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ON HOMOGENEOUS IMAGES OF COMPACT ORDERED SPACES

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ABSTRACT We answer a 1975 question of G R Gordh by showing that if X is a homogeneous compactum which is the continuous image of a compact ordered space then at least one of the following holds

(i) X is metrizable, (ii) $\dim X = 0$ or (iii) X is a union of finitely many pairwise disjoint generalized simple closed curves

We begin to examine the structure of homogeneous 0-dimensional spaces which are continuous images of ordered compacta

1. **Introduction.** The aim of this paper is to investigate homogeneous spaces which are continuous images of ordered compacta. In 1975, G. R. Gordh proved that if a homogeneous and hereditarily unicoherent continuum is the continuous image of an ordered compactum, then it is metrizable, and so indecomposable [7, Theorem 3]. Further, he asked if, in general, every homogeneous continuum which is the continuous image of an ordered compactum must be either metrizable or a generalized simple closed curve.

Our Theorem 1 provides an affirmative answer to Gordh's question. Moreover, in Theorem 2, we prove that a homogeneous space which is not 0-dimensional and which is the continuous image of an ordered compactum is either metrizable or a union of finitely many pairwise disjoint generalized simple closed curves. Our methods of proof involve characterizations of continuous images of arcs obtained in [16] in terms of cyclic elements and T-sets.

When dealing with the class A of all homogeneous and 0-dimensional spaces which are the continuous images of ordered compacta, the situation becomes less clear. By a recent theorem of M. Bell, each member of A is first countable. Moreover, by a result of [18], each member of A can be embedded into a dendron. We give a rather simple construction leading to a wide subclass of A. In particular, we show that not all members of A are orderable, and that there exists a strongly homogeneous space X which is the continuous image of an ordered compactum and which is not first countable. It follows that $X \notin A$. Our investigations of the class A led to some natural questions which are stated at the end of the paper.

All spaces considered in this paper are Hausdorff.

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A *continuum* is a compact connected (Hausdorff) space. A point x is a *separating point* of a connected space X if $X - \{x\}$ is not connected. A point x is an *end-point* of a continuum X if X has a neighbourhood basis at x of open sets with one point boundaries. If X is a locally connected continuum then by a *cyclic element* of X is meant a separating point of X, an end-point of X or a maximal connected subset of X which contains no separating point of itself. Each cyclic element of X is itself a locally connected continuum. We say that a locally connected continuum X is *cyclic* if it contains no separating points. It follows that a point $x \in X$ is an end-point of X if and only if $X - \{x\}$ is connected and x is contained in no non-degenerate cyclic element of X.

Let X be a locally connected continuum and $x, y \in X$ with $x \neq y$. Let

$$E_{xy} = \{x, y\} \cup \{z \in X : X - \{z\} \text{ can be separated between } x \text{ and } y\}.$$

Then E_{xy} is a compact naturally ordered set (see *e.g.* [26, III 4.2]). Let Y be an irreducible continuum in X from x to y. Let C_{xy} denote the union of Y and the cyclic elements of X which meet Y in at least two points. Then C_{xy} is a locally connected continuum called the *cyclic chain* from x to y (We recall that it does not depend on any particular choice of Y). The set of separating points of C_{xy} is $E_{xy} - \{x, y\}$.

A space X is *homogeneous* if for each $x, y \in X$ there is a homeomorphism h of X onto itself such that $h(x) = y$.

By an *ordered space* is meant a space X which admits a linear ordering such that the order topology on X coincides with the given topology. An ordered space X has a *gap* if $X = A \cup B$ where A and B are sets (one of them possibly void) such that A has no largest element, B has no smallest element and $x < y$ for each $x \in A$ and each $y \in B$. We say X has a *jump* if there exist a and b in X such that $a < b$ and if $x \in X$ with $a \leq x \leq b$ then either $a = x$ or $b = x$. We call b the *immediate successor* of a in X.

A compact ordered space (often called: ordered compactum) has no gaps. In particular, it has first and last elements. An *arc* is a non-degenerate, compact, connected, ordered space. Alternatively, an arc is a non-degenerate, compact, ordered space with no jumps. A (generalized) *simple closed curve* is a space obtained from an arc by identifying the first and last elements of the arc.

A continuum J is said to be *hereditarily equivalent* if it is homeomorphic to each of its non-degenerate subcontinua. There exist many hereditarily equivalent arcs (see *e.g.* [8] or the recent paper [14], where more references can be found). It is well-known that each hereditarily equivalent arc is first countable, [1], [11] and [22]. The reader should be warned that there exists an hereditarily equivalent arc which does not admit an order reversing (*i.e.*, end-points exchanging) homeomorphism onto itself (see [8]; more perverse examples can be found in [4]). Fortunately, if $[a, b]$ is an hereditarily equivalent arc and $[p, q]$ is a subarc of $[a, b]$ such that $a \leq p < q \leq b$ in the natural ordering of $[a, b]$, then there is a homeomorphism $h: [a, b] \longrightarrow [p, q]$ such that $h(a) = p$ (see [22, p. 1417] or [14, Lemma 2.2]). The latter fact implies that each simple closed curve obtained by identifying the end-points of an hereditarily equivalent arc is a homogeneous space. However, it is unclear to the authors if an arc obtained by splitting a point of a

