Proceedings of the Edinburgh Mathematical Society (2014) **57**, 145–173
DOI:10.1017/S0013091513000898

DOUBLE SOLIDS, CATEGORIES AND NON-RATIONALITY

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Dedicated to Slava—teacher and friend—with admiration

Abstract This paper suggests a new approach to questions of rationality of 3-folds based on category theory. Following work by Ballard *et al.*, we enhance constructions of Kuznetsov by introducing Noether–Lefschetz spectra: an interplay between Orlov spectra and Hochschild homology. The main goal of this paper is to suggest a series of interesting examples where the above techniques might apply. We start by constructing a sextic double solid X with 35 nodes and torsion in $H^3(X,\mathbb{Z})$. This is a novelty: after the classical example of Artin and Mumford, this is the second example of a Fano 3-fold with a torsion in the third integer homology group. In particular, X is non-rational. We consider other examples as well: V_{10} with 10 singular points, and the double covering of a quadric ramified in an octic with 20 nodal singular points. After analysing the geometry of their Landau–Ginzburg models, we suggest a general non-rationality picture based on homological mirror symmetry and category theory.

Keywords: Fano varieties; rationality questions; Landau–Ginzburg model

2010 *Mathematics subject classification:* Primary 14E08; 14F05
Secondary 14J45, 14J33

1. Introduction

This paper suggests a new approach to questions of rationality of 3-folds based on category theory. It was inspired by recent work of Shokurov and by Kuznetsov's idea about the Griffiths component (see [**28**]). This work is a natural continuation of ideas developed in [**12**, **23**] and of ideas of Kawamata and his school.

We first extend a classical example of Artin and Mumford to construct a sextic double solid X with 35 nodes and torsion in $H^3(X,\mathbb{Z})$. The construction is based on an approach by Gross and suggests a close relation between the Artin and Mumford example and the sextic double solid X with 35 nodes. This example, a novelty on its own, opens up the possibility of a series of interesting examples: V_{10} with 10 singular points and the double covering of a quadric ramified in an octic with 20 nodal singular points.

In this paper we start by investigating these examples from the point of view of homological mirror symmetry (HMS). We consider the mirrors of the sextic double solid X with 35 nodes, of the Fano variety V_{10} with 10 singular points in general positions, and of the double covering of a quadric ramified in an octic with 20 nodal singular points. We note that the monodromy around the singular fibre over 0 of the Landau–Ginzburg models is strictly unipotent in all these examples, which suggests that the categorical behaviour should be very similar to that of the Artin–Mumford example. We conjecture that the reason for categorical similarity in all these examples is that they contain the category of an Enriques surface as a semi-orthogonal summand in their derived categories. This is done in § 5, where we introduce Landau–Ginzburg models and compare their singularities.

In § 6 we introduce several new rationality invariants coming out of the notions of spectra and the enhanced Noether–Lefschetz spectra of categories. We give a conjectural categorical explanation of the examples from §§ 2–5. The novelty (conjecturally) is that non-rationality of these examples cannot be predicted by Orlov spectra, but it is detected by the Noether–Lefschetz spectrum.

The paper has the following structure. In §§ 2–4 we describe classical calculations of a sextic double solid X with 35 nodes. In § 5 we look at some mirror considerations for studying some Landau–Ginzburg models. In § 6 we suggest a general categorical framework for studying the phenomena in §§ 2–5.

The paper is based on examples we have analysed in [**3**,**11**,**24**,**25**,**27**]. All these suggest a direct connection between the monodromy of Landau–Ginzburg models, spectra and wall crossings in the moduli space of stability conditions, which was partially explored in [**17**]. This paper is a humble attempt to shed some light on this connection. We expect that further application of this method will be the theory of three-dimensional conic bundles (a small part of the huge algebro–geometric heritage of Shokurov; see Remark 4.3). In particular, we expect that the Noether–Lefschetz spectra of categories will allow us to prove non-rationality of new classes of conic bundles: classes where the method of the intermediate Jacobian does not work.

All varieties considered in this paper are defined over the field of complex numbers $\mathbb{C}$. The torsion subgroup of a given group G is denoted by $\mathrm{Tors}(G)$; the n-torsion subgroup is denoted by $\mathrm{Tors}_n(G)$. We denote the Du Val singularities of ADE type by A_n, D_n and E_n. We denote a Landau–Ginzburg model of a variety X by $\mathrm{LG}(X)$.

2. Determinantal double solids and Brauer–Severi varieties

2.1. The classical Artin–Mumford example

A double solid is an irreducible double covering $\pi\colon X \to \mathbb{P}^3$. The branch locus of such a π is a surface $S \subset \mathbb{P}^3$ of even degree. In 1972 Artin and Mumford gave an example of a special singular quartic double solid X (i.e. $\deg S = 4$) that is non-rational because of the existence of a non-zero 2-torsion in its integer cohomology group $H^3(X, \mathbb{Z})$ (see [**1**]). Since quartic double solids are unirational (see, for example, [**21**, Example 10.1.3 (iii)]),

https://doi.org/10.1017/S0013091513000898 Published online by Cambridge University Press

this gives (together with the examples presented at the same time by Iskovskikh and Manin and Clemens and Griffiths) an example of a non-rational unirational 3-fold.

In [**2**] Aspinwall *et al*. present a special case of a singular Calabi–Yau 3-fold: an octic double solid X (i.e. $\deg S = 8$) with 80 nodes on S and a non-zero 2-torsion in $H^3(X,\mathbb{Z})$.

In this section we adapt an approach used in [**2**] to check again the existence of the 2-torsion in $H^3(X,\mathbb{Z})$ for the Artin–Mumford quartic double solid X, and present an example of a sextic double solid X with 35 nodes and a non-zero torsion in $H^3(X,\mathbb{Z})$. In particular, this special nodal sextic double solid is not rational. Other examples are presented in the sections to follow.

2.2. Quadric bundles and determinantal double solids

Let X_0 be a smooth complex projective variety, let L be an invertible sheaf on X_0, and let $E \to X_0$ be a vector bundle of rank $r \geqslant 2$ over X_0.

A *quadric bundle* in E parametrized by L is the $\mathcal{O}_{X_0}$-map

$$\varphi\colon L^{-1} \to \mathrm{Sym}^2 E^*.$$

The determinantal loci of φ are the subvarieties

$$D_{r-k} = D_{r-k}(\varphi) = \{x \in X_0\colon \operatorname{rank}\varphi_x \leqslant r-k\}, \quad k = 0,1,2,\dots.$$

Geometrically, a quadric bundle φ represents the bundle of quadrics

$$\mathcal{Q} = \{Q_x \subset \mathbb{P}(E_x)\colon x \in X_0\},$$

and

$$D_{r-k} = \{x \in X_0\colon \operatorname{rank} Q_x \leqslant r-k\}.$$

If $D_{r-k} \subset X_0$ are non-empty and have the expected codimensions $k(k+1)/2$, then their classes in $A_*(X_0)$ can be computed by the formulae in [**15**, **22**]. For our purposes, we need only know explicit formulae for the first two determinantals D_{r-1} and D_{r-2}, which can be computed formally as follows. Rewrite φ in the form

$$\varphi\colon \mathcal{O}_{X_0} \to \mathrm{Sym}^2(E^* \otimes L^{1/2}),$$

and compute $c(E^* \otimes L^{1/2}) = 1 + c_1 + c_2 + \cdots + c_r$. Then

$$D_{r-1} = 2c_1 \quad \text{and} \quad D_{r-2} = 4(c_1c_2 - c_3).$$

In the particular case when the base $X_0 = \mathbb{P}^n$ is a projective space, the determinantal locus D_{r-1} is a hypersurface in $\mathbb{P}^n$ of even degree; therefore, D_{r-1} defines the double covering

$$\pi\colon X \to \mathbb{P}^n$$

branched along D_{r-1}. We call such an X a *determinantal double solid*.

2.3. Cohomological Brauer groups and Brauer–Severi varieties

Let X be a complex algebraic variety, let $\mathcal{O}_X$ be the structure sheaf of X, and let $\mathcal{O}_X^*$ be the sheaf of units in $\mathcal{O}_X$. The Picard group and the (cohomological) Brauer group of X are, respectively, the first and second cohomology groups

$$\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*) \quad \text{and} \quad \operatorname{Br}(X) = H^2(X, \mathcal{O}_X^*).$$

There exists an exact sequence

$$\operatorname{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(X, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Br}(X) \to 0$$

(see [**14**, Part II, (3.1)]). If, in addition, X is non-singular and it fulfils the conditions

$$\operatorname{Pic}(X) = H^2(X, \mathbb{Z}) \quad \text{and} \quad H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0, \tag{2.1}$$

then, by the universal coefficient theorem, $\operatorname{Br}(X) \cong \operatorname{Tors}(H^3(X, \mathbb{Z}))$ (see, for example, [**1**]). For any X as above, *a Brauer–Severi variety* over X is a variety $\mathcal{P}$ with the structure of a $\mathbb{P}^n$-bundle $f\colon \mathcal{P} \to X$ over X.

Not every Brauer–Severi variety is a projectivization of a vector bundle over X, and the Brauer group gives obstructions for a Brauer–Severi variety to be a presented as a projectivization of such. On X, we consider the exact sequence

$$0 \to \mathcal{O}_X^* \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 0,$$

where $\mathcal{O}_X^*$ is the multiplicative group of X.

The corresponding long exact sequence is

$$0 \to \operatorname{Pic}(X) \to H^1(X, \mathrm{GL}_{n+1}) \xrightarrow{j} H^1(X, \mathrm{PGL}_{n+1}) \xrightarrow{\delta} \operatorname{Br}(X) \to \cdots.$$

The vector bundles $E \to X$ of rank $(n+1)$ are elements of the cohomology group $H^1(X, \mathrm{GL}_{n+1})$, while the $\mathbb{P}^n$-bundles $\mathcal{P} \to X$ are elements of $H^1(X, \mathrm{PGL}_{n+1})$.

