Similarly we get for $1\le r\le m$, $1\le m\le p$, and $\omega>\lambda_{m+1}$, since $\omega^{-(\alpha-p)}(\omega-u)^{\alpha-p-1}$ is a decreasing function of $\omega$, that

$$(17\text{a})\qquad \left|\omega^{-\alpha}\int_{\lambda_{n+r}}^{\lambda_{n,r+1}}\sum_{\nu=n+1}^{n+r}\beta_\nu^p(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\right|\le K\,|t_\nu^{(p)}(x)|\,\Lambda_{n+r}^{-(\alpha-p)}\to 0$$

as $n\to\infty$ uniformly in $\omega>\lambda_{m+1}$. By (16), (17) and (17a) we see that

$$(18)\qquad J_{n,q}(x,\omega)\to 0$$

as $n\to\infty$ uniformly in $\lambda_{n+q}<\omega\le\lambda_{n+q+1}$, $1\le q\le p$; and that

$$(19)\qquad \omega^{-\alpha}\left\{\sum_{r=1}^{p}\int_{\lambda_{n+r}}^{\lambda_{n+r+1}}\sum_{\nu=n+1}^{n+r}\beta_\nu^p(u)t_\nu^{(p)}(x)(\omega-u)^{\alpha-p-1}\,du\right\}\to 0$$

as $n\to\infty$ uniformly in $\omega>\lambda_{n+p+1}$. We have for $\omega>\lambda_{n+p+1}$

$$(20)\qquad \omega^{-\nu}\int_{\lambda_{n+p+1}}^{\omega}A^p(\lambda,x,u)(\omega-u)^{\alpha-p-1}\,du$$
$$=\omega^{-\alpha}\left\{\int_0^{\omega}-\int_0^{\lambda_{n+p+1}}\right\}A^p(\lambda,x,u)(\omega-u)^{\alpha-p-1}\,du$$

(and by [23, Lemma 1.42 for $l=\alpha-p$, $k=p$ and $\varphi(x)=x^\alpha$] we get)

$$\to 0$$

as $n\to\infty$ uniformly in $\omega>\lambda_{n+p+1}$. By (16), (18), (19) and (20) we see that

$$\left\|x-\sum_{j=0}^{n}t_j^{(n)}(x)\,\delta^{(p,j)}\right\|_{\lambda\alpha}\to 0$$

as $n\to\infty$, which completes the proof.

**Proof of Theorem 1. Necessity** Since for $\alpha>0$ $(R,\lambda,\alpha)$ is regular, (4) is necessary and we may assume $x\in R_{\lambda\alpha}(c^0)$. By the argument in Peyerimhoff [12, §8] and Maddox [9, p. 166] with minor modifications, the general continuous linear functional on $R_{\lambda\alpha}(c^0)$, $\alpha>0$, is of the form

$$f(x)=\int_{\lambda_0}^{\infty}A^\alpha(\lambda,x,\omega)dg(\omega),\qquad \int_{\lambda_0}^{\infty}\omega^\alpha\,|dg(\omega)|<\infty,$$

and $\|f\|=\int_{\lambda_0}^{\infty}\omega^\alpha\,|dg(\omega)|$. The proof that for $\alpha>0$ (3) and (5) are necessary, is due to Peyerimhoff [12], Maddox [9] and Russell [20]. Briefly if $\sum_\nu b_{\rho\nu}x_\nu$ converges for each $\rho$ whenever $x\in R_{\lambda\alpha}(c^0)$, then $f_\rho(x)=\sum_\nu b_{\rho\nu}x_\nu$ is a continuous linear

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functional on $R_{\lambda\alpha}(c^0)$ and hence

$$(21)\quad f_\rho(x)=\sum_\nu b_{\rho\nu}x_\nu=\int_{\lambda_0}^\infty A^\alpha(\lambda,x,\omega)\,dg_\rho(\omega),\ \|f_\rho\|=\int_{\lambda_0}^\infty \omega^\alpha|dg_\rho(\omega)|<\infty.$$

Choosing $x=e^n (n\geq 0)$ in (21) we get (5)(i) in the form

$$(22)\qquad b_{\rho\nu}=\int_{\lambda_0}^\infty (R,\lambda,\alpha,e^\nu,\omega)\,dg_\rho(\omega).$$

Since $\lim_\rho f_\rho\ (x)$ exists for each $x\in R_{\lambda\alpha}(c^0)$ it follows by the uniform boundedness principle that (5)(ii) is necessary. Now if $p<\alpha\leq p+1$ we get by (12), (22) and Theorem 5 for each $\rho$ and each $x\in R_{\lambda\alpha}(c^0)$

$$(23)\quad \begin{aligned} f_\rho(x)&=\lim_{n\to\infty} f_\rho\left(\sum_{j=0}^n t_j^{(p)}(x)\,\delta^{(p,j)}\right)\\ &=\lim_{n\to\infty}\left\{f_\rho\left(\sum_{j=0}^{n+p}x_je^j\right)-\sum_{r=1}^p\left(\sum_{k=1}^r T'^{(p)}_{n+r,n+k}t^{(p)}_{n+k}(x)\right)f_\rho(e^{n+r})\right\}\\ &=\lim_{n\to\infty}\left\{\sum_{j=0}^{n+p}b_{\rho j}x_j-\sum_{r=1}^p\left(\sum_{k=1}^r T'^{(p)}_{n+r,n+k}t^{(p)}_{n+k}(x)\right)b_{\rho,n+r}\right\}.\end{aligned}$$

Since $\sum_j b_{\rho j}x_j$ is assumed convergent for each $\rho$ and each $x\in R_{\lambda\alpha}(c^0)$ it follows by the definition $f_\rho\ (x)=\sum_j b_{\rho j}x_j$ that (6) is necessary.

