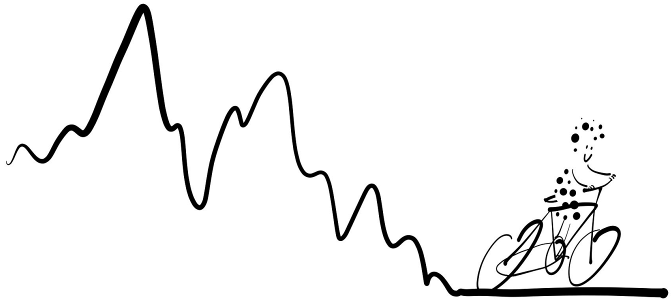
2

Martingales



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2.1 Conditional Expectation

Martingales are a central object in probability. To define them properly we need to develop the notion of *conditional expectation*.

Proposition 2.1.1 (Existence of conditional expectation) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let $X : \Omega \to \mathbb{R}$ be an integrable random variable (i.e. $\mathbb{E}|X| < \infty$). Then, there exists an a.s. unique $\mathcal{G}$-measurable and integrable random variable Y such that for all $A \in \mathcal{G}$ we have $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A]$.*

Definition 2.1.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let $X : \Omega \to \mathbb{R}$ be an integrable random variable (i.e. $\mathbb{E}|X| < \infty$). Denote $\mathbb{E}[X \mid \mathcal{G}]$ to be the (a.s. unique) random variable such that for all $A \in \mathcal{G}$, we have $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]\mathbf{1}_A]$.

For an event $A \in \mathcal{F}$, we denote $\mathbb{P}[A \mid \mathcal{G}] := \mathbb{E}[\mathbf{1}_A \mid \mathcal{G}]$.

If $\mathbb{P}[A] > 0$, we also define

$$\mathbb{E}[X \mid A, \mathcal{G}] := \frac{\mathbb{E}[X\mathbf{1}_A \mid \mathcal{G}]}{\mathbb{P}[A]} \qquad \text{and} \qquad \mathbb{P}[B \mid A, \mathcal{G}] := \frac{\mathbb{P}[B \cap A \mid \mathcal{G}]}{\mathbb{P}[A]}.$$

It is important to note that conditional expectation produces a random variable and not a number. One may think of $\mathbb{E}[X \mid \mathcal{G}]$ as the "best guess" for X given the information $\mathcal{G}$.

Uniqueness is a simple exercise:

Exercise 2.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let X be an integrable random variable.

Let $Y, Z : \Omega \to \mathbb{R}$ be $\mathcal{G}$-measurable random variables, and assume that for any $A \in \mathcal{G}$ the expectations $\mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[Z\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A]$ exist and are equal.

Show that Y, Z are integrable, and that $Y = Z$ a.s. ▷ solution ◁

We now prove the existence of conditional expectation.

Proof of Proposition 2.1.1 The existence of conditional expectation utilizes a powerful theorem from measure theory: the *Radon–Nykodim theorem*. It states that if μ, ν are σ-finite measures on a measurable space (M, Σ), and if $\nu \ll \mu$ (i.e. for any $A \in \Sigma$, if $\mu(A) = 0$ then $\nu(A) = 0$), then there exists a measurable function $\frac{d\nu}{d\mu}$ such that for any ν-integrable function f, we have that $f\frac{d\nu}{d\mu}$ is μ-integrable and $\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu$. (See Theorem A.4.6 in Durrett, 2019.)

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This is a deep theorem, but from it the existence of conditional expectation is straightforward.

We start with the case where $X \geq 0$. Let $\mu = \mathbb{P}$ on $(\Omega, \mathcal{F})$ and define $\nu(A) = \mathbb{E}[X\mathbf{1}_A]$ for all $A \in \mathcal{G}$. One may easily check that ν is a measure on $(\Omega, \mathcal{G})$ and that $\nu \ll \mathbb{P}|_{\mathcal{G}}$. Thus, there exists a $\mathcal{G}$-measurable function $\frac{d\nu}{d\mu}$ such that for any $A \in \mathcal{G}$, we have $\mathbb{E}[X\mathbf{1}_A] = \nu(A) = \int \mathbf{1}_A d\nu = \int \mathbf{1}_A \frac{d\nu}{d\mu} d\mu$. A $\mathcal{G}$-measurable function is just a random variable measurable with respect to $\mathcal{G}$. So we may take $Y = \frac{d\nu}{d\mu}$, and we have $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A]$ for all $A \in \mathcal{G}$.