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homogeneous simple closed curve into two points must always be an hereditarily equivalent continuum (Note that in a less general situation considered in [22, Theorem 2] that was the case)

We remark that a closed subspace of a space which is the continuous image of an ordered compactum must be also the continuous image of an ordered compactum Furthermore, since arcs are locally connected continua, each space which is the continuous image of an arc is a locally connected continuum as well

If A, B and C are arcs with a common end-point x and $\{x\} = A \cap B = A \cap C = B \cap C$ then the continuum $A \cup B \cup C$ is said to be a *simple triod*

A continuum is said to be *rim-finite* if it has a basis of open sets with finite boundaries Clearly, every subcontinuum of a rim-finite continuum is a rim-finite, locally connected continuum If a and b are two points of a rim-finite continuum X every subcontinuum of X which is irreducible with respect to containing a and b is an arc It is also well-known that each rim-finite continuum is the continuous image of an arc (see *e g* [20, p 179])

A continuum X is said to be a *dendron* if each pair of distinct points of X is separated by a third point of X It is easy to see that each dendron is rim-finite If X is a dendron we let $R(X)$ denote the set of ramification points of X *i e* the set of $x \in X$ such that $X - \{x\}$ has at least three components Recall that *dendrite* is synonymous with metrizable dendron

The reader is referred to the survey papers [20] and [13] for more results and references concerning continuous images of ordered compacta and dendrons, respectively

2 **Background on images of compact ordered spaces.** A subset A of a continuum X is said to be a *T-set* in X if A is closed and each component of $X - A$ has a two point boundary A T-set A in X is said to be a *strong T-set* in X if each component of $X - A$ is homeomorphic to the open interval $]0, 1[$ of real numbers

It is well-known and easy to prove that if A is a T-set in a cyclic, locally connected continuum X and J is a component of $X - A$ with $\mathrm{bd}(J) = \{a, b\}$ then $c\ell(J)$ is a locally connected continuum which is a cyclic chain from a to b

Let X be a cyclic, locally connected continuum A sequence $\{A_n\}_{n=1}^{\infty}$ of T sets in X *T-approximates* X (resp *deeply T approximates* X) if the following conditions (i)–(iv) (resp (i)–(v)) are satisfied

(i) A_1 is metrizable,

(ii) $A_1 \subset A_2 \subset \ldots$,

(iii) if J is a component of $X - A_n$ then A_{n+1} contains the set of separating points of $c\ell(J)$,

(iv) if J is a component of $X - A_n$ and M is a non-degenerate cyclic element of $c\ell(J)$, then $M \cap A_{n+1}$ is a metrizable T-set in $c\ell(M)$ and $M \cap A_{n+1}$ contains at least three points,

(v) if J is a component of $X - A_n$ then the set of separating points of $c\ell(J)$ is not metrizable

THEOREM [2]. *Let X be a Hausdorff space which is the continuous image of a compact ordered space. If some Cartesian power X^{α} of X is a homogeneous space, then X is first countable.*

THEOREM B (SEE [16, THEOREM 1.1] AND [17, THEOREM 4]). *If X is a cyclic, locally connected continuum, then the following conditions are equivalent:*

(i) *X is the continuous image of a compact ordered space,*

(ii) *there is a sequence $\{A_n\}_{n=1}^{\infty}$ of T-subsets of X which T-approximates X,*

(iii) *there is a sequence $\{A_n\}_{n=1}^{\infty}$ of T-subsets of X which deeply T-approximates X.*

THEOREM C. *Let X be a locally connected continuum which is the continuous image of a compact ordered space. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of T-sets in X which T-approximates X and let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is a dense subset of X and $X - A$ is 0-dimensional. Moreover, if $x \in X - A$ and, for each n, J_n is the component of $X - A_n$ which contains x, then $\{J_n\}_{n=1}^{\infty}$ is a local basis of X at x.*

PROOF. That A is dense in X was proved in [16, Lemma 3.4]. Let $x \in X - A$ and, for each n, let J_n be the component of $X - A_n$ which contains x. By [24, Theorem 8], $\bigcap_{n=1}^{\infty} c\ell(J_n) = \{x\}$. Hence, $\{J_n\}_{n=1}^{\infty}$ is a local basis of X at x and, therefore, $X - A$ is 0-dimensional.

LEMMA 1 (SEE [15, THEOREM 2] OR [10, LEMMA 8]). *If a Hausdorff space X is the continuous image of an ordered compactum, then there is a continuum Y such that Y is the continuous image of an arc, X is a subset of Y and X is a strong T-set in Y. Further, if X is a continuum with no separating point then Y is cyclic.*

LEMMA 2. *A homogeneous continuum has no separating point.*

PROOF. Every continuum has at least two non-separating points, see *e.g.* [26] or [9].