Therefore, by the above sequence the $\mathbb{P}^n$-bundle $\mathcal{P}$ is not a projectivization of a vector bundle on X if and only if $\delta(\mathcal{P}) \neq 0$. Since $(n+1)\delta = 0$, any $\mathcal{P}$ with $\delta(\mathcal{P}) \neq 0$ gives rise to a non-zero $(n+1)$-torsion element $\delta(\mathcal{P}) \in \operatorname{Br}(X)$. If, moreover, X fulfils (2.1), then $\mathcal{P}$ represents a non-zero $(n+1)$-torsion element of $H^3(X, \mathbb{Z}) \cong \operatorname{Br}(X)$. In the particular case we consider below, $\mathcal{P}$ is a $\mathbb{P}^1$-bundle that is not a projectivization of a vector bundle, thus representing a non-zero 2-torsion element of $H^3(X, \mathbb{Z})$.

In the next sections we use the following.

Lemma 2.1 (torsion criterion for non-rationality). *For the smooth complex variety Y, the torsion subgroup $\operatorname{Tors}(H^3(Y, \mathbb{Z}))$ is a birational invariant of X. In particular, if Y is rational, then $\operatorname{Tors}(H^3(Y, \mathbb{Z})) = 0$.*

Proof. See [**1**, Proposition 1] or [**5**, § 9]. □

3. A determinantal sextic double solid X with a non-zero 2-torsion in $H^3(X,\mathbb{Z})$

3.1. The double solids of Artin and Mumford, and Aspinwall *et al.*, and a determinantal sextic double solid

The Artin–Mumford 3-fold from [**1**] is a special double solid with a branch locus: a quartic surface S with 10 nodes and with a torsion in the third integer cohomology group $H^3 = H^3(\tilde{X},\mathbb{Z})$, where $\tilde{X} \to X$ is the blow-up of X at its nodes. As was shown later by Endrass, the group H^3 of a double solid X branched over a nodal quartic surface S can have a non-zero torsion only in the case when S has 10 nodes (see [**10**]). Therefore, the branch loci of eventual further examples of nodal 3-fold double solids with a non-zero torsion in the third integer cohomology group H^3 should be of degree d either equal to 2 or greater than or equal to 6. If, in addition, we require such an X to be a Fano 3-fold, then d must be less than or equal to 6, i.e. if there exists such an X, it must be a sextic double solid or a double quadric. Note that non-singular Fano 3-folds X have a zero torsion in $H^3 = H^3(X,\mathbb{Z})$, so the requirement that X be singular (and nodal, for simplicity) is substantial.

In [**2**] Aspinwall *et al*. study a special case of a Calabi–Yau 3-fold that is a double solid X with a torsion in H^3 and with a branch locus S of degree 8 (an octic double solid). The similarity between the Artin–Mumford quartic double solid and the octic double solid from [**2**] is that they are both *determinantal* double solids. Both these varieties X are singular: in the Artin–Mumford case X has 10 ordinary double points (nodes), while the octic double solid from [**2**] has 80 nodes.

Below, we describe an example of a determinantal nodal sextic double solid X with a torsion in H^3. After the example of Artin and Mumford, this is the second example of a (necessary) singular nodal Fano 3-fold (see above) with a torsion in the third integer cohomology group. In particular, our X must be non-rational (see Lemma 2.1).

It was shown by Iskovskikh [**20**] that the general sextic double solid is non-rational due to the small group $\mathrm{Bir}(X)$ of birational automorphisms of X. This argument was later extended by Cheltsov and Park, proving the non-rationality of certain singular sextic double solids (see [**7**]).

From this point of view, the example studied below is a non-rational sextic double solid X with 35 ordinary double points. According to Cheltsov (V. Przyjalkowski, private communication, 2010), the non-rationality of this X cannot be derived, at least for now, from the results of [**7**].

The proof of the non-rationality of X presented below follows ideas from [**2**, Appendix].

3.2. The determinantal sextic double solid

Let $\mathbb{P}^3 \times \mathbb{P}^4 \subset \mathbb{P}^{19}$ be a Segre variety of $\mathbb{C}^*$-classes of non-zero 4×5 matrices, and let

$$W = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H \cap F$$

be a general complete intersection of $\mathbb{P}^3 \times \mathbb{P}^4$ with a hyperplane $H = \mathbb{P}^{18} \subset \mathbb{P}^{19}$ and a divisor F of bidegree $(1,2)$. Let $Z = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H$, and denote by p_Z and p_W the restrictions

https://doi.org/10.1017/S0013091513000898 Published online by Cambridge University Press

of the projection $p\colon \mathbb{P}^3\times\mathbb{P}^4\to\mathbb{P}^3$ to Z and to W, respectively. The projection p_W defines a structure of a quadric bundle

$$p_W\colon W\to\mathbb{P}^3$$

on W with fibres: quadrics $Q_x=p_W^{-1}(x)$ in the 3-spaces

$$\mathbb{P}^3_x=p_Z^{-1}(x)=(x\times\mathbb{P}^4)\cap H,\quad x\in\mathbb{P}^3.$$

The $\mathbb{P}^3$-bundle $p_Z\colon Z\to\mathbb{P}^3$ is a projectivization of the rank 4 vector bundle E on $\mathbb{P}^3$, defined by

$$0\to E\to\mathcal{O}_{\mathbb{P}^3}^{\oplus 5}\to\mathcal{O}_{\mathbb{P}^3}(1)\to 0;$$

therefore, $c(E^*)=1+h+h^2+h^3$ in $A_*(\mathbb{P}^3)=\mathbb{C}[h]/(h^4)$. Since W is an intersection of $Z=\mathbb{P}(E)\to\mathbb{P}^3$ with a bidegree $(1,2)$ divisor, the bundle of quadrics defining a quadric bundle $p_W\colon W\to\mathbb{P}^3$ is given by the map

$$\varphi\colon\mathcal{O}_{\mathbb{P}^3}(-1)\to S^2E^*.$$

So

$$c(E^*(\tfrac{1}{2}))=1+c_1+c_2+c_3=1+3h+4h^2+\tfrac{13}{4}h^3,$$

and hence

$$[D_3(\varphi)]=2c_1=6h\quad\text{and}\quad[D_2(\varphi)]=4(c_1c_2-c_3)=35.$$

For a general choice of a bidegree $(1,2)$ divisor F, the branch locus

$$S=D_3(\varphi)$$

is a sextic surface in $\mathbb{P}^3$ with 35 nodes: the 35 points of

$$\delta=D_2(\varphi)=\{p_1,\dots,p_{35}\}.$$

Let

$$\pi\colon X\to\mathbb{P}^3$$

be a double covering branched along the sextic surface $S=D_3$. Since $\operatorname{Sing}(S)=\delta$, and the points $p_i\in\delta$ are nodes of S, the sextic double solid X has 35 nodes: the preimages of the 35 points $p_1,\dots,p_{35}$ of δ.

Proposition 3.1. *Let $W=(\mathbb{P}^3\times\mathbb{P}^4)\cap H\cap F$ be a general complete intersection of $\mathbb{P}^3\times\mathbb{P}^4$ with a hyperplane and a divisor of bidegree $(1,2)$. The following conditions then hold.*

(1) *The degeneration locus $S=D_3$ of the quadric fibration $p_W\colon W\to\mathbb{P}^3$, induced by the projection $p\colon\mathbb{P}^3\times\mathbb{P}^4\to\mathbb{P}^3$, is a sextic surface with 35 nodes.*

(2) *Let $\pi\colon X\to\mathbb{P}^3$ be a double covering branched along the sextic surface $S=D_3$. The group $H^3(X,\mathbb{Z})$ then contains a non-zero 2-torsion element; in particular, X is non-rational.*

3.3. Proof of Proposition 3.1

Part (1) follows from previous considerations. It remains to verify (2). Following an approach from [**2**], we find, below, a non-zero 2-torsion element of $H^3(X,\mathbb{Z})$, by representing it as a Brauer–Severi variety over the smooth part of X. Together with Lemma 2.1, this completes the proof.

We consider the quadric bundle $p_W\colon W\to\mathbb{P}^3=\mathbb{P}^3(x)$, and restrict it over the open subset

$$\mathbb{P}^3_0=\mathbb{P}^3-\delta.$$

We define

$$S_0=S-\delta,\qquad X_0=X-\delta_X\quad\text{and}\quad W_0=W-\delta_W,$$

where $\delta_X=\pi^{-1}(\delta)$ is an isomorphic preimage of $\delta=\{p_1,\dots,p_{35}\}$ on X, and $\delta_W=p^{-1}(\delta)$ is the set of 35 rank 2 quadric surfaces $Q_i=p^{-1}(p_i)$, $i=1,\dots,35$. Outside δ_W, the projection p restricts to a quadric bundle

$$p_{W_0}\colon W_0\to\mathbb{P}^3_0$$

with degeneration locus S_0.

Let $\pi\colon X_0\to\mathbb{P}^3_0$ be an induced determinantal double covering branched along S_0. As follows from our construction, the fibres of the quadric bundle $p_{W_0}\colon W_0\to\mathbb{P}^3_0$ are the quadrics $Q_x\subset\mathbb{P}^3_x$, $x\in\mathbb{P}^3-\delta$.

Let $\mathcal{P}$ be the family of lines $l\subset W_0$ in the quadrics Q_x, $x\in\mathbb{P}^3-\delta$, and let $\mathcal{P}_0\subset\mathcal{P}$ be the family of these lines $l\in\mathcal{P}$, which lie on the quadrics Q_x, $x\in\mathbb{P}^k_0-\delta$.

We denote by

$$f_P\colon\mathcal{P}\to\mathbb{P}^3$$

the map sending a line $l\subset Q_x$ to a point $x\in\mathbb{P}^3$, and we denote by $f_P\colon\mathcal{P}_0\to\mathbb{P}^3_0$ its restriction over $\mathbb{P}^3_0$.

We also define

$$\pi_0\colon X_0\to\mathbb{P}^3_0$$

to be the restriction of the double covering $\pi\colon X\to\mathbb{P}^3$ to $X_0=X-\delta_X$.

For any point $x\in\mathbb{P}^3_0-S_0=\mathbb{P}^3-S$ the quadric $Q_x\subset\mathbb{P}^3_x$ is smooth, while for any $x\in S_0=S-\delta$ the quadric Q_x is a quadratic cone of rank 3 in $\mathbb{P}^3_x$.