For a general X (not necessarily nonnegative) we may write $X = X^+ - X^-$ for $X^{\pm}$ nonnegative. One may check that $Y := \mathbb{E}[X^+ \mid \mathcal{G}] - \mathbb{E}[X^- \mid \mathcal{G}]$ has the required properties. □

The uniqueness property described in Exercise 2.1 is a good tool for computing the conditional expectation in many cases; usually one "guesses" the correct random variable and verifies it by showing that it admits the properties guaranteeing it is equal to the conditional expectation a.s.

Let us summarize some of the most basic properties of conditional expectation with the following exercises.

Exercise 2.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X an integrable random variable, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra.

Show that if X is $\mathcal{G}$-measurable, then $\mathbb{E}[X \mid \mathcal{G}] = X$ a.s.

Show that if X is independent of $\mathcal{G}$, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$ a.s.

Show that if $\mathbb{P}[X = c] = 1$, then $\mathbb{E}[X \mid \mathcal{G}] = c$ a.s. ▷ solution ◁

Exercise 2.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X an integrable random variable, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra.

Show that $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$. ▷ solution ◁

Recall that for $A \in \mathcal{F}$, we defined $\mathbb{P}[A \mid \mathcal{G}] = \mathbb{E}[\mathbf{1}_A \mid \mathcal{G}]$.

Exercise 2.4 Prove Bayes' formula for conditional probabilities:

Show that for any $B \in \mathcal{G}$ and $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$, we have

$$\mathbb{P}[B \mid A] = \frac{\mathbb{E}[\mathbf{1}_B \, \mathbb{P}[A \mid \mathcal{G}]]}{\mathbb{P}[A]}.$$

▷ solution ◁

Exercise 2.5 Show that conditional expectation is linear; that is,

$$\mathbb{E}[aX + Y \mid \mathcal{G}] = a\,\mathbb{E}[X \mid \mathcal{G}] + \mathbb{E}[Y \mid \mathcal{G}] \qquad \text{a.s.}$$

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Show that if $X \leq Y$ a.s., then $\mathbb{E}[X \mid \mathcal{G}] \leq \mathbb{E}[Y \mid \mathcal{G}]$ a.s.

Show that if $X_n \nearrow X$ a.s., $X_n \geq 0$ for all n a.s., and X is integrable, then $\mathbb{E}[X_n \mid \mathcal{G}] \nearrow \mathbb{E}[X \mid \mathcal{G}]$. ▷ solution ◁

Exercise 2.6 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X an integrable random variable, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra.

Show that if Y is $\mathcal{G}$-measurable and $\mathbb{E}|XY| < \infty$, then $\mathbb{E}[XY \mid \mathcal{G}] = Y\,\mathbb{E}[X \mid \mathcal{G}]$ a.s. ▷ solution ◁

Exercise 2.7 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X an integrable random variable. Suppose that $(A_n)_n$ is a sequence of pairwise disjoint events such that $\sum_n \mathbb{P}[A_n] = 1$ (i.e. $(A_n)_n$ form an *almost-partition* of Ω). Let $\mathcal{G} = \sigma((A_n)_n)$. Show that for all n,

$$\mathbb{E}[X \mid \mathcal{G}]\mathbf{1}_{A_n} = \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{\mathbb{P}[A_n]}\mathbf{1}_{A_n} \qquad \text{a.s.}$$

Use this to conclude that

$$\mathbb{E}[X \mid \mathcal{G}] = \sum_n \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{\mathbb{P}[A_n]} \cdot \mathbf{1}_{A_n} \qquad \text{a.s.}$$

▷ solution ◁

Definition 2.1.3 Let X be an integrable (real-valued) random variable, and Y another random variable, not necessarily real-valued. Define $\mathbb{E}[X \mid Y] := \mathbb{E}[X \mid \sigma(Y)]$.