**Sufficiency.** The sufficiency of (3), (4) and (5) if $0\leq\alpha\leq 1$ is due to Russell [20, Theorem 1]. We assume $p<\alpha\leq p+1$ and prove that (3), (4), (5) and (6) are sufficient. The functions $g_\rho$ existing by (5) define continuous linear functionals

$$f_\rho(x)=\int_{\lambda_0}^\infty A^\alpha(\lambda,x,\omega)\,dg_\rho(\omega)$$

on $R_{\lambda\alpha}(c^0)$. The norms of these continuous linear functionals are uniformly bounded by (5)(ii), and by (3) $\lim_\rho f_\rho\ (\delta^{(p,j)})$ exists for each $j\geq 0$ (since $\{\delta^{(p,j)}\}_{j\geq 0}$ is a finite linear combination of $e^j,\ldots,e^{j+p}$) where, by Theorem 5, $\{\delta^{(p,j)}\}_{j\geq 0}$ is a Schauder-basis for $R_{\lambda\alpha}(c^0)$. Hence $\lim_\rho f_\rho\ (x)$ exists for each $x\in R_{\lambda\alpha}(c^0)$. Now by (23) and (6) we have

$$f_\rho(x)=\lim_{n\to\infty}\left\{\sum_{j=0}^{n+p}b_{\rho j}x_j-\sum_{r=1}^p\left(\sum_{k=1}^r T'^{(p)}_{n+r,n+k}t^{(p)}_{n+k}(x)\right)b_{\rho,n+r}\right\}=\sum_j b_{\rho j}x_j,$$

for each $\rho$ and each $x\in R_{\lambda\alpha}(c^0)$. The existence of $\lim_\rho f_\rho\ (x)$ for each $x\in R_{\lambda\alpha}(c^0)$ implies that $\lim_\rho \sum_j b_{\rho j}x_j$ exists for each $x\in R_{\lambda\alpha}(c^0)$ which completes the proof.

**Proof of Theorem 2.** Define the integer $p$ by $p<\alpha\leq p+1$. By (7), Lemma 6 and Lemma 7, we have for each $x\in c^0_{(R,\lambda,\alpha)}$ and each $k$, $1\leq k\leq p$:

$$\begin{aligned}\left|b_{\rho,n+r}\left(\sum_{k=1}^r T'^{(p)}_{n+r,n+k}t^{(p)}_{n+k}(x)\right)\right| &\leq\sum_{k=1}^r |b_{\rho,n+r}|\cdot|T'^{(p)}_{n+r,n+k}|\cdot|t^{(p)}_{n+k}(x)|\\ &\leq\sum_{k=1}^r (H_\rho\Lambda^-_{n+r})(G_\rho\Lambda^p_{n+r})\cdot o\left(\min_{n+k\leq q<n+k+p}\Lambda_q^{\alpha-p}\right)\\ &\to 0 \text{ as } n\to\infty \text{ (since } n+k\leq n+p<n+k+p) \text{ uniformly in } 1\ \leq r\leq p.\end{aligned}$$

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Hence (6) is satisfied; the proof follows now by Theorem 1.

**Proof of Theorem 3.** The proof follows by Theorem 2 and Lemma 3.

**Proof of Theorem 4.** For the necessity of (8), (9) and (10), if $\alpha>0$ and for the sufficiency of (8), (9) and (10) if $0<\alpha\leq 1$ see Russell [20, Theorem 2]. For the sufficiency of (8), (9) and (10) if $\alpha=2, 3, 4, \ldots$ see Russell [21, p. 300]. Assume $p<\alpha\leq p+1$ and that (8), (9) and (10) are satisfied. Define $b_{\rho\nu}=\bar{b}_{\rho\nu}-\bar{b}_{\rho,\nu+1}$. Then (3) holds with $\beta_\nu=\bar{\beta}_\nu-\bar{\beta}_{\nu+1}$. By (10), (5) holds and $\bar{b}_{\rho\nu}\to 0$ as $\nu\to\infty$ for each $\rho$. Hence $\sum_\nu b_{\rho\nu}=\bar{b}_{\rho 0}$ and (4) holds with $\beta=\bar{\beta}_0$. The assumption $\Lambda_{n-1}=0(\Lambda_n)$ and (9) imply (7). Thus, the conditions of Theorem 2 hold for the method $B$ and $B$ includes $(R, \lambda, \alpha)$. Now given any series $\sum_\nu c_\nu$ with partial sums $s_\nu$, we have

$$\sum_{\nu=0}^{N}\bar{b}_{\rho\nu}c_\nu=\sum_{\nu=0}^{N-1}b_{\rho\nu}s_\nu+\bar{b}_{\rho N}s_N. \tag{24}$$

If $\sum c_\nu$ is summable $(R, \lambda, \alpha)$ to $\dot{s}$, then we may assume without loss of generality that $\dot{s}=0$ (since $(R, \lambda, \alpha)$ is regular and $\lim_\rho \sum b_{\rho\nu}$ exists) and then, by the limitation theorem for Riesz means [5, Theorem 21], $s_N=o(\Lambda_N^\alpha)$. Hence by (9) and (24) we get $\sum_\nu \bar{b}_{\rho\nu}c_\nu=\sum_\nu b_{\rho\nu}s_\nu$ whenever either side exists. Since $B$ includes $(R, \lambda, \alpha)$, also $\bar{B}$ include $(R, \lambda, \alpha)$.

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