We then have that

$$f_P^{-1}(x)\cong P^1\vee P^1\quad\text{for } x\in\mathbb{P}^3_0-S_0=\mathbb{P}^3-S$$

and

$$f_P^{-1}(x)\cong\mathbb{P}^1\qquad\quad\text{for } x\in S_0=S-\delta.$$

Since S_0 is also a branch locus of the double covering $\pi_0\colon X_0\to\mathbb{P}^3_0$, we identify points of X_0 with generators of the quadrics Q_x, $x\in\mathbb{P}^3_0$. Therefore, the mapping $\mathcal{P}_0\to\mathbb{P}^3_0$ is represented as a composition

$$\mathcal{P}_0\xrightarrow{f_0}X_0\xrightarrow{\pi_0}\mathbb{P}^3_0,$$

https://doi.org/10.1017/S0013091513000898 Published online by Cambridge University Press

where

$$f_0\colon \mathcal{P}_0 \to X_0$$

is a $\mathbb{P}^1$-fibration sending the sets of lines l on the quadrics Q_x to the generators of Q_x containing l. Let

$$\tilde{X} \to X$$

be the blow-up of X at the 35 nodes of X identified with the 35 double points $p_1, \dots, p_{35}$ of the surface S. Following [**2**], we see that $\mathcal{P}_0$ is not a projectivization of a vector bundle over X_0. This yields that the Brauer group $\mathrm{Br}(\tilde{X})$ has a non-zero element of order 2, representing a non-zero 2-torsion element in $H^3(\tilde{X}, \mathbb{Z})$.

Suppose that $f_0\colon \mathcal{P}_0 \to X_0$ is a projectivization of a rank 2 vector bundle $E \to X_0$. Up to a twist by a line bundle, we can always assume that E has sections. Then, any section of E gives rise to a rational section of $f_0\colon \mathcal{P}_0 = \mathbb{P}(E) \to X_0$. The following lemma concludes the argument.

Lemma 3.2. *The $\mathbb{P}^1$-fibration $f_0\colon \mathcal{P}_0 \to X_0$ has no rational sections. In particular, $\mathcal{P}_0$ is not a projectivization of a rank 2 vector bundle on X_0.*

Proof (see [2] for more details). Suppose that f_0 has a rational section, i.e. a rational map $\sigma\colon X_0 \to \mathcal{P}_0$ defined over an open dense subset $U \subset X_0$ and such that $f_0(\sigma(u)) = u$ for any $u \in U$. By definition, the points of $\mathcal{P}_0$ are the lines l that lie on the quadrics Q_t, $t \in \mathbb{P}^3_0$. Denote by $l_u \in \mathcal{P}_0$ the line $l_u = \sigma(u)$ for points $u \in U$, i.e.

$$\begin{aligned} \sigma\colon U &\to \mathcal{P}_0 \\ x &\mapsto l_u. \end{aligned}$$

Let $\pi\colon X \to \mathbb{P}^3$ be the double covering, and let $i\colon X \to X$ be the involution interchanging two possibly coincident π-preimages of the points $x \in \mathbb{P}^3$. Without any loss of generality (e.g. by replacing U by $U \cap i(U)$), we may assume that $U = i(U)$. Let $D \subset W$ be the Zariski closure of the set

$$\{l_u \cap l_{i(u)}\colon u \in U \text{ and } u \neq i(u)\}.$$

The variety D is a 3-fold in W that intersects the general quadric $Q_x \subset \mathbb{P}^3_x = x \times \mathbb{P}^3$, $x = \pi(u)$ at *a unique* point, the point $y(u) = l_u \cap l_{i(u)}$, i.e. $DQ_x = 1$.

The 5-fold $W = (\mathbb{P}^3 \times \mathbb{P}^4) \cap H \cap (F(x;y) = 0)$ is an ample divisor in the 6-fold $Z = (\mathbb{P}^3 \times \mathbb{P}^3) \cap H$, which in turn is an ample divisor in $\mathbb{P}^3 \times \mathbb{P}^4$.

By the Lefschetz hyperplane section theorem, the restriction map then defines the isomorphism

$$H^4(\mathbb{P}^3 \times \mathbb{P}^4, \mathbb{Z}) \to H^4(Z, \mathbb{Z}) \to H^4(W, \mathbb{Z}).$$

In particular, the codimension 2 subvariety $D \subset W$ is a restriction of a codimension 2 subvariety of $\mathbb{P}^3 \times \mathbb{P}^4$ to W.

In the Chow ring

$$A_*(\mathbb{P}^3 \times \mathbb{P}^4) = \mathbb{Z}[h_1, h_2]/(h_1^4, h_2^5),$$

the class of the fibre Q_x of $p\colon W \to X$ is $2h_1^3h_2^2$. Since codimension 2 cycles on $\mathbb{P}^3 \times \mathbb{P}^4$ are generated over $\mathbb{Z}$ by h_1^2, h_2^2 and h_1h_2, the intersection number of any codimension 2 cycle on W with general quadric Q_x is *even*, which contradicts the equality $DQ_x = 1$. $\square$

Note also that the varieties X_0 and $\tilde{X}$ fulfil (2.1) from 2.3, so $\mathrm{Br}(X_0)$ and $\mathrm{Br}(\tilde{X})$ are isomorphic to $H^3(X_0, \mathbb{Z})$ and $H^3(\tilde{X}, \mathbb{Z})$.

Theorem 3.3. *The $\mathbb{P}^1$-bundle $\mathcal{P}_0$ represents a non-zero 2-torsion element in* $\mathrm{Br}(X) = H^3(\tilde{X}, \mathbb{Z})$. *In particular, $\tilde{X}$, and hence X, is non-rational.*

Proof. Let E_i, $i = 1, \dots, 35$, be the exceptional divisors of the blow-up $\tilde{X} \to X$ at the nodes $p_1, \dots, p_{35}$. Then, by [**14**], for the Brauer groups of $X_0 = X - \{p_1, \dots, p_{35}\} \cong \tilde{X} - \bigcup\{E_i\colon i = 1, \dots, 35\}$, there exists an exact sequence

$$0 \to \mathrm{Br}(\tilde{X}) \to \mathrm{Br}(X_0) \to \bigoplus_{i=1}^{35} H^1(E_i, \mathbb{Q}/\mathbb{Z}),$$

and since, for surfaces $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$, one has that $H^1(E_i, \mathbb{Q}/\mathbb{Z}) = 0$, $i = 1, \dots, 35$, and thus $\mathrm{Br}(\tilde{X}) \cong \mathrm{Br}(X_0)$. $\square$

It follows from Lemmas 3.2 and 2.3 that $\mathcal{P}_0$ represents a non-zero 2-torsion element of $H^3(\tilde{X}, \mathbb{Z})$. Combining this with Lemma 2.1, we get the non-rationality of $\tilde{X}$, and hence *the non-rationality* of X. This proves Proposition 3.1.

4. An Artin–Mumford quartic double solid

4.1. Quadrics in $\mathbb{P}^3$ and an Artin–Mumford quartic double solid

Let $\mathbb{P}^3 = \mathbb{P}^3(y)$, $(y) = (y_0 : \cdots : y_3)$, be the three-dimensional complex projective space. In the space $\mathbb{P}^9 = \mathbb{P}(H^0(\mathcal{O}_{\boldsymbol{P}^3}(2)))$ of quadrics in $\mathbb{P}^3$ consider the determinantals

$$\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \mathbb{P}^9,$$

where

$$\Delta_k = \{Q \in \mathbb{P}^9\colon\ \operatorname{rank} Q \leqslant k,\ k = 1, 2, 3\}.$$

The elements of $\mathbb{P}^9$ are $\mathbb{C}^*$-classes of symmetric 4×4 matrices $Q = (q_{ij})$, $0 \leqslant i, j \leqslant 3$, and the determinantals Δ_k, $1 \leqslant k \leqslant 3$, defined by vanishing $(k+1) \times (k+1)$ minors of Q, have the following properties (for more details see, for example, [**9**, § 1]):

- $\Delta_3 \subset \mathbb{P}^9$ is a quartic hypersurface,
- $\Delta_2 = \operatorname{Sing} \Delta_3$ has dimension 6 and degree 10,
- $\Delta_1 = \operatorname{Sing} \Delta_2 = v_2(\mathbb{P}^3)$ is the Veronese image of $\mathbb{P}^3$ in $\mathbb{P}^9$,
- the determinantal quartic Δ_3 has an ordinary double singularity along $\Delta_2 - \Delta_1$.

Consider the *general* 3-space $\mathbb{P}^3 = \mathbb{P}^3(x) \subset \mathbb{P}^9$. As follows from previous considerations,

- $S = \mathbb{P}^3 \cap \Delta_3$ is a quartic surface with the only singularities being the 10 points of intersection $\delta = \mathbb{P}^3 \cap \Delta_2 = \{p_1, \dots, p_{10}\}$, and any p_k, $k = 1, \dots, 10$, is an ordinary double point (a node) of S.

(The quartic surfaces defined as determinantal loci of 3-spaces of quadrics in projective 3-space have appeared in the works of Cayley under the name *quartic symmetroids* since the 1980s.)

Since $\deg(S) = 4$ is an even number, there exists the double covering

$$\pi\colon X \to \mathbb{P}^3$$

branched along S, i.e. X is a determinantal quartic double solid.

The double solid X has 10 nodes: isomorphic preimages of the 10 nodes $p_1, \dots, p_{10}$ of the branch locus S, which we also denote by $p_1, \dots, p_{10}$. Let $\tilde{X}$ be the blow-up of X at these 10 points. In the same way as in § 3, we get the following.

Proposition 4.1. *The group $H^3(\tilde{X}, \mathbb{Z})$ contains a non-zero 2-torsion element; in particular, X is non-rational.*

Remark 4.2. In [**1**], Artin and Mumford prove a stronger result, $\mathrm{Tors}(H^3(X, \mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, by using splitting of the discriminant curve for the natural conic bundle structure on X (see also [**33**, Theorem 2]).