LEMMA 3. *A compact, homogeneous space X which contains a non-empty open and metrizable subset is metrizable.*

PROOF. By homogeneity and compactness, X has a finite cover by closed metrizable sets, and so X is metrizable by [5, Theorem 4.4.19].

3. **On Gordh's question.**

THEOREM 1. *If X is a homogeneous continuum which is the continuous image of an ordered compactum, then either X is metrizable or X is a simple closed curve.*

PROOF. We may assume that X is not metrizable. By Lemma 2, X has no separating point. By Lemma 1, there is a cyclic locally connected continuum Y such that $X \subset Y, X$ is a strong T-set in Y, and Y is the continuous image of an arc. By Theorem B, there is a sequence $\{A_n\}_{n=1}^{\infty}$ of T-subsets of Y which T-approximates Y. Let $A = \bigcup_{n=1}^{\infty} A_n$. By Theorem C, A is dense in Y and $Y - A$ is 0-dimensional.

Suppose first that X is not contained in A. Let $x \in X - A \subset Y - A$. For each n, let J_n be the component of $Y - A_n$ such that $x \in J_n$ and let $U_n = X \cap J_n$. By Theorem C, the collection $\{U_n : n = 1, 2, \ldots\}$ is a local basis for X at x. Obviously, $\mathrm{bd}_X(U_n) \subset \mathrm{bd}_Y(J_n)$ and so $\mathrm{bd}_X(U_n)$ consists of at most two points, for $n = 1, 2, \ldots$. It follows that X is a rim-finite continuum which has a basis of open sets with at most two point boundaries. Therefore, X contains no simple triod. Thus X is either an arc or a simple closed curve. By Lemma 2, X is a simple closed curve.

Now, suppose that $X \subset A$. Let $B_n = X \cap A_n$ for $n = 1, 2, \ldots$. Then each B_n is a closed subset of X and $X = \bigcup_{n=1}^{\infty} B_n$. Hence, there is a positive integer n such that the interior of B_n in X is non-empty. Let $m = \min\{n : B_n$ has non-empty interior in $X\}$. If $m = 1$, then $\mathrm{int}_X(B_1)$ is a metrizable open subset of X (because A_1 is metrizable), and X is metrizable by Lemma 3. Thus it suffices to consider the case when $m > 1$.

By the choice of m, there is a component J of $Y - A_{m-1}$ such that $J \cap B_m$ has non-empty interior in X. Let $V = \mathrm{int}_X(J \cap B_m)$. Let $\mathrm{bd}(J) = \{a, b\}$ and $E = \{x \in J : x$ separates $J\} \cup \{a, b\}$. Then $c\ell(J)$ is a cyclic chain from a to b, and E is a compact ordered space. Let C be a non-degenerate cyclic element of $c\ell(J)$. Then $C \cap E$ consists of exactly two points which will be denoted by 0_C and 1_C. Moreover, $C - E$ is an open subset of J and of Y. We have two cases to consider.

First, assume that there is a non-degenerate cyclic element C of $c\ell(J)$ such that $V \cap (C - E) \neq \emptyset$. Let $W = V \cap (C - E)$. Then W is a non-empty open subset of X. Since $W \subset B_m \cap (C - E) \subset A_m \cap C$ and $A_m \cap C$ is metrizable, it follows that W is metrizable. By Lemma 3, X is metrizable.

Finally, consider the case when $V \subset E$. Since V is a non-empty open subset of a continuum X, it contains a non-degenerate continuum K. Thus, K is a non-degenerate subcontinuum of the compact linearly ordered space E. Therefore, K is an arc. Let $W = K -$(end-points of K). It is easy to see that W is an open subset of J. Therefore W is an open subset of X. This proves that each point of X has a neighbourhood which is an arc, whence, X is a simple closed curve. This completes the proof of Theorem 1.

Let X be a locally connected continuum. Let $E(X)$ denote the set of end-points of X. Recall also that a subset Z of X is said to be a *node* of X provided Z is an end-point of X or Z is a non-degenerate cyclic element of X such that $\mathrm{bd}(Z)$ is a single point. It is well-known, [26, IV.8.2], that every locally connected continuum which is not cyclic has at least two nodes (The argument given in [26] can be easily extended to the non-metric case; see also [27]).

THEOREM 2. *Let X be a homogeneous space which is the continuous image of an ordered compactum. If X has a non-degenerate component, then either X is metrizable or X has only finitely many components.*

PROOF. Let K denote a component of X. Then K is homogeneous and each component of X is homeomorphic to K. By Theorem 1, either K is metrizable or K is a homogeneous and non-metrizable simple closed curve. Hence, we have two totally different cases to consider.

CASE 1: THE COMPONENTS OF X ARE NON-METRIZABLE. By Theorem 1, each component K of X is a non-metrizable simple closed curve. If A is a subcontinuum of K then

A has a non-empty interior in K. By Lemma 3, A is non-metric. Thus, X contains no non-degenerate metrizable subcontinuum. We are going to prove that X has only finitely many components.