Exercise 2.8 Show that if X is an integrable random variable, and Y is a random variable taking on countably many values, then

$$\mathbb{E}[X \mid Y] = \sum_{y \in R_Y} \frac{\mathbb{E}\left[X\mathbf{1}_{\{Y=y\}}\right]}{\mathbb{P}[Y = y]}\mathbf{1}_{\{Y=y\}},$$

where $R_Y = \{y \colon \mathbb{P}[Y = y] > 0\}$.

Exercise 2.9 Prove Chebychev's inequality for conditional expectation: Show that if $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then a.s.

$$\mathbb{P}[|X| \geq a \mid \mathcal{G}] \leq a^{-2} \cdot \mathbb{E}\left[X^2 \mid \mathcal{G}\right].$$

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Exercise 2.10 Prove Cauchy–Schwarz for conditional expectation: Show that if $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then XY is integrable and a.s.

$$(\mathbb{E}[XY \mid \mathcal{G}])^2 \leq \mathbb{E}\left[X^2 \mid \mathcal{G}\right] \cdot \mathbb{E}\left[Y^2 \mid \mathcal{G}\right].$$

Proposition 2.1.4 (Jensen's inequality) *If φ is a convex function such that $X, \varphi(X)$ are integrable, then a.s.*

$$\mathbb{E}[\varphi(X) \mid \mathcal{G}] \geq \varphi(\mathbb{E}[X \mid \mathcal{G}]).$$

Proof As in the usual proof of Jensen's inequality, we know that $\varphi(x) = \sup_{(a,b)\in S}(ax+b)$ where $S = \left\{(a,b) \in \mathbb{Q}^2 : \forall\, y\,,\ ay + b \leq \varphi(y)\right\}$. If $(a,b) \in S$, then monotonicity of conditional expectation gives

$$\mathbb{E}[\varphi(X) \mid \mathcal{G}] \geq a\,\mathbb{E}[X \mid \mathcal{G}] + b \qquad \text{a.s.}$$

Taking the supremum over $(a,b) \in S$, since S is countable, we have that

$$\mathbb{E}[\varphi(X) \mid \mathcal{G}] \geq \varphi(\mathbb{E}[X \mid \mathcal{G}]) \qquad \text{a.s.}$$ □

Proposition 2.1.5 (Tower property) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X an integrable random variable, and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ sub-σ-algebras.*

Then, $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}]$ a.s.

Proof Note that $\mathbb{E}[X \mid \mathcal{H}]$ is $\mathcal{H}$-measurable and thus $\mathcal{G}$-measurable. So $\mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}]$ a.s.

For the other assertion, since $\mathbb{E}[X \mid \mathcal{H}]$ is $\mathcal{H}$-measurable, we only need to show the second property. That is, for any $A \in \mathcal{H}$, since $A \in \mathcal{G}$ as well,

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X \mid \mathcal{H}]\mathbf{1}_A] &= \mathbb{E}[\mathbb{E}[X\mathbf{1}_A \mid \mathcal{H}]] = \mathbb{E}[X\mathbf{1}_A] \\ &= \mathbb{E}[\mathbb{E}[X\mathbf{1}_A \mid \mathcal{G}]] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]\mathbf{1}_A].\end{aligned}$$ □

2.2 Martingales: Definition and Examples

Definition 2.2.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **filtration** is a sequence $(\mathcal{F}_t)_t$ of nested sub-σ-algebras $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$.

Example 2.2.2 The basic example of a filtration is one induced by a sequence of random variables. If $(X_t)_t$ is a sequence of random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathcal{F}_t = \sigma(X_0, \ldots, X_t)$ is easily seen to be a filtration. △▽△

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Definition 2.2.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\mathcal{F}_t)_t$ be a filtration. Let $(M_t)_t$ be a sequence of complex-valued random variables.