4.2. Artin–Mumford quartic double solids and Enriques surfaces

We start by recalling the well-known connection between Artin–Mumford double solids and Enriques surfaces, defined by Reye congruences (see, for example, [**9**]). In the above notation, the Artin–Mumford double solids are defined by the general 3-spaces $\mathbb{P}^3(x)$ in the space $\mathbb{P}^9 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3(y)}(2)))$ of quadrics in $\mathbb{P}^3(y)$, $(y) = (y_0 : \dots : y_3)$. Let

$$\{Q_x\} = \{Q_x \subset \mathbb{P}^3(y)\colon x \in \mathbb{P}^3(x) = \mathbb{P}^3(x_0 : \dots : x_3)\}$$

be the set of quadrics in $\mathbb{P}^3(y)$ defined by the 3-space $\mathbb{P}^3(x)$. Let G be the Grassmannian of the lines $l \subset \mathbb{P}^3(y)$.

It is known that the general line $l \subset \mathbb{P}^3(y)$ lies on a unique quadric from the family $\{Q_x\}$, and the set of lines

$$R = \{l \in G\colon \text{the line } l \subset \mathbb{P}^3(y) \text{ lies in a } \mathbb{P}^1\text{-family of quadrics } Q_x\}$$

is an Enriques surface in $G = G(2, 4)$, classically called a *Reye congruence* (see [**9**]). Let τ be an involution

$$(x, y) \overset{\tau}{\longleftrightarrow} (y, x)$$

on $\mathbb{P}^3(x) \times \mathbb{P}^3(y)$. The fixed point set of τ is the diagonal Δ defined by $\{x = y\}$ in $\mathbb{P}^3(x) \times \mathbb{P}^3(y)$. For a quadratic form

$$Q(y) = \sum_{0 \leqslant i,\, j \leqslant 3} q_{ij} y_i y_j, \quad q_{ij} = q_{ji},$$

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let

$$B(x,y)=\sum_{0\leqslant i,\,j\leqslant 3} q_{ij}x_iy_j$$

be its corresponding bilinear form. A basis $Q_0(y),\dots,Q_3(y)$ of $\mathbb{P}^3(x)\subset\mathbb{P}^9$ then defines a quadruple of bilinear forms $B_0(x,y),\dots,B_3(x,y)$, and hence a linear section

$$\tilde{S}=(\mathbb{P}^3(x)\times\mathbb{P}^3(y))\cap H_0\cap\cdots\cap H_3,$$

where $H_i=(B_i(x,y)=0)$. For a general choice of

$$\mathbb{P}^3(x)=\langle Q_0,\dots,Q_3\rangle,$$

the set $\tilde{S}$ is a smooth complete intersection of four hyperplane sections of $\mathbb{P}^3(x)\times\mathbb{P}^3(y)$, and hence $\tilde{S}$ is a smooth K3 surface: a *Steiner* K3 surface in a 3-space of quadrics $\mathbb{P}^3(y)$. Since all B_i are invariant under the involution τ, $\tilde{S}$ is also invariant under τ, i.e. $\tau(\tilde{S})=\tilde{S}$. Therefore, τ restricts to an involution $\tau\colon\tilde{S}\to\tilde{S}$; and since, for general $\mathbb{P}^3(x)$, the surface $\tilde{S}$ does not intersect diagonal Δ, we conclude that τ is without fixed points on $\tilde{S}$. The K3 surface $\tilde{S}$ has the following properties (see [**9**, **29**]).

Let $\mathbb{P}^3(x)$ be a general 3-space of the 9-space $\mathbb{P}^9$ of quadrics in $\mathbb{P}^3(y)$, and let $S=D_3\subset\mathbb{P}^3(x)$, $R\subset G(2,4)$ and $\tilde{S}$ be, respectively, the quartic symmetroid, the Enriques surface (the Reye congruence) and the Steiner K3 surface defined by $\mathbb{P}^3(x)$. It then holds that

(i) $\tilde{S}$ is the blow-up of S at its 10 nodes $\delta=\{p_1,\dots,p_{10}\}$,

(ii) $R\subset G=G(2,4)$ is isomorphic to the quotient $\tilde{S}/\tau$ of $\tilde{S}$ by the involution τ.

Let $\pi\colon X\to\mathbb{P}^3(x)$ be the Artin–Mumford double solid, defined by the general 3-space $\mathbb{P}^3(x)\subset\mathbb{P}^9$, let $G=G(1:\mathbb{P}^3(y))$ be as above and let

$$\tilde{G}=\{(x,l)\in\mathbb{P}^3(x)\times G\colon l\subset Q_x\}.$$

The following (see [**5**, § 9]) then hold.

(iii) $\tilde{G}=\mathcal{P}$ (see the proof of Proposition 3.1), and the projection $\tilde{G}\to G$, $(x,l)\mapsto l$ is a blow-up of the Enriques surface $R\subset G=G(2,4)$.

(iv) The projection $\sigma\colon\tilde{G}\to\mathbb{P}^3$, $(x,l)\mapsto x$ factorizes into

$$\tilde{G}\xrightarrow{f}X\xrightarrow{\pi}\mathbb{P}^3(x),$$

and the restriction $\tilde{G}_0\to X_0$ of f over $X_0\subset X$ coincides with the $\mathbb{P}^1$-bundle $f_0\colon\mathcal{P}_0\to X_0$:

$$\begin{array}{ccc}
\mathcal{P}_0 \subset \mathcal{P}\cong\tilde{G} & \xrightarrow{\sigma} & G(2,4)\supset R\\
{\scriptstyle f_0}\downarrow \quad \downarrow{\scriptstyle f} & & \\
X_0\subset X & & \\
{\scriptstyle \pi_0}\downarrow \quad \downarrow{\scriptstyle \pi} & & \\
\mathbb{P}^3\subset\mathbb{P}^3 & &
\end{array}$$

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4.3. The non-rationality of X by 2.1 (see [5])

We observe that, since $\sigma\colon \mathcal{P} = \tilde{G} \to G(2,4)$ is a blow-up of the surface R in the 4-fold $G(2,4)$,

$$H^4(\mathcal{P},\mathbb{Z}) = \sigma^* H^4(G(2,4),\mathbb{Z}) \oplus \sigma^{-1} H^2(R,\mathbb{Z})$$
$$\cong H^4(G(2,4),\mathbb{Z}) \oplus H^2(R,\mathbb{Z}).$$

Furthermore, since R is an Enriques surface, $c_1(R) \in H^2(R,\mathbb{Z})$ is an element of order 2. Therefore, $Z = \sigma^{-1}c_1(R)$ is an element of order 2 in $H^4(\mathcal{P},\mathbb{Z})$. After restriction, we get an element $Z_0 \in H^4(\mathcal{P}_0,\mathbb{Z})$ of order 2.

Since $f_0\colon \mathcal{P}_0 \to X_0$ is a $\mathbb{P}^1$-bundle, all fibres of f_0 are isomorphic to the two-dimensional spheres S^2. Therefore, the integral cohomology of $\mathcal{P}_0$ and X_0 fit in the Gysin sequence for S^2-fibration:

$$\cdots \to H^3(\mathcal{P}_0,\mathbb{Z}) \to H^1(X_0,\mathbb{Z}) \xrightarrow{e} H^4(X_0,\mathbb{Z}) \xrightarrow{f_0^*} H^4(\mathcal{P}_0,\mathbb{Z}) \xrightarrow{f_{0*}} H^2(X_0,\mathbb{Z}) \to \cdots .$$

Here, e is the cup product with the Euler class $e(f_0) \in H^3(X_0,\mathbb{Z})$ of f_0 (see [**6**, Chapter III, § 14] and [**16**, 4.11]).

If $\mathrm{Im}(e) \neq 0$, then any non-zero element of $\mathrm{Im}(e) \in H^4(X_0,\mathbb{Z})$ is a 2-torsion element, since $2e = 0$ (see [**16**, Theorem 4.11.2 (I)]).

In the case when $\mathrm{Im}(e) = 0$, $Z_0 \in \mathrm{Tors}_2(H^4(\mathcal{P}_0,\mathbb{Z}))$ must be an image $Z_0 = f_0^*(C_0)$ of an element $C_0 \in H^4(X_0,\mathbb{Z})$, since $\mathrm{Tors}(H^2(X_0,\mathbb{Z})) = 0$ (see [**5**, p. 30]). Since in this case f_0^* is an embedding and Z_0 is a non-zero 2-torsion element of $H^4(\mathcal{P}_0,\mathbb{Z})$, C_0 is also a non-zero 2-torsion element of $H^4(X_0,\mathbb{Z})$.

Thus, in both cases there exists a 2-torsion element $C_0 \in H^4(X_0,\mathbb{Z})$.

Let $\sigma_X\colon \tilde{X} \to X$ be the blow-up of X at the 10 nodes $p_1,\dots,p_{10}$ of X, and let $E_i = \sigma_X^{-1}(p_i) \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the 10 exceptional divisors on $\tilde{X}$. Since $\tilde{X}$ is isomorphic to a disjoint union of X_0 and E_i, $i = 1,\dots,10$, and $H^3(E_i,\mathbb{Z}) = H^3(\mathbb{P}^1 \times \mathbb{P}^1,\mathbb{Z}) = 0$, $H^4(X_0,\mathbb{Z})$ is embedded isomorphically in $H^4(\tilde{X},\mathbb{Z})$. In particular, $C_0 \in H^4(X_0,\mathbb{Z})$ is embedded as an element C of order 2 in $H^4(\tilde{X},\mathbb{Z})$.

Since for a smooth projective complex 3-fold $\tilde{X}$ one has that

$$\mathrm{Tors}(H^4(\tilde{X},\mathbb{Z})) \cong \mathrm{Tors}(H^3(\tilde{X},\mathbb{Z}))$$

(see [**1**, § 1]), the 2-torsion element $C \in H^4(\tilde{X},\mathbb{Z})$ equally represents a 2-torsion element $Z_C \in H^3(\tilde{X},\mathbb{Z})$. By Lemma 2.1, the latter yields that $\tilde{X}$ (and hence X) is non-rational.