Let I be an arc which contains no subset homeomorphic to $[0,1]$ (*e.g.* one can take I to be $[0,1]^\infty$ with the order topology induced by the lexicographic ordering). Then I contains no nondegenerate metrizable subcontinuum. Let 0_I and 1_I denote the end-points of I, $\leq_I$ denote the natural ordering of I from 0_I to 1_I, and $I^0 = I - \{1_I\}$.

Let $f\colon C \longrightarrow X$ be a continuous surjection of an ordered compactum C onto X. Let $\leq_C$ denote a natural ordering of C. Since C is compact, $(C, \leq_C)$ has no gaps.

We let D denote an arc which is formed from C by inserting a copy of $I - \{0_I, 1_I\}$ into each jump of $(C, \leq_C)$. Namely, let $B = \{x \in C :$ there is an immediate successor to x in $(C, \leq_C)\}$, let $D = (C \times \{0_I\}) \cup (B \times I^0)$ and place a lexicographic ordering $\leq$ on $D : (c,i) < (c',i')$ provided either $c <_C c'$, or $c = c'$ and $i <_I i'$. Then $(D, \leq)$ has no gaps and no jumps, and so D with its order topology is an arc. It contains a homeomorphic copy $C' = C \times \{0_I\}$ of C and each component of $D - C'$ is homeomorphic to $I - \{0_I, 1_I\}$.

Let G denote the decomposition of D into sets $f^{-1}(x) \times \{0_I\}, x \in X$, and single points. Since f is continuous, G is upper semi-continuous. Let Y be the quotient space $Y = D/_G$. Then Y is a continuous image of the arc D. Hence, Y is a locally connected continuum. Observe that Y contains X and each component of $Y - X$ is homeomorphic to $I - \{0_I, 1_I\}$.

Let M be a non-degenerate subcontinuum of Y. Then either M is contained in a component of X or M meets a component K of $Y - X$. In the latter case $M \cap K$ contains a non-degenerate continuum (because K is an open subset of Y). Recall that neither X nor K contains a non-degenerate metrizable subcontinuum. It follows that no non-degenerate subcontinuum of Y is metrizable.

We have proved that Y is a continuous image of an arc and Y contains no non-degenerate metric continuum. By [23], Y is a rim-finite continuum.

Let $\{J_\alpha : \alpha < \delta\}$ be a well-enumeration of the collection of all components of $Y - X$. Define sets Z_α, $\alpha < \delta$, by transfinite induction:

(a) $Z_0 = Y$,

(b) $Z_\lambda = \bigcap_{\alpha<\lambda} Z_\alpha$ if λ is a limit ordinal number, $0 < \lambda < \delta$,

and

(c) $Z_{\alpha+1} = \begin{cases} Z_\alpha & \text{if } Z_\alpha - J_\alpha \text{ is not connected} \\ Z_\alpha - J_\alpha & \text{otherwise.} \end{cases}$

It follows that each Z_α, $\alpha < \delta$, is a compact space which contains X. Since the intersection of a nested collection of continua is a continuum again, it follows that each Z_α is a continuum. Let $Z = \bigcap_{\alpha<\delta} Z_\alpha$. Then $X \subset Z$, and each component of $Z - X$ is a component of $Y - X$ homeomorphic to $I - \{0_I, 1_I\}$. Also, $Z - K$ is not connected for each component K of $Z - X$, and Z is a subcontinuum of Y. Since Y is rim-finite, Z is rim-finite as well. Being a rim-finite continuum, Z is locally connected.

If Z is cyclic then $X = Z$ because each point of $Z - X$ disconnects Z by the definition of Z. In particular, X is connected.

Suppose, therefore, that Z is not cyclic and let Q be a node in Z. Let K be a component of $Z - X$. Since $c\ell(K)$ is an arc such that each point of K disconnects Z and Q is a node of

Z it follows that $K \cap Q = \emptyset$. Hence, there is a component L of X such that $Q \subset L$. Recall that, in the case under consideration, the components of X are simple closed curves. Thus, $Q = L$. Since Q is a node, let x be the unique point in the boundary of Q in Z. Then $L - \{x\}$ is an open set of Z and, hence, in X. Since X is homogeneous, each point of X has an open neighbourhood which meets exactly one component of X. Hence, by homogeneity, each component of X is open in X. Since X is compact, X has only finitely many components. This completes the proof when each component of X is non-metrizable.

CASE 2: X HAS A METRIZABLE COMPONENT. Then all components of X are non-degenerate and metrizable continua. Let Y be a locally connected continuum such that $X \subset Y$, X is a strong T-set in Y, and Y is the continuous image of an arc.

Below, we shall consider the case when Y is not cyclic. If Y is cyclic, the argument is somewhat simpler and is left to the reader.

Assume that Y is not cyclic. First, we shall show that Y has no end-point. Suppose on the contrary that x is an end-point of Y. It follows that $x \in X$. Let M be the component of X which contains x. Then M is a non-degenerate homogeneous continuum. By Lemma 2, M has no separating point. Therefore, there exists a cyclic element Y' of Y such that $x \in M \subset Y'$, and so x is not an end-point of Y, a contradiction.