The sequence $(M_t)_t$ is said to be a **martingale with respect to the filtration** $\mathcal{F}_t$ if the following conditions hold: For all t,

- M_t is measurable with respect to $\mathcal{F}_t$,
- $\mathbb{E}\,|M_t| < \infty$, and
- $\mathbb{E}[M_{t+1} \mid \mathcal{F}_t] = M_t$ a.s.

Exercise 2.11 Let μ be a probability measure on $\mathbb{Z}$ such that for $(U_t)_{t\geq 1}$ i.i.d.-μ, we have $\mathbb{E}[U_t] = 0$ and $\mathbb{E}\,|U_t| < \infty$. Let $M_0 = 0$ and $M_t := \sum_{k=1}^{t} U_k$. Show that $(M_t)_t$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(U_1, \ldots, U_t)$.

What about the filtration $\mathcal{F}_t{}' = \sigma(M_0, \ldots, M_t)$? ▷ solution ◁

Exercise 2.12 Let $(M_t)_t$ be a martingale with respect to a filtration $(\mathcal{F}_t)_t$. Show that $(M_t)_t$ is also a martingale with respect to the *canonical filtration* $\mathcal{F}_t{}' = \sigma(M_0, \ldots, M_t)$. ▷ solution ◁

In light of Exercise 2.12, we do not really need to specify the filtration when speaking about a martingale $(M_t)_t$, since we can always refer to the *canonical filtration* $\sigma(M_0, \ldots, M_t)$. Thus, whenever we speak of a martingale without specifying the filtration, we are referring to the canonical filtration.

Exercise 2.13 Let $(X_t)_t$ be the simple random walk on $\mathbb{Z}^d$. That is, $X_t = \sum_{j=1}^{t} U_j$, where U_j are i.i.d. uniform on the standard basis of $\mathbb{Z}^d$ and the inverses.

Show that $M_t = \langle X_t, v \rangle$ is a martingale, where $v \in \mathbb{R}^d$.

Show that $M_t = ||X_t||^2 - t$ is a martingale. ▷ solution ◁

Definition 2.2.4 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_t$ a filtration. A **stopping time** with respect to the filtration $(\mathcal{F}_t)_t$ is a random variable T with values in $\mathbb{N} \cup \{\infty\}$ such that $\{T \leq t\} \in \mathcal{F}_t$ for every t.

A stopping time with respect to a process $(X_t)_t$ is defined as being a stopping time with respect to the canonical filtration of the process $\sigma(X_0, \ldots, X_t)$.

Usually we will not specify the filtration, since it will be obvious from the context.

Example 2.2.5 Some examples: If $(X_t)_t$ is a process and we take the canonical filtration,

- $T = \inf\{t : X_t \in A\}$ is a stopping time;

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- $E = \sup\{t\colon X_t \in A\}$ is typically *not* a stopping time.

△ ▽ △

Exercise 2.14 Show that if $(M_t)_t$ is a martingale and T is a stopping time, then $(M_{T\wedge t})_t$ is also a martingale. ▷ solution ◁

Exercise 2.15 Show that if T, T' are both stopping times with respect to a filtration $(\mathcal{F}_t)_t$, then so is $T \wedge T'$. ▷ solution ◁

The relation of probability and harmonic functions is via martingales as the following exercise shows.

Exercise 2.16 Let G be a finitely generated group. Let μ be an adapted probability measure on G. Let $(X_t)_t$ be the μ-random walk.

Show that $f\colon G \to \mathbb{C}$ is μ-harmonic if and only if $(f(X_t))_t$ is a martingale (with respect to the canonical filtration $\mathcal{F}_t = \sigma(X_0, \ldots, X_t)$). ▷ solution ◁

2.3 Optional Stopping Theorem

It follows from the definition that $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for a martingale $(M_t)_t$. We would like to conclude that this also holds for *random* times. However, this is not true in general.

Example 2.3.1 Let $M_t = \sum_{j=1}^t X_j$, where $(X_j)_j$ are all i.i.d. with distribution $\mathbb{P}[X_j = 1] = \mathbb{P}[X_j = -1] = \frac{1}{2}$ (i.e. $(M_t)_t$ is the simple random walk on $\mathbb{Z}$). Let $T = \inf\{t\colon M_t = 1\}$.