Remark 4.3. It was suggested to us by Shramov that the methods of [**2**] can be applied to the double covering of a quadric ramified in an octic with 20 singular points. More precisely, we consider a divisor of bidegree $(1,2)$ in $Q \times \mathbb{P}^3$, where Q is a quadric 3-fold. In this case we get a quadric fibration given by a map $\mathcal{O}_Q(-1) \to S^2(E^*)$, where E is a trivial vector bundle of rank 4. We get a 2-torsion (and hence non-rationality) in a middle cohomology of a double quadric with 20 nodal singular points. Using the fact

Table 1. *Classical homological mirror symmetry.*

A-models (symplectic)	B-models (algebraic)
$X = (X, \omega)$ is a closed symplectic manifold	X is a smooth projective variety
Fukaya category $\mathsf{Fuk}(X)$. The objects are Lagrangian submanifolds L, which may be equipped with flat line bundles. The morphisms are given by the Floer cohomology $HF^*(L_0, L_1)$.	*Derived category* $D^b(X)$. The objects are complexes of coherent sheaves $\mathcal{E}$. The morphisms are $\mathrm{Ext}^*(\mathcal{E}_0, \mathcal{E}_1)$.
Y is a non-compact symplectic manifold with a proper map $W\colon Y \to \mathbb{C}$, which is a symplectic fibration with singularities.	Y is a smooth quasi-projective variety with a proper holomorphic map $W\colon Y \to \mathbb{C}$.
Fukaya–Seidel category of the Landau–Ginzburg model $\mathrm{FS}(\mathrm{LG}(Y))$. The objects are Lagrangian submanifolds $L \subset Y$, which, at ∞, are fibred over $\mathbb{R}^+ \subset \mathbb{C}$. The morphisms are $HF^*(L_0^+, L_1)$, where the superscript $+$ indicates a perturbation removing intersection points at ∞.	The category $D^b_{\mathrm{sing}}(W)$ of algebraic B-branes, which is obtained by considering the singular fibres $Y_z = W^{-1}(z)$, dividing $D^b(Y_z)$ by the subcategory of perfect complexes $\mathrm{Perf}(Y_z)$, and then taking the direct sum over all such z.

that the double covering of a quadric ramified in an octic with 20 singular points is a degeneration of a three-dimensional quartic, we study its Landau–Ginzburg model in § 5.

5. Mirror side

In this section we turn to homological mirror symmetry in an attempt to show that the phenomena observed in previous sections are a part of a much more general scheme. We briefly outline in Table 1 a schematic picture of classical homological mirror symmetry, in a version relevant to our purpose (for more details see [**23**]).

In what follows we describe the fibrewise compactifications of weak Landau–Ginzburg models of a quartic double solid, a Fano 3-fold V_{10} and a sextic double solid (see [**18**,**31**]). We conjecture that these compactifications are Landau–Ginzburg models of the Artin–Mumford example V_{10} and the sextic double solid, respectively, in the sense of HMS.

Throughout this section we use the following standard notation for blow-up. Consider the affine variety

$$\{F(x_1, \ldots, x_n) = 0\} \subset \mathbb{A}(x_1, \ldots, x_n).$$

We blow up the affine space $\{x_1 = \cdots = x_k = 0\}$. The blown-up hypersurface is given by the system of equations

$$F(x_1, \ldots, x_n) = 0,$$
$$x_i x'_j = x'_i x_j, \quad 1 \leqslant i,\ j \leqslant k,$$

in

$$\mathbb{A}(x_1, \ldots, x_n) \times \mathbb{P}(x'_1 : \cdots : x'_n).$$

Consider the local chart $x'_1 \neq 0$. We choose the coordinates

$$x_1, \frac{x'_2}{x'_1}, \ldots, \frac{x'_k}{x'_1}, x_{k+1}, \ldots, x_n.$$

In these coordinates, the blown-up variety is the zero locus of a polynomial given by the division of

$$F(x_1, x_1 x'_2, \ldots, x_1 x'_k, x_{k+1}, \ldots, x_n)$$

by the maximal possible power of x_1. We denote by x_i the coordinates in this local chart, instead of x'_i/x'_1, for simplicity. We denote this local chart by $x_1 \neq 0$.

We embed fibrewise the above pencil in a projective space or a product of projective spaces and then resolve singularities. All Calabi–Yau compactifications (see [**31**]) are birational in codimension 1.

5.1. The Landau–Ginzburg model of a quartic double solid

The weak Landau–Ginzburg model for a quartic double solid is given by

$$f = \frac{(x + y + 1)^4}{xyz} + z \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

We compactify the pencil $\{f = \lambda,\ \lambda \in \mathbb{C}\}$ in the neighbourhood of $\lambda = 0$ in $\mathbb{P}(x : y : z : t) \times \mathbb{A}(\lambda)$ and get the hypersurface

$$\{(x + y + t)^4 + xyz(z - \lambda t) = 0\} \subset \mathbb{P}(x : y : z : t) \times \mathbb{A}(\lambda).$$

Its singularities are the seven lines

$$\begin{aligned} &l_0 = \{x + y + t = z = \lambda = 0\}, && l_1 = \{x = y = t = 0\}, \\ &l_2 = \{x + y = z = t = 0\}, && l_3 = \{x = y + t = z = 0\}, \\ &l_4 = \{x = y + t = z + \lambda y = 0\}, && l_5 = \{x + t = y = z = 0\}, \\ &l_6 = \{x + t = y = z + \lambda x = 0\}. && \end{aligned}$$

Generically, the above singularities are locally products of Du Val singularities of type A_3 by the affine line. The 'horizontal' lines l_2 to l_6 intersect the 'vertical' line l_0; moreover, the pairs of lines l_3 and l_4, l_5 and l_6 intersect l_0 at one point (see Figure 1).

We resolve the singularities by blowing up these lines. First we blow up the vertical line l_0 twice. After this the singularities are proper transforms of the lines l_1 to l_6 and the

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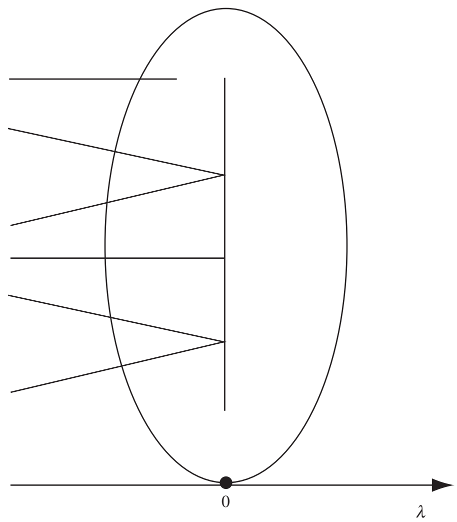

Figure 1. Singularities for a quartic double solid.

five lines lying on the exceptional divisors. Each of them intersects the proper transform of one of the lines l_2 to l_6. After blowing up these five lines we get a 3-fold with six lines of singularities coming from l_1 to l_6, which are of type A_3 along a horizontal affine line globally. Blowing them up fibrewise, we get the final resolution. We apply this procedure in the following steps.

Step 0. The line l_1 is of type A_3 along the affine line globally. Blowing it up twice we get horizontal exceptional fibres, so they do not give an additional component for the fibre over $\lambda = 0$. We proceed to a resolution in the neighbourhood of the line l_0.

Step 1. Let $a = x + y + t$. Our variety is then given by

$$\{a^4 + xyz^2 = \lambda xyz(a - x - y)\} \subset \mathbb{P}(x : y : z : a) \times \mathbb{A}(\lambda)$$

and $l_0 = \{a = z = \lambda = 0\}$. There exist two similar local charts: $x \neq 0$ and $y \neq 0$. Consider the local chart $y \neq 0$. It contains the lines of singularities l_0, l_2, l_3, l_4. We study the resolution in this chart and double the picture over the lines l_3, l_4. In this local chart we have an affine hypersurface

$$a^4 + xz^2 = \lambda xz(a - x - 1)$$

and we need to blow up the line $l_0 = \{a = z = \lambda = 0\}$.

The local chart 1a ($a \neq 0$)*.* We have the hypersurface

$$a^2 + xz^2 = \lambda xz(a - x - 1).$$

The exceptional divisor is given by the equation $a = 0$, so it consists of the three components

$$E_1^a = \{a = x = 0\}, \qquad E_2^a = \{a = z + (x + 1)\lambda = 0\}, \qquad E_3^a = \{a = z = 0\}.$$

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The proper transform of the fibre over $\lambda = 0$ is $E_0 = \{\lambda = a^2 + xz^2 = 0\}$. The singularities are

$$l_1^a = \{x = z = a = 0\}, \qquad l_2^a = \{x = \lambda + z = a = 0\},$$
$$l_3^a = \{z = a = \lambda = 0\}, \qquad l_4^a = \{x + 1 = z = a = 0\}.$$

We have that

$$E_2^a \cap E_3^a = l_3^a \cup l_4^a, \qquad E_1^a \cap E_3^a = l_1^a, \qquad E_0 \cap E_2^a \cap E_3^a = l_3^a.$$

All proper transforms of the lines l_2 to l_6 do not lie in this chart.

The local chart 1z ($z \neq 0$). There is nothing new in this chart: all we are interested in is contained in the chart 1a.

The local chart 1λ ($\lambda \neq 0$). We have the hypersurface

$$\lambda^2 a^4 + xz^2 = xz(\lambda a - x - 1).$$

The exceptional divisor is given by the equation $\lambda = 0$, so it consists of the three components

$$E_1^\lambda = \{\lambda = x = 0\}, \qquad E_2^\lambda = \{\lambda = z + x + 1 = 0\}, \qquad E_3^\lambda = \{\lambda = z = 0\}.$$

The proper transform of the fibre over $\lambda = 0$ does not lie in this chart. We have that

$$E_1^\lambda = E_1^a, \qquad E_2^\lambda = E_2^a, \qquad E_3^\lambda = E_3^a.$$

The singularities are

$$\begin{aligned}
l_1^\lambda &= \{x = z = \lambda = 0\} = E_1^\lambda \cap E_3^\lambda, \\
l_2^\lambda &= \{a = x = z = 0\} && \text{(proper transform of } l_3\text{)}, \\
l_3^\lambda &= \{x + 1 = z = \lambda = 0\} = E_2^\lambda \cap E_3^\lambda, \\
l_4^\lambda &= \{a = z = x + 1 = 0\} && \text{(proper transform of } l_2\text{)}, \\
l_5 &= \{x = z + 1 = \lambda = 0\} = E_1^\lambda \cap E_2^\lambda, \\
l_6^\lambda &= \{x = z + 1 = a = 0\} && \text{(proper transform of } l_4\text{)}.
\end{aligned}$$

So, after the first blow-up we get a configuration of the components of the central fibre as shown in Figure 2.