Let Z be a node of Y. Since Y contains no end-point, Z is a non-degenerate cyclic element of Y. Then bd(Z) consists of a single point z. Let $X' = X \cap (Z - \{z\})$. Since each component of $Y - X$ is a copy of $]0,1[$, it follows that $z \in X$. Let M denote the component of X which contains z. By Lemma 2, M has no separating point and, consequently, either $M \subset Z$ or $M \cap Z = \{z\}$. In each case, X' is a non-empty open subset of X. Moreover, if $M \cap Z = \{z\}$ then each component of X which meets X' is contained in X', and if $M \subset Z$ then each component of X which meets X' is contained in $X' \cup \{z\}$.

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of T-subsets of Z which deeply T-approximates Z. Let $A = \bigcup_{n=1}^{\infty} A_n$. By Theorem C, A is dense in Z and $Z - A$ is 0-dimensional.

Now, we shall show that $X' \subset A$ (our argument shows also that $Z - \{z\} \subset A$). Let L be a component of X which meets X'. Then $L \subset Z$. Since L is non-degenerate and $Z - A$ is 0-dimensional, there is a positive integer m such that $L \cap A_m$ consists of at least two points. Suppose that L is not contained in A_m. Then there is a component J of $Z - A_m$ such that $L \cap J \neq \emptyset$. Let bd(J) $= \{a, b\}$. Since L is a continuum which meets A_m, it follows that $L \cap \{a, b\} \neq \emptyset$. First, assume that $L \cap \{a, b\}$ consists of a single point, say a. Recall that $L \cap A_m$ has more than one point, and so L is not a subset of $c\ell(J)$. It follows that a is a separating point of L. However, the continuum L, being non-degenerate and homogeneous, has no separating points, a contradiction. Now, assume that $a, b \in L$. Then there is a subcontinuum L' of L such that $a, b \in L' \subset c\ell(J)$. Since $c\ell(J)$ is a cyclic chain from a to b, it follows that each separating point of J belongs to L'. However, L and L' are metrizable and the set of all separating points of J is non-metrizable (because $\{A_n\}_{n=1}^{\infty}$ is assumed to deeply T-approximate Z), a contradiction. We have proved that $L \subset A_m$. This concludes the proof that $X' \subset A$.

Recall that X' is a non-empty open subset of X. Let V be a non-empty open subset of X such that $c\ell(V) \subset X'$. Then $c\ell(V) \subset A$. Since $c\ell(V) = \bigcup_{n=1}^{\infty}\big(c\ell(V) \cap A_n\big)$, there is an integer n such that $c\ell(V) \cap A_n$ has a non-empty interior in $c\ell(V)$.

Let $m = \min\{n : c\ell(V) \cap A_n$ has a non-empty interior in $c\ell(V)\}$. Let U' denote the interior of $c\ell(V) \cap A_m$ in $c\ell(V)$, and let $U = U' \cap V$. Then U is a non-empty open subset of X. We shall show that U contains a non-empty open and metrizable subset W of X. Then Lemma 3 will imply that X is metrizable.

If $m = 1$, then $A_m = A_1$ is metrizable and it suffices to let $W = U$. Suppose that $m > 1$. By the choice of m, there is a component J of $Z - A_{m-1}$ such that $J \cap U \neq \emptyset$. Let $W' = J \cap U$. Then W' is a non-empty open subset of X. Let $\mathrm{bd}(J) = \{a, b\}$ and $E = \{a, b\} \cup \{x \in J : x$ separates $J\}$. Recall that $c\ell(J)$ is a cyclic chain from a to b. Hence, if C is a non-degenerate cyclic element of $c\ell(J)$, then $C \cap E$ consists of exactly two points and $C - E$ is an open subset of Z. Suppose that $W' \cap (C - E) \neq \emptyset$ for some non-degenerate cyclic element C of $c\ell(J)$. Let $W = W' \cap (C - E)$. Recall that $A_m \cap C$ is metrizable and $W \subset U' \subset A_m$. It follows that $W \subset A_m \cap C$ is a non-empty, open and metrizable subset of X.

Finally, suppose that $W' \subset E$. Recall that E is a compact linearly ordered space. Since W' is an open subset of X and each component of X is non-degenerate, it follows that W' contains a non-degenerate subcontinuum W'' of X. Then W'' is a non-degenerate subcontinuum of E, and therefore W'' is an arc. Obviously, W'' is metrizable (because it is a subset of a single component of X). Let $W = W''$-(end-points of W''). It is easy to see that W is an open subset of J. Consequently, W is a metrizable open subset of $Z - \{z\}$ and of X. By Lemma 3, the proof of Theorem 2 is complete.