We have seen that $(M_t)_t$ is a martingale and T is a stopping time.

In Section 2.4, we will prove that $T < \infty$ a.s. (i.e. the simple random walk on $\mathbb{Z}$ is recurrent). So M_T is well defined, and actually, by definition $M_T = 1$ a.s. However, $M_0 = 0$ a.s., so $\mathbb{E}[M_T] = 1 \neq 0 = \mathbb{E}[M_0]$. △ ▽ △

In contrast to the general case, *uniform integrability* is a condition under which $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ for stopping times.

Definition 2.3.2 (Uniform integrability) Let $(X_\alpha)_{\alpha\in I}$ be a collection of random variables. We say that the collection $(X_\alpha)_\alpha$ is **uniformly integrable** if

$$\lim_{K\to\infty} \sup_\alpha \mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|>K\}}\right] = 0.$$

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Exercise 2.17 Show that if X is integrable, then the collection $X_\alpha := X$ is uniformly integrable. ▷ solution ◁

Exercise 2.18 Show that if $(X_\alpha)_\alpha$ is uniformly integrable, then $\sup_\alpha \mathbb{E}\,|X_\alpha| < \infty$.

▷ solution ◁

Exercise 2.19 Show that if for some $\varepsilon > 0$ we have $\sup_\alpha \mathbb{E}\,|X_\alpha|^{1+\varepsilon} < \infty$, then $(X_\alpha)_\alpha$ is uniformly integrable. ▷ solution ◁

Exercise 2.20 Show that if $(\mathcal{F}_\alpha)_\alpha$ is a collection of σ-algebras and X is an integrable random variable, then $(\mathbb{E}[X \mid \mathcal{F}_\alpha])_\alpha$ is uniformly integrable. ▷ solution ◁

Exercise 2.21 Show that if $(X_n)_n$ is uniformly integrable and $X_n \to X$ a.s., then X is integrable. ▷ solution ◁

The following is *not* the strongest form of optional stopping theorems that are possible to prove, but it is sufficient for our purposes.

Theorem 2.3.3 (Optional stopping theorem) *Let $(M_t)_t$ be a martingale and T a stopping time, both with respect to a filtration $(\mathcal{F}_t)_t$.*

We have that $\mathbb{E}\,|M_T| < \infty$ and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ if one of the following holds:

- *The stopping time T is a.s. bounded; that is, there exists $t \geq 0$ such that $T \leq t$ a.s.*
- *$T < \infty$ a.s. and $(M_t)_t$ is uniformly integrable and $\mathbb{E}\,|M_T| < \infty$.*

We will actually see (in Exercise 2.25) that in the last condition, the requirement $\mathbb{E}\,|M_T| < \infty$ is redundant.

Proof For the first case, if $T \leq t$ a.s., then

$$\mathbb{E}[M_T] = \mathbb{E} \sum_{j=0}^{t-1} \mathbf{1}_{\{T>j\}} \cdot (M_{j+1} - M_j) + \mathbb{E}[M_0].$$

Since $\{T > j\} = \{T \leq j\}^c \in \mathcal{F}_j$, we get that

$$\begin{aligned} \mathbb{E}\left[(M_{j+1} - M_j)\mathbf{1}_{\{T>j\}}\right] &= \mathbb{E}\,\mathbb{E}[(M_{j+1} - M_j)\mathbf{1}_{\{T>j\}} \mid \mathcal{F}_j] \\ &= \mathbb{E}\left[\mathbf{1}_{\{T>j\}}\,\mathbb{E}[M_{j+1} - M_j \mid \mathcal{F}_j]\right] = 0. \end{aligned}$$

So $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ in this case.

For the second case, note that since $\mathbb{E}\,|M_T| < \infty$, then $\mathbb{E}\left[|M_T|\mathbf{1}_{\{|M_T|>K\}}\right] \to 0$ as $K \to \infty$.