We then blow up the line l_3^a. It is enough to consider it in the chart 1a. That is, we blow up the line

$$\{z = a = \lambda = 0\}$$

at

$$\{a^2 + xz^2 - \lambda xz(a - x - 1) = 0\}.$$

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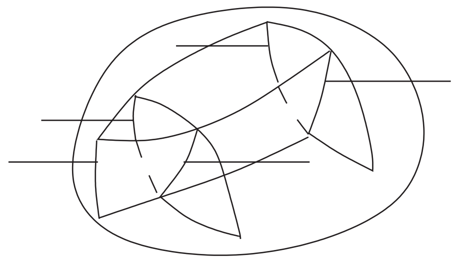

Figure 2. The picture after the first blow-up.

The only meaningful local chart is $\lambda \neq 0$. In this chart, we get the hypersurface

$$\{a^2 + xz^2 - xz(\lambda a - x - 1) = 0\}.$$

The exceptional divisor is

$$E^{a,\lambda} = \{\lambda = a^2 - xz(z + x + 1) = 0\}.$$

The singularities in its neighbourhood are

$$\begin{aligned} &\{a = x = z = 0\} && \text{(proper transform of } l_1^a),\\ &\{a = x = z + 1 = 0\} && \text{(proper transform of } l_2^a),\\ &\{a = x + 1 = z = 0\} && \text{(proper transform of } l_4^a). \end{aligned}$$

All of them lie on the exceptional divisor. So we did not get 'new' singularities after this blow-up. The divisors E_2^a, E_3^a now intersect only by the proper transform of l_4^a; the divisor E_1^a intersects E_2^a and E_3^a in two separated lines both intersecting the line $E_1^a \cap E^{a,\lambda} = \{x = a = \lambda = 0\}$. The proper image of E_0 intersects only $E^{a,\lambda}$ by a line lying far from the rest of the exceptional divisors.

We now blow up the line $l_4^a = l_3^\lambda$. The line l_3^a does not lie in the chart 1λ, so we can consider this blow-up only in the chart 1λ. We make the change of variables $x \to x - 1$. We then get a hypersurface

$$\{\lambda^2 a^4 + (x - 1)z^2 = (x - 1)z(\lambda a - x) = 0\},$$

and we then need to blow up the line

$$\{x = z = \lambda = 0\}.$$

We get one exceptional divisor, proper images of the lines l_5^λ and l_6^λ that lie far from the exceptional divisor, proper images of l_1^λ and l_2^λ (we discuss them later) and a proper image of l_4^λ (in other words, of l_2). It is globally of type A_3 along a line, so it resolves horizontally and does not give an exceptional divisor over $\lambda = 0$.

So, after this blow-up the divisors E_2^λ and E_3^λ are separated.

We now blow up the line l_1^λ. As before, we can do it in the chart 1λ. We have the hypersurface

$$\{\lambda^2 a^4 + xz(z + x + 1 - \lambda a) = 0\},$$

and need to blow up the line $\{x = z = \lambda = 0\}$.

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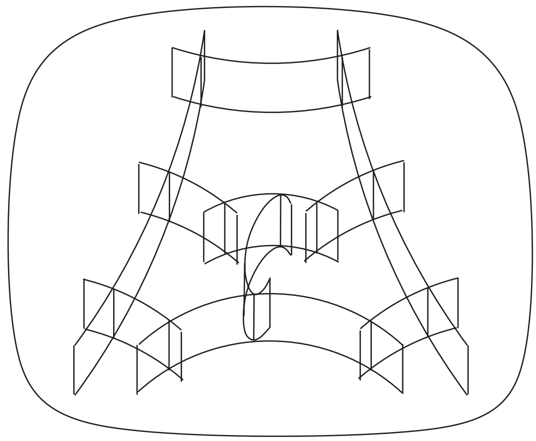

Figure 3. The fibre over 0 in the Landau–Ginzburg model for a quartic double solid.

We get proper transforms of l_3^λ and l_4^λ, as already discussed, proper transforms of l_5^λ and l_6^λ, mentioned in the next paragraph, and a proper transform of l_2^λ (in other words, of l_3). It is globally of type A_3 along a line, so it resolves horizontally and does not give an exceptional divisor in the central fibre.

Finally, the picture we get after blowing up the line l_5^λ is very similar to the picture we get after blowing up the line l_1^λ.

We summarize the final picture of resolved singularities (see Figure 3).

Via direct calculations (see [**25**, **27**]), we get the following.

Proposition 5.1. *The monodromy of the singular fibre at* 0 *of the Landau–Ginzburg model for a quartic double solid with* 10 *singular points is strictly unipotent.*

The proof of the above proposition is based on the analysis of monodromy change under a conifold transition.

5.2. The Landau–Ginzburg model of V_{10}

The weak Landau–Ginzburg model for a Fano variety V_{10} is

$$f = \frac{(x^2+x+y+z+xy+xz+yz)^2}{xyz} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

Compactifying the pencil $\{f = \lambda, \lambda \in \mathbb{C}\}$ in the neighbourhood of $\lambda = 0$ in $\mathbb{P}(x:y:z:t) \times \mathbb{A}(\lambda)$, we get the hypersurface

$$\{(x^2+xt+zt+xz+yt+yz+xy)^2 = \lambda xyzt\} \subset \mathbb{P}(x:y:z:t) \times \mathbb{A}(\lambda).$$

Its singularities are the 12 lines

$$\begin{aligned}
&l_1 = \{x+z=t=\lambda=0\}, && l_2 = \{x=z=t=0\}, && l_3 = \{x+z=y=t=0\},\\
&l_4 = \{x+y=t=\lambda=0\}, && l_5 = \{x=y=t=0\}, && l_6 = \{x+y=z=t=0\},\\
&l_7 = \{x=y=z=0\}, && l_8 = \{x+z=y=\lambda=0\}, && l_9 = \{x+t=y=z=0\},\\
&l_{10} = \{x=y+t=z=0\}, && l_{11} = \{x+t=y=\lambda=0\}, && l_{12} = \{x+t=z=\lambda=0\},
\end{aligned}$$

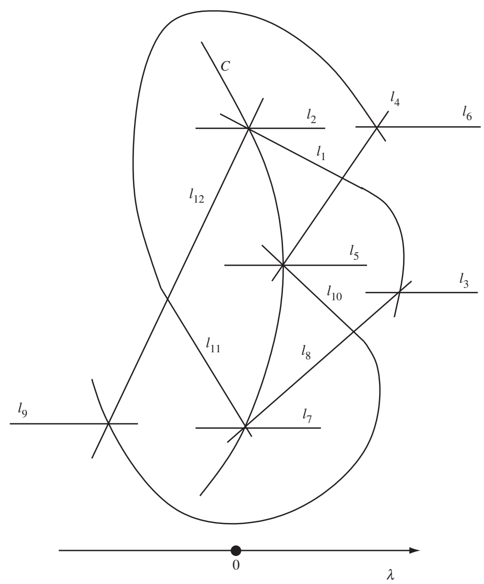

Figure 4. Singularities for V_{10}.

and the conic

$$C = \{x = yt + zt + yz = \lambda = 0\}$$

(see Figure 4).

There is a symmetry $x \leftrightarrow y \leftrightarrow z$, so we have three types of singular lines: two horizontal line types and one vertical line type.

We blow up l_6, we set $a = x + y$, and we consider a local chart $x = 1$. In this chart, the coordinates of our family can be written as

$$\{(a + at + zt + az)^2 = \lambda(a - 1)zt\}.$$

In the neighbourhood of l_6 it is analytically equivalent to a hypersurface $\{a^2 = \lambda zt\}$. In this local chart l_6, l_{11} and l_4 are given by the equations $a = z = t = 0$, $a = z = \lambda = 0$ and $a = t = \lambda = 0$, respectively. They are intersecting transversally the lines of singularities of type A_1. So, blowing l_6 up we get one horizontal exceptional divisor. In its neighbourhood the singularities (proper images of l_{11} and l_4) are the lines of singularities of type A_1. Similarly, by symmetry, the same holds in a neighbourhood of the lines l_3 and l_9. After performing the blow-ups described above, the singularities can be seen in Figure 5.

We blow up l_7 in the local chart $t = 1$. We have the hypersurface

$$\{(x^2 + x + z + xz + y + yz + xy)^2 = \lambda xyz\}.$$

Analytically, in a neighbourhood of l_7 it is isomorphic to a hypersurface

$$\{(x + y + z + yz)^2 = \lambda xyz\}.$$

The lines l_7, l_8, l_{11} and C are given by the equations $x = y = z = 0$, $x + z = y = \lambda = 0$, $x + y = z = \lambda = 0$ and $y + z + yz = x = \lambda = 0$, respectively.

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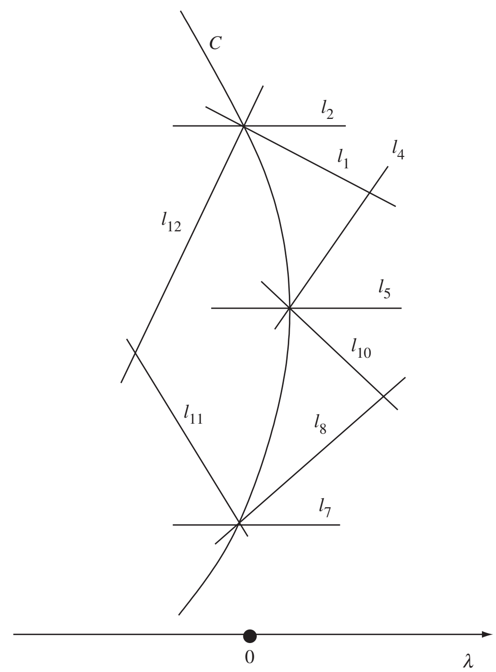

Figure 5. Singularities for V_{10}.