4. **0-dimensional homogeneous spaces.** Now, it is natural to investigate the homogeneous 0-dimensional spaces which are continuous images of ordered compacta. Let X be a homogeneous and 0-dimensional space which is the continuous image of an ordered compactum. By Theorem A, X is first countable. In the case when X is metrizable we have a very simple classification: either X is finite or it is a Cantor set. However, the case when X is not metrizable is still unclear. The following fact is now much appreciated:

THEOREM D, [18, THEOREM 2.1]. *A 0-dimensional space which is the continuous image of an ordered compactum is homeomorphic to a strong T-set in a dendron.*

We remark that, by [19, Lemma 6.3], each compact subset of a dendron is homeomorphic to a strong T-set in another dendron.

A family F of subsets of X is said to be *cross-free* if for all $U, V \in F$ one of the following conditions holds: $U \subset V$, $V \subset U$, $U \cap V = \emptyset$ or $U \cup V = X$.

THEOREM E ([2], SEE ALSO [13, THEOREM 6.6, P. 79]). *A Hausdorff space X can be embedded into a dendron if and only if X admits a subbasis which is a cross-free family of subsets of X.*

By Theorems D and E, we have the following immediate corollary:

COROLLARY 1. *The following conditions are equivalent for a compact 0-dimensional space X:*

(i) *X is the continuous image of an ordered compactum,*
(ii) *X admits a subbasis which is a cross-free family.*

By Theorem D, we may restrict our attention to homogeneous closed subsets of dendrons. One might be tempted to conjecture that such sets are orderable. However, the example below shows that such a conjecture fails.

Following [12], we say that a 0-dimensional space X is *strongly homogeneous* provided each non-empty closed-open subset of X is homeomorphic to X. It is easy to see that a first countable, strongly homogeneous and 0-dimensional space is homogeneous. It would be interesting to know if each homogeneous and closed subset of a dendron or even of an arc is strongly homogeneous (clearly, such a subset must be 0-dimensional).

The following fact is well-known (see *e.g.* [13]):

LEMMA 4. *If X is a dendron, then the collection*

$$\big\{P : P \text{ is a component of } X - \{x\} \text{ for some } x \in X\big\}$$

is a subbasis for the topology of X.

Let X be a non-degenerate dendron. If $x \in X$, we let $K(x)$ denote the collection of all components of $X - \{x\}$, and we let $k(x)$ denote the cardinality of the collection $K(x)$. Recall that $E(X)$ denotes the set of end-points of X, *i.e.*, $E(X) = \{x \in X : k(x) = 1\}$, and $R(X) = \{x \in X : k(x) \geq 3\}$ denotes the set of ramification points of X.

Let C be a subset of X such that $k(x)$ is finite for each $x \in C$. We are going to construct a space $s(X, C)$ which is formed from X by splitting each $x \in C$ into $k(x)$ points (we remark that our construction is a particular form of more general constructions described in [6]). Let

$$s(X, C) = (X - C) \cup \{(x, P) : x \in C \text{ and } P \in K(x)\}$$

and define $\pi\colon s(X, C) \longrightarrow X$ by

$$\pi(t) = \begin{cases} t & \text{if } t \in X - C \\ x & \text{if } t = (x, P) \text{ for some } x \in C \text{ and } P \in K(x). \end{cases}$$

Let J be the collection which consists of all sets Q such that

$$Q = \begin{cases} \pi^{-1}(P) & \text{for some } x \in X \text{ and } P \in K(x), \text{ or} \\ \pi^{-1}(P) \cup \{(x, P)\} & \text{for some } x \in C \text{ and } P \in K(x). \end{cases}$$

We take J as a subbasis for open sets in $s(X, C)$.

The following fact has a rather easy and straightforward proof:

PROPOSITION 1. *If X is a non-degenerate dendron and $C \subset X$ such that $k(x)$ is finite for each $x \in C$ then*

1. $s(X, C)$ is a compact space,

2. $\pi\colon s(X, C) \longrightarrow X$ is continuous and irreducible,
3. $s(X, C)$ can be embedded into a dendron, and so it is the continuous image of an ordered compactum,
4. $s(X, C)$ is orderable provided X is an arc, and
5. $s(X, C)$ is 0-dimensional if and only if, for each arc $I \subset X$, $C \cap I$ is a dense subset of I.

By Lemma 4, one easily gets the following:

PROPOSITION 2. *If X is a dendron, then X is first countable if and only if each arc contained in X is first countable and $k(x) \leq \aleph_0$ for each $x \in X$.*

Moreover, it is not difficult to see that

PROPOSITION 3. *If X is a dendron and $C \subset X$ such that $k(x)$ is finite for each $x \in C$, then $s(X, C)$ is first countable if and only if each arc contained in X is first countable and $k(x) \leq \aleph_0$ for each $x \in X - C$.*

Observe that $s([0,1],[0,1])$ is canonically homeomorphic to $s([0,1],]0,1[)$, and it is homeomorphic to the Alexandrov double arrow space. Let $\mathbb{Q}$ denote the set of rational numbers in $]0,1[$. Then $s([0,1],\mathbb{Q})$ is homeomorphic to the Cantor set. It is also easy to see that $s([0,1],[0,1])$, $s([0,1],\mathbb{Q})$ and $s([0,1],[0,1]-\mathbb{Q})$ are strongly homogeneous spaces. That observation can be generalized as follows:

PROPOSITION 4. *Let I be an arc with end-points a and b and let C be a dense subset of I. If, for each subarc J of I such that the end-points of J belong to C, there exists a homeomorphism $h_J\colon I \longrightarrow J$ such that*

$$h_J(C - \{a, b\}) = C \cap J - (\textit{end-points of } J),$$

then $s(I, C)$ is strongly homogeneous.