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Now, $(M_t)_t$ is uniformly integrable so $\sup_t \mathbb{E}\left[|M_t|\mathbf{1}_{\{|M_t|>K\}}\right] \to 0$ as $K \to \infty$. Thus,

$$\begin{aligned}\mathbb{E}\left[|M_{T\wedge t}|\mathbf{1}_{\{|M_{T\wedge t}|>K\}}\right] &\le \mathbb{E}\left[|M_T|\mathbf{1}_{\{|M_T|>K\}}\mathbf{1}_{\{T\le t\}}\right] + \mathbb{E}\left[|M_t|\mathbf{1}_{\{|M_t|>K\}}\mathbf{1}_{\{T>t\}}\right] \\ &\le \mathbb{E}\left[|M_T|\mathbf{1}_{\{|M_T|>K\}}\right] + \mathbb{E}\left[|M_t|\mathbf{1}_{\{|M_t|>K\}}\right],\end{aligned}$$

so $\sup_t \mathbb{E}\left[|M_{T\wedge t}|\mathbf{1}_{\{|M_{T\wedge t}|>K\}}\right] \to 0$ as $K \to \infty$.

Let

$$\varphi_K(x) = \begin{cases} K & \text{if } x > K, \\ x & \text{if } |x| \le K, \\ -K & \text{if } x < -K. \end{cases}$$

Note that $|\varphi_K(x) - x| \le |x|\mathbf{1}_{\{|x|>K\}}$.

Since $M_{T\wedge t} \to M_T$ a.s. as $t \to \infty$ (because we assumed that $T < \infty$ a.s.), we also have that $\varphi_K(M_{T\wedge t}) \to \varphi_K(M_T)$ a.s. as $t \to \infty$. Since $\varphi_K(M_{T\wedge t}), \varphi_K(M_T)$ are uniformly bounded by K, we can apply dominated convergence to obtain that

$$\lim_{t\to\infty} \mathbb{E}\,|\varphi_K(M_{T\wedge t}) - \varphi_K(M_T)| = 0.$$

Thus,

$$\begin{aligned}&\mathbb{E}\,|M_T - M_{T\wedge t}| \\ &\le \mathbb{E}\,|\varphi_K(M_T) - M_T| + \mathbb{E}\,|\varphi_K(M_{T\wedge t}) - M_{T\wedge t}| + \mathbb{E}\,|\varphi_K(M_{T\wedge t}) - \varphi_K(M_T)| \\ &\le \mathbb{E}\left[|M_T|\mathbf{1}_{\{|M_T|>K\}}\right] + \sup_t \mathbb{E}\left[|M_{T\wedge t}|\mathbf{1}_{\{|M_{T\wedge t}|>K\}}\right] + \mathbb{E}\,|\varphi_K(M_{T\wedge t}) - \varphi_K(M_T)|.\end{aligned}$$

Taking $t \to \infty$ and then $K \to \infty$, we get that $\mathbb{E}\,|M_T - M_{T\wedge t}| \to 0$.

Since $T\wedge t$ is an a.s. bounded stopping time, $\mathbb{E}[M_{T\wedge t}] = \mathbb{E}[M_0]$. In conclusion,

$$|\,\mathbb{E}[M_T - M_0]| = |\,\mathbb{E}[M_T - M_{T\wedge t}]| \to 0. \qquad \square$$

Exercise 2.22 Show that if a stopping time T is a.s. finite and a martingale $(M_t)_t$ is a.s. uniformly bounded (i.e. there exists m such that $|M_t| \le m$ a.s. for all t), then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. ▷ solution ◁

Exercise 2.23 Assume that $(X_n)_n, X$ are random variables such that $X_n \to X$ a.s. Show that if $(X_n)_n$ is uniformly integrable, then $X_n \to X$ also in L^1. ▷ solution ◁

Exercise 2.24 Show that for a martingale $(M_t)_t$ and for any a.s. finite stopping time T, we have $\mathbb{E}\,|M_{T\wedge t}| \le \mathbb{E}\,|M_t|$. ▷ solution ◁

Exercise 2.25 A specific case of the *martingale convergence theorem* states that if $(M_t)_t$ is a martingale with $\sup_t \mathbb{E}|M_t| < \infty$, then there exists a random variable M_∞ such that $M_t \to M_\infty$ a.s., and $\mathbb{E}|M_\infty| < \infty$. (We will prove the martingale convergence theorem in Theorem 2.6.3.)