Consider a local chart $x \neq 0$ in the above blow-up. We get the hypersurface

$$\{(1 + y + z + xyz)^2 = \lambda xyz\}.$$

The exceptional divisor is given by

$$\{x = y + z + 1 = 0\},$$

and the singularities in its neighbourhood are given by

$$l_1^x = \{x = y = z + 1 = 0\}, \qquad l_2^x = \{x = y + 1 = z = 0\},$$
$$l^x = \{x = y + z + 1 = \lambda = 0\}, \qquad l_8 = \{y = z + 1 = \lambda = 0\},$$
$$l_{11} = \{y + 1 = z = \lambda = 0\}.$$

In the neighbourhood of l_1^x the singularities are three intersecting lines: one horizontal line l_1^x and two vertical lines l^x and l_8. They are analytically equivalent to singular lines on the hypersurface $\{a^2 = \lambda xy\}$. We blow up l_1^x first and then l^x and l_8. We get two non-intersecting exceptional divisors in the central fibre coming from l^x and l_8.

Consider now a local chart $y \neq 0$ in the blow-up. We get the hypersurface

$$\{(x + 1 + z + yz)^2 = \lambda xyz\}.$$

The exceptional divisor is given by

$$\{y = x + z + 1 = 0\},$$

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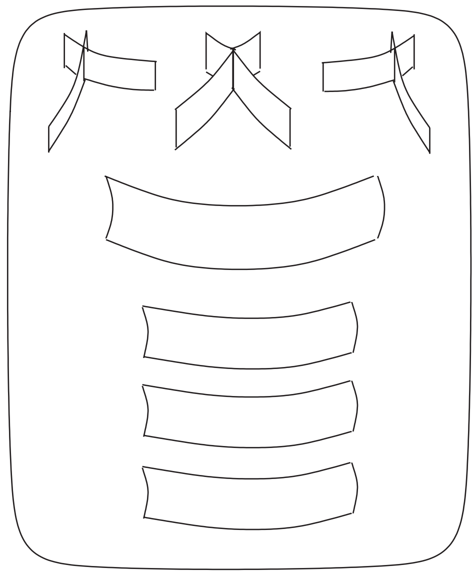

Figure 6. Singularities for V_{10}.

and the singularities in its neighbourhood are given by

$$l^x = \{y = x + z + 1 = \lambda = 0\}, \qquad l^y = \{x = y = z + 1 = 0\},$$
$$l_2^x = \{y = z = x + 1 = 0\}, \qquad l_{11} = \{x + 1 = z = \lambda = 0\},$$
$$C = \{x = 1 + z + yz = \lambda = 0\}.$$

In the neighbourhood of l^y the singularities form three intersecting lines of ordinary double points l^y, l_{11} and C, as before, so we can resolve them in a similar way.

Finally, we repeat the same procedure in the last local chart $z \neq 0$. The lines l_1 and l_4 intersect transversally and are of type A_1. Blowing them up one by one, we get, in the central fibre, two exceptional divisors intersecting in a line.

The central fibre of resolution is shown in Figure 6. There are 11 surfaces.

As before, direct calculations based on [**25**, **27**] give the following.

Proposition 5.2. *The monodromy of the singular fibre at* 0 *of the Landau–Ginzburg model for* V_{10} *with* 10 *singular points is strictly unipotent.*

5.3. The Landau–Ginzburg model of a sextic double solid

The weak Landau–Ginzburg model for a sextic double solid is

$$\frac{(x + y + z + 1)^6}{xyz} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

We are compactifying it in a projective space. The singularities are shown in Figure 7. They are three vertical lines, three horizontal lines and a horizontal plane (lines are symmetric with respect to changing coordinates $x \leftrightarrow y \leftrightarrow z$).

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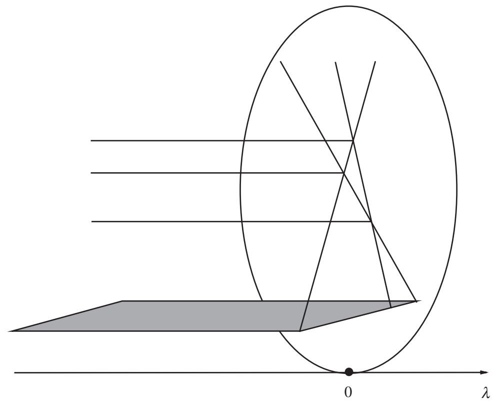

Figure 7. The singularities for a sextic double solid.

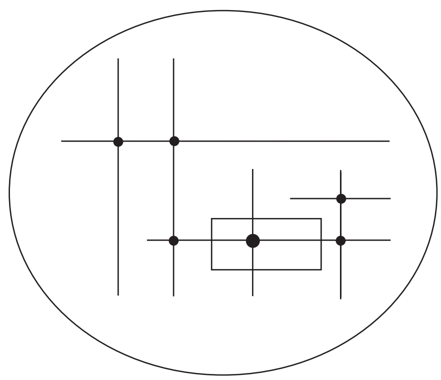

Figure 8. The final picture after a resolution of singularities in the neighbourhood of the non-normality locus.

We normalize the plane of singularities by blowing it up (twice). We then resolve the horizontal vertical singularities. We record the structure of the central fibre and the vertical singularities in Figures 8, 9 and 10, glued in a way demonstrated in Figure 11.

The lines on Figure 8 are surfaces (we look on them 'from above'). Bold ones intersect the 'base' surface. The rectangle is a surface lying 'over' the 'base'. It intersects the two remaining surfaces in two curves (which do not intersect the base surface). The point of intersection of these lines and a 'vertical' line of intersection of two other planes is denoted by a fat point. Eventually, we have nine surfaces and twelve lines recorded in the picture.

We follow the procedure for resolving singularities as in previous examples. The final picture is obtained by gluing the configurations of surfaces shown in Figures 8–10 along Figure 11. A more detailed description of the Landau–Ginzburg model for a sextic double solid can be found in [**8**]. Direct calculations (see [**25**, **27**]) yield the following.

Proposition 5.3. *The monodromy of the singular fibre at* 0 *of the Landau–Ginzburg model for a sextic double solid with* 35 *singular points is strictly unipotent.*

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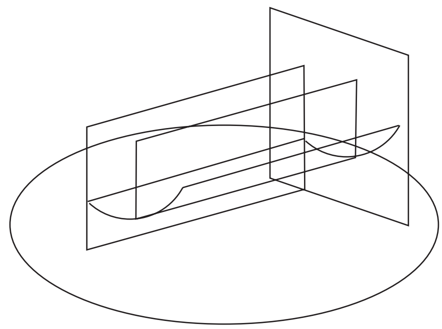

Figure 9. The resolution in the neighbourhood of the ‘first sheet’ of Figure 11.

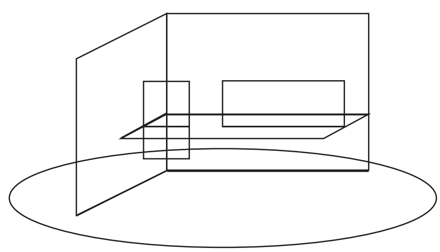

Figure 10. The resolution in the neighbourhood of the ‘deep sheet’ of Figure 11.

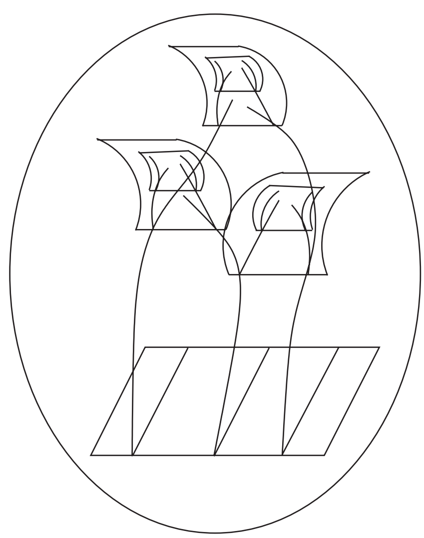

Figure 11. After blowing up the horizontal singularities.

The results from [**25**] suggest that the double covering of a quadric ramified in an octic with 20 nodal singular points will also have strictly unipotent monodromy of the singular fibre at 0 of its Landau–Ginzburg model. Indeed, this double covering is nothing more than a three-dimensional quartic deformation, and its monodromy was computed in [**25**].

We extract categorical information from this common phenomenon: strict unipotency of monodromy in the following theorems and conjectures.

We denote by $H(\mathrm{LG}(X), \mathcal{F})$ the hypercohomologies of the perverse sheaf of vanishing cycles on the Landau–Ginzburg model. $H(\mathrm{LG}(X), \mathcal{F})$ measure the cohomologies of X and the monodromy of $\mathrm{LG}(X)$ (see [**13**, **23**]).

Theorem 5.4. *Let X be a smooth Fano variety. Let $\mathrm{LG}(X)$ be its Landau–Ginzburg model (in particular, HMS for X and $\mathrm{LG}(X)$ holds). The Hochschild homology of the Fukaya–Seidel category of $\mathrm{LG}(X)$ is then $H(\mathrm{LG}(X), \mathcal{F})$.*

Proof. The proof follows from [**26**]. □

According to homological mirror symmetry, the Hochschild homology of the Fukaya–Seidel category of the Landau–Ginzburg model is isomorphic to the Hochschild homology of category $D^b(X)$.

Combining results from §§ 5.1 and 5.2 with the conifold transition change described in [**23**], we get the following.

Proposition 5.5. *The Hochschild homologies of category $D^b(X)$ of the Artin–Mumford example, and of the resolved V_{10} with 10 singular points, is isomorphic.*

Proof. The proof follows from direct calculations of the cohomology of the resolved V_{10} with 10 singular points. □

In fact, this homology looks like cohomology of a projective space.

Using the above analysis of the monodromy of the Landau–Ginzburg models of the Artin–Mumford example, of V_{10} with 10 singular points, of the double covering of a quadric ramified in an octic with 20 nodal singular points, and of a double solid with ramification in a sextic with 35 singular points (see [**25**]), we arrive at the following.

Conjecture 5.6. *The categories $D^b(X)$ of the Artin–Mumford example, of V_{10} with 10 singular points, of the double covering of a quadric ramified in an octic with 20 nodal singular points, and of a double solid with ramification in a sextic with 35 singular points, contain the category of a nodal Enriques surface as a semi-orthogonal summand.*

Remark 5.7. While this paper was being written, Ingalls and Kuznetsov, familiar with our work, stated the above conjecture for the Artin–Mumford example, and proved it for the minimal resolution of this example (see [**19**]). The first two authors are collaborating with Kuznetsov in order to prove this conjecture for V_{10} with 10 singular points.