PROOF. It is easy to prove by induction that, for each finite discrete space D, the product $s(I, C) \times D$ is homeomorphic to $s(I, C)$. Now, every closed-open subset of $s(I, C)$ consists of finitely many convex closed-open subsets of $s(I, C)$ equipped with its canonical ordering induced from I.

Proposition 4 admits the following generalization to dendrons:

PROPOSITION 5. *Let X be a non-degenerate dendron and let C be a subset of X such that $C \cap I$ is dense in I for each subarc I of X and $k(x)$ is finite for each $x \in C$.*

Suppose that

(a) for each finite collection $\{P_1, \ldots, P_n\}$ such that $P_1 \cap \cdots \cap P_n \neq \emptyset$ and each P_i is a component of $X - \{x_i\}$ where $x_i \in C$, there is a homeomorphism h_Y of X onto the set $Y = c\ell(P_1 \cap \cdots \cap P_n)$ such that $h_Y\big(C - E(X)\big) = Y \cap C - E(Y)$, and

(b) $s(X, C)$ is homeomorphic to $s(X, C) \times \{0, 1\}$.

Then $s(X, C)$ is strongly homogeneous.

The authors do not know whether the condition (b) is really necessary in Proposition 5. Note that (b) holds when there is $x \in C$ such that $k(x) = 2$.

Let $\alpha \in \{3, 4, \ldots, \aleph_0\}$. It is well-known (see *e.g.* [3]) that there exists a unique up to a homeomorphism dendrite X_α with respect to the following properties (α_1) and (α_2):

(α_1) if $x \in X_\alpha$ then $k(x) \in \{1, 2, \alpha\}$,

(α_2) if $J \subset X_\alpha$ is an arc in X, then $R(X_\alpha) \cap J$ is a dense subset of J.

The dendrite X_α has also the following properties:

(α_3) $R(X_\alpha)$ is a countable dense subset of X_α,

(α_4) $E(X_\alpha)$ is a dense subset of X_α and $E(X_\alpha)$ is homeomorphic to the space of all irrational numbers,

(α_5) if Y is a dendrite such that $k(y) \leq \alpha$ for each $y \in Y$, then Y can be embedded into X_α.

In particular, $X_{\aleph_0}$ is a universal dendrite (it was constructed in 1923 by T. Ważewski, [25]).

EXAMPLE 1. Let $\alpha \in \{3, 4, \ldots\}$ and let X_α be the dendrite defined above.

(a) Observe that $s\big(X_\alpha, R(X_\alpha)\big)$ is a compact, metrizable and 0-dimensional space with no isolated point. Hence, it is homeomorphic to the Cantor set.

(b) Let $C_\alpha = X_\alpha - \big(E(X_\alpha) \cup R(X_\alpha)\big)$. Then the hypotheses of Proposition 5 are satisfied. Consequently, $s(X_\alpha, C_\alpha)$ is a strongly homogeneous, compact and 0-dimensional space. Since $\pi\colon s(X_\alpha, C_\alpha) \longrightarrow X_\alpha$ is irreducible, $s(X_\alpha, C_\alpha)$ is separable. Moreover, it is easy to see that $s(X_\alpha, C_\alpha)$ is first countable (by Proposition 3), non-metrizable and homogeneous. We are going to show that $s(X_\alpha, C_\alpha)$ is not orderable.

Let A denote the Alexandrov double arrow space and $\leq$ be the standard linear ordering on A. Let B be a dense countable subset of A which does not contain the $\leq$-first and $\leq$-last elements of A. Let P denote the collection of all ordered pairs $(t_1, t_2) \in A \times A$ such that $t_1 < t_2$ and t_2 is the immediate successor of t_1. Moreover, let $Q = \{(t_1, t_2) \in P : t_1 \in B$ or $t_2 \in B\}$, $Z = (A \times \{0\}) \cup \{(t_1, 1)$: there is $t_2 \in A$ such that $(t_1, t_2) \in Q\}$ and let $\prec$ denote the lexicographic ordering on Z, *i.e.*, $(a, i) \prec (b, j)$ if and only if $a < b$, or $a = b$ and $i \leq j$. We take Z with its order topology induced by $\prec$. Then Z is a compact ordered space. Clearly, A is homeomorphic to a subspace of Z, and Z is 0-dimensional and separable (roughly speaking, Z is formed from A by inserting an isolated point in each jump of a countable dense set of jumps in A).