Use this to show that if $(M_t)_t$ is a uniformly integrable martingale and T is an a.s. finite stopping time, then $\mathbb{E}|M_T| < \infty$ (so this last condition is redundant in the optional stopping theorem). ▷ solution ◁

2.4 Applications of Optional Stopping

Let us give some applications of the optional stopping theorem (OST) to the study of random walks on $\mathbb{Z}$.

We consider $\mathbb{Z} = \langle -1, 1 \rangle$. This is the usual Cayley graph on $\mathbb{Z}$, with neighbors given by adjacent integers. We take the measure $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$. That is, uniform on $\{-1, 1\}$.

Thus, the μ-random walk $(X_t)_t$ can be represented as $X_t = \sum_{j=1}^t U_j$ where $(U_j)_j$ are i.i.d. and $\mathbb{P}[U_j = 1] = \mathbb{P}[U_j = -1] = \frac{1}{2}$.

First, it is simple to see that $(X_t)_t$ is a martingale (we have already seen this above). Now, let $T_z := \inf\{t : X_t = z\}$. Note that T_z is a stopping time. Also, for $a < 0 < b$ we have the stopping time $T_{a,b} := T_a \wedge T_b$, which is the first exit time of (a, b).

Now, the martingale $\left(M_t = X_{T_{a,b} \wedge t}\right)_t$ is a.s. uniformly bounded (by $|a| \vee |b|$). As an exercise to the reader it is left to show that $T_{a,b} < \infty$ $\mathbb{P}_0$-a.s. Thus,

$$
\begin{aligned}
0 = \mathbb{E}[M_{T_{a,b}}] &= \mathbb{P}[T_a < T_b] \cdot a + \mathbb{P}[T_b < T_a] \cdot b \\
&= \mathbb{P}[T_a < T_b](a - b) + b.
\end{aligned}
$$

We deduce that

$$
\mathbb{P}_0[T_a < T_b] = \frac{b}{b - a}.
$$

Now, note that $(x + X_t)_t$ has the distribution of a random walk started at x. Thus, for all $n > x > 0$,

$$
\mathbb{P}_x[T_0 < T_n] = \mathbb{P}_0[T_{-x} < T_{n-x}] = \frac{n - x}{n} = 1 - \frac{x}{n}.
$$

This is the probability that a gambler starting with x dollars will go bankrupt before reaching n dollars in wealth, and is known as the *gambler's ruin estimate*.

One of the extraordinary facts (albeit classical) about the random walk on $\mathbb{Z}$ is now obtained by taking $n \to \infty$:

$$
\mathbb{P}_x[T_0 = \infty] = \lim_{n \to \infty} \mathbb{P}_x[T_0 > T_n] = 0.
$$

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That is, no matter how much money you have entering the casino, you always eventually reach 0 (and this is in the case of a fair game!)

In other words, the random walk on $\mathbb{Z}$ is *recurrent*: it reaches 0 a.s. But how long does it take to reach 0?

Note that since the random walk takes steps of size 1, we have that for $n > x > 0$, under $\mathbb{P}_x$, the event $T_0 > T_n$ implies that $T_0 \geq 2n - x$. Thus,

$$\mathbb{E}_x[T_0] = \sum_{n=0}^{\infty} \mathbb{P}_x[T_0 > n] \geq \sum_{n>x} \mathbb{P}_x[T_0 \geq 2n - x]$$
$$\geq \sum_{n>x} \mathbb{P}_x[T_0 > T_n] = \sum_{n>x} \frac{x}{n} = \infty.$$

So $\mathbb{E}_x[T_0] = \infty$. Indeed the walker reaches 0 a.s., but the time it takes is infinite in expectation. That is, the random walk on $\mathbb{Z}$ is *null-recurrent*. We will expand on the notions of recurrence and null-recurrence in Chapter 3 (and specifically in Section 3.8).