In the next section we look at the above observations from the perspective of the theory of the spectra of categories.

6. The spectrum, the enhanced spectrum and applications

6.1. The classical spectrum

In this subsection we review the notions of spectra and gaps following [**4**].

Non-commutative Hodge structures were introduced in [**26**], as a means of bringing the techniques and tools of Hodge theory into the categorical and non-commutative realm. In the classical setting, much of the information about an isolated singularity is recorded by means of the Hodge spectrum, a set of rational eigenvalues of the monodromy operator. A categorical analogue of this Hodge spectrum appears in the works of Orlov and Rouqier [**30**, **32**]. We call this *the Orlov spectrum*. Recent work in [**4**] suggests an intimate connection with the classical singularity theory.

We recall the definitions of the Orlov spectrum and discuss some of the main results in [**4**]. Let $\mathcal{T}$ be a triangulated category. For any $G \in \mathcal{T}$ denote by $\langle G\rangle_0$ the smallest full subcategory containing G that is closed under isomorphisms, shifting and taking finite direct sums and summands. Now, inductively define $\langle G\rangle_n$ as the full subcategory of objects B such that there exists a distinguished triangle $X \to B \to Y \to X[1]$, with $X \in \langle G\rangle_{n-1}$ and $Y \in \langle G\rangle_0$, and direct summands of such objects.

Definition 6.1. Let G be an object of a triangulated category $\mathcal{T}$. If there exists an n with $\langle G\rangle_n = \mathcal{T}$, we set

$$t(G) = \min\{n \geqslant 0 \mid \langle G\rangle_n = \mathcal{T}\}.$$

Otherwise, we set $t(G) = \infty$. We call $t(G)$ the *generation time* of G. If $t(G)$ is finite, we say that G is a *strong generator*. The *Orlov spectrum* of $\mathcal{T}$ is the union of all possible generation times for strong generators of $\mathcal{T}$. The Rouqier dimension is the smallest number in the Orlov spectrum. We say that a triangulated category $\mathcal{T}$ has a *gap* of length s if a and $a+s$ are in the Orlov spectrum but r is not in the Orlov spectrum for $a < r < a+s$. We denote the maximum (finite) gap of the Orlov spectrum of $\mathcal{T}$ by $\mathrm{Gap}(\mathcal{T})$.

The following three conjectures are from [**4**].

Conjecture 6.2. *If X is a smooth variety, then any gap of $D^b(X)$ is at most the Krull dimension of X.*

Conjecture 6.3. *The maximal gap in Orlov's spectrum is a birational invariant.*

In particular, this conjecture states that if X is a smooth projective rational 3-fold, then the gap of $D^b(X)$ is equal to 1.

We now apply the theory of gaps to the observations from the previous sections. We first formulate the following.

Conjecture 6.4. *Let X be a smooth algebraic surface. Then, $h^{2,0}(X) = 0$ is equivalent to* $\mathrm{Gap}(D^b(X)) = 1$.

Combining this with Conjecture 5.6, we get the following.

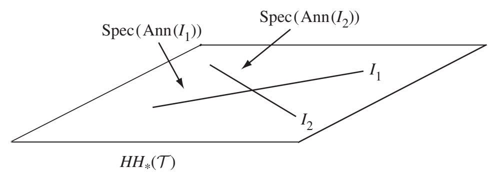

Figure 12. Noether–Lefschetz spectra.

Table 2. *HMS and Noether–Lefschetz spectra.*

A category $\mathcal{T}$	NLSpec($\mathcal{T}$)
$D^b(X)$	$\mathrm{NLSpec}(\mathcal{T}) \subset \mathrm{Spec}_{dg-gr}(HH^*(D^b(X))) \times \mathrm{Spec}(\mathcal{T})$
FS(LG(X))	$\mathrm{NLSpec}(\mathcal{T}) \subset \mathrm{Spec}_{dg-gr}(H^*(\mathrm{LG}(X), \mathcal{F})) \times \mathrm{Spec}(\mathcal{T})$

Conjecture 6.5. *The gap of the category $D^b(X)$ for the Artin–Mumford example, of V_{10} with* 10 *singular points, of the double covering of a quadric ramified in an octic with* 20 *nodal singular points, and of the double solid with ramification in a sextic with* 35 *singular points, is equal to* 1.

In other words, the gap of the Orlov spectra is too weak a categorical invariant to distinguish the rationality of these examples. In the next section we introduce more advanced Noether–Lefschetz spectra.

6.2. Enhanced Noether–Lefschetz spectra

Let $\mathcal{T}$ be an enhanced triangulated category and let $HH^*(\mathcal{T})$ be its Hochschild cohomology.

Definition 6.6. We denote by *Noether–Lefschetz spectra* $\mathrm{NL}(\mathcal{T})$ the ordered collection of sets over $HH^*(\mathcal{T})$ defined as follows. For any graded ideal I in $HH^*(\mathcal{T})$ we consider the differential graded (DG) subcategory $\mathrm{Ann}(I)$ in $\mathcal{T}$: the annihilator of I. The set $\mathrm{Spec}(\mathrm{Ann}(I))$ is the set of generators of $\mathcal{T}$ in the DG subcategory $\mathrm{Ann}(I)$. We denote the maximum gap of $\mathrm{Spec}(\mathrm{Ann}(I))$ over all subsets I by $\mathrm{NLGap}(\mathcal{T})$ (see Figure 12).

Clearly, $\mathrm{Spec}(\mathcal{T})$ embeds in the set $(I, \mathrm{Spec}(\mathrm{Ann}(I)))$, but the behaviour of the gaps in $\mathrm{NL}(\mathcal{T})$ is much more complex (for more examples see [**3**]).
We make the following conjecture.

Conjecture 6.7. *Let X be a three-dimensional smooth projective variety. If X is rational, then the gaps in $\mathrm{NL}(D^b(X))$ are equal to* 1.

The above conjecture suggests a new invariant of rationality. It is based on our studies of Landau–Ginzburg models from previous sections. Proposition 5.1 together with HMS suggests that $\mathrm{NL}(D^b(X))$ are completely determined by the monodromy and vanishing cycles of Landau–Ginzburg models (see Table 2). Still, it is possible that $\mathrm{NL}(D^b(X))$ has all gaps equal to 1 and X is not rational.

Table 3. *Summarizing the conjectures.*

A Fano variety X	$D^b(X)$ and $HH_0(X)$	$\mathrm{Gap}(D^b(X))$	$\mathrm{NLGap}(X)$
A double covering of $\mathbb{P}^3$ ramified in a K3 surface with 10 nodal singular points (Artin–Mumford variety).	$D^b(X) = \langle D^b(E), E_1, \dots, E_{10} \rangle$, where E is a nodal Enriques surface. $\dim(HH_0(X)) = 4$	1	$\geqslant 2$
Double covering $V_{10} \xrightarrow{2:1} V_5$	$D^b(X) = \langle D^b(E), \dots \rangle$, where E is a nodal Enriques surface. $\dim(HH_0(X)) = 4$	1	$\geqslant 2$
$\mathbb{P}^3$	$\dim(HH_0(X)) = 4$	1	1
A sextic double solid with 35 nodal singular points.	$D^b(X) = \langle D^b(E), \dots \rangle$, where E is a nodal Enriques surface. $\dim(HH_0(X)) = 4$	1	$\geqslant 2$
The double covering of a quadric ramified in an octic with 20 nodal singular points.	$D^b(X) = \langle D^b(E), \dots \rangle$, where E is a smooth Enriques surface. $\dim(HH_0(X)) = 4$	1	$\geqslant 2$

In what follows we give conjectural examples of three-dimensional varieties that have gaps equal to 1 in $\mathrm{Spec}(D^b(X))$ and have gaps equal to 2 or higher in $\mathrm{NL}(D^b(X))$. Following Conjecture 6.3, homological mirror symmetry and examples in § 5, we make the following conjecture.

Conjecture 6.8. *In all examples, the Artin–Mumford example, V_{10} with 10 singular points, the double covering of a quadric ramified in an octic with 20 nodal singular points and the double solid with ramification in a sextic with 35 singular points, $\mathrm{NLGap}(D^b(X))$ is equal to 2 or higher.*

This conjecture is based on the fact that Landau–Ginzburg models, for the Artin–Mumford example, for V_{10} with 10 singular points, for the double covering of a quadric ramified in an octic with 20 nodal singular points and for the double solid with ramification in a sextic with 35 singular points, have the same monodromies (see also [**25**]).

We record all our findings and conjectures in Table 3.

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Remark 6.9. It is quite possible that derived categories of the Artin–Mumford example and of V_{10} are related via deformation, in which case it is not surprising that their spectra are equal.

Remark 6.10. The considerations in the last two sections suggest a strong correlation between spectra, monodromy and walls in moduli spaces of stability conditions. We pose the following two questions.

Question 1. Do Noether–Lefschetz spectra define a stratification on the moduli space of stability conditions?

Question 2. Are classical Noether–Lefschetz loci connected to this stratification?

Remark 6.11. The Artin–Mumford example is an example of a conic bundle. We expect that the technique discussed here will lead to many examples of conic bundles for which the gap of Orlov's spectrum is equal to 1 and whose non-rationality can be established using gaps in the Noether–Lefschetz spectra.

Acknowledgements. The authors thank D. Favero, G. Kerr, M. Kontsevich, A. Kuznetsov, D. Orlov and T. Pantev for useful discussions, the referee for proofreading the paper, and C. Shramov for pointing out the example of the double covering of a quartic ramified in an octic with 20 nodal points. A.I. was funded by the FWF (Grant P20778). L.K. was funded by the NSF (Grant DMS0600800), the NSF FRG (Grant DMS-0652633), the FWF (Grant P20778) and the ERC (Grant GEMIS). V.P. was funded by the NSF FRG (Grant DMS-0854977), the NSF (Grants DMS-0854977 and DMS-0901330), by the FWF (Grants P24572-N25 and P20778), by the RFFI (Grants 11-01-00336-a, 11-01-00185-a, 12-01-33024 and 12-01-31012; Grants MK-1192.2012.1, NSh-5139.2012.1) and by the AG Laboratory GU-HSE RF (Grant ag. 11 11.G34.31.0023).

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