Let G denote the decomposition of Z into the sets $\{(t, 0), (t', 0)\}$ such that $(t, t') \in Q$, and single points. Then G is upper semi-continuous. Let $Y = Z/_G$ denote the quotient space. In [21] it was proved that Y is not orderable.

It is not difficult to see that Y can be embedded into $s(X_\alpha, C_\alpha)$. Therefore $s(X_\alpha, C_\alpha)$ is not orderable.

(c) Clearly, $s(X_\alpha, X_\alpha)$ is homeomorphic to $s\big(X_\alpha, X_\alpha - E(X_\alpha)\big)$. Note that $s(X_\alpha, X_\alpha)$ is a compact and 0-dimensional space and the hypotheses of Proposition 5 are satisfied. Hence, it is strongly homogeneous. Moreover, $s(X_\alpha, X_\alpha)$ is separable, because $\pi\colon s(X_\alpha, X_\alpha) \longrightarrow X_\alpha$ is irreducible. It is also clear that $s(X_\alpha, X_\alpha)$ is first countable, non-metrizable and homogeneous. We remark that $s(X_\alpha, X_\alpha)$ is orderable and, in fact, one can prove that $s(X_\alpha, X_\alpha)$ is homeomorphic to $s([0, 1], C)$, where C is a dense subset of $[0, 1]$ such that C is a union of countably many Cantor sets.

EXAMPLE 2. Now, we shall construct a space X such that X is the continuous image of an ordered compactum and, moreover, it is 0-dimensional, strongly homogeneous and not first countable. By Theorem A, X is not homogeneous. We let $X = s(Z, C)$ be the space obtained by splitting certain points of a dendron Z.

First, we claim that there is a dendron Z which is unique with respect to possessing the following properties:

(1) there exist a (necessarily non-continuous) function $j: Z \longrightarrow [0,1]$ and a point $z_0 \in Z$ such that $j^{-1}(0) = \{z_0\}$ and $j|_{[z_0,z]}$ is a homeomorphism of $[z_0, z]$ onto $[0,1]$ for each $z \in E(Z) - \{z_0\}$,

(2) if $z \in j^{-1}(q)$ for some $q \in \mathbb{Q}$ then $k(z) = 2$ (recall that $\mathbb{Q}$ is the set of all rationals in $]0,1[$),

(3) if $z \in j^{-1}(t)$ for some $t \in]0,1[-\mathbb{Q}$, then $k(z) = \aleph_1$, and

(4) $k(z_0) = 1$ and if $z \in j^{-1}(1)$, then $k(z) = 1$.

In fact, Z is a pseudo-tree compactification (see [19, Theorem 4.8]) of the space

$$\left(G\big((]0,1[-\mathbb{Q}, \leq), \aleph_1\big), T'_{\leq_G}\right)$$

of [19, Theorem 7.6]. The uniqueness of Z follows from [19, Theorem 7.7 (c)]. We let $C = j^{-1}(\mathbb{Q})$. It is not difficult to see that all the assumptions of Proposition 5 are satisfied ((a) holds by the uniqueness of Z). We let $X = s(Z, C)$. Then X is a compact subset of some dendron, and so it is the continuous image of an ordered compactum. Moreover, X has all the properties listed above.

5. **Problems.** Now, we summarize several problems which arose in the present paper.

1. Let X be an arc formed by splitting a point of a homogeneous simple closed curve into two points. Is then X hereditarily equivalent? (Compare with [22, Theorem 2]).

2. Let X be a homogeneous space which is a closed subset of an arc or more generally of a dendron. Does it follow that X is strongly homogeneous?

3. Let X be an infinite, strongly homogeneous, 0-dimensional, closed subset of an arc. Does it follow that X is first countable and, hence, homogeneous? (Compare with Example 2).

4. Is the assumption (b) necessary in Proposition 5?

5. Let X be an infinite strongly homogeneous and 0-dimensional closed subset of a dendron. Can X be obtained as $s(Y, C)$ for some dendron Y and some $C \subset Y$ such that $k(x) < \aleph_0$ for each $x \in C$?

6. The following is (loosely) related to Example 1 (c). Let C and D be dense subsets of $[0,1]$ each of which is a union of countably many Cantor sets. It is easy to see that there exists a homeomorphism $h: [0,1] \longrightarrow [0,1]$ such that $h(0) = 0$ and $h(C - \{0,1\}) = D - \{0,1\}$. Therefore, C and $[0,1] - C$ are homogeneously embedded into $[0,1]$ in the sense of [14]. The inverse limit construction given in [14] provides us with two hereditarily equivalent arcs $([0,1], C)_\omega$ and $([0,1], [0,1] - C)_\omega$. It would be interesting to know if

those arcs are not homeomorphic to the ones considered in [14].

7. Let X be a compact 0-dimensional space such that $X \times \{0, 1, \ldots, n\}$ is homeomorphic to X for some positive integer n. Does it follow that $X \times \{0, 1, \ldots, n-1\}$ is homeomorphic to X? More generally, what is the set of all positive integers m such that $X \times \{0, 1, \ldots, m\}$ is homeomorphic to X?

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