Exercise 2.26 Show that for $a < 0 < b$, we have that $\mathbb{P}_0[T_{a,b} < \infty] = 1$.

In fact, strengthen this to show that for all $a < 0 < b$ there exists a constant $c = c(a, b) > 0$ such that for all t, and any $a < x < b$,

$$\mathbb{P}_x[T_{a,b} > t] \leq e^{-ct}.$$

Conclude that $\mathbb{E}_x[T_{a,b}] < \infty$ for all $a < x < b$. ▷ solution ◁

Let us now consider a different martingale.

$$\mathbb{E}\left[X_{t+1}^2 \mid X_t\right] = \tfrac{1}{2}(X_t + 1)^2 + \tfrac{1}{2}(X_t - 1)^2 = X_t^2 + 1.$$

So $\left(M_t := X_t^2 - t\right)_t$ is a martingale.

If we apply the OST (Theorem 2.3.3) we get

$$-1 = \mathbb{E}_{-1}[M_{T_0}] = \mathbb{E}_{-1}\left[X_{T_0}^2\right] - \mathbb{E}_{-1}[T_0] = -\mathbb{E}_{-1}[T_0].$$

So $\mathbb{E}_{-1}[T_0] = 1$, a contradiction!

The reason is that we applied the OST in the case where we could not, since $(M_t)_t$ is not necessarily bounded, and this in fact shows that $(M_t)_t$ is not uniformly integrable.

One may note that for $-n < x < n$, under $\mathbb{P}_x$, the martingale $\left(M_{t\wedge T_{-n,n}}\right)_t$ admits

$$|M_{t\wedge T_{-n,n}}| \leq \left|X_{T_{-n,n}}^2 - T_{-n,n}\right| + |X_t^2 - t|\mathbf{1}_{\{T_{-n,n}>t\}} \leq 2n^2 + T_{-n,n} + t\mathbf{1}_{\{T_{-n,n}>t\}}.$$

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Thus $\left(\text{using } (a+b)^2 \le 2a^2 + 2b^2\right)$,

$$\begin{aligned}\mathbb{E}_x\left[\left|M_{t\wedge T_{-n,n}}\right|^2\right] &\le 2\,\mathbb{E}_x\left[\left|2n^2 + T_{-n,n}\right|^2\right] + 2t^2\,\mathbb{P}_x[T_{-n,n} > t] \\ &\le 2\,\mathbb{E}_x\left[\left|2n^2 + T_{-n,n}\right|^2\right] + 2t^2 \cdot e^{-ct},\end{aligned}$$

for some $c = c(n) > 0$. This implies that $\sup_t \mathbb{E}_x\left[|M_{t\wedge T_{-n,n}}|^2\right] < \infty$, so $\left(M_{t\wedge T_{-n,n}}\right)_t$ is a uniformly integrable martingale.

Given this, we may apply the OST to get that for any $-n < x < n$,

$$x^2 = \mathbb{E}_x[M_{T_{-n,n}}] = \mathbb{E}_x\left[X^2_{T_{-n,n}}\right] - \mathbb{E}_x[T_{-n,n}] = n^2 - \mathbb{E}_x[T_{-n,n}],$$

so $\mathbb{E}_x[T_{-n,n}] = n^2 - x^2$. Specifically, $\mathbb{E}[T_{-n,n}] = n^2$. This property is sometimes referred to as the random walk on $\mathbb{Z}$ being *diffusive*.

Similarly to the above, one may easily see that the martingale $(M_{t\wedge T_{0,n}})_t$ is $\mathbb{P}_x$-a.s. bounded for any $0 < x < n$. So

$$x^2 + \mathbb{E}_x[T_{0,n}] = \mathbb{E}_x\left[|X_{T_{0,n}}|^2\right] = \mathbb{P}_x[T_0 > T_n]\cdot n^2 = xn.$$

Thus, $\mathbb{E}_x[T_{0,n}] = (n-x)x$.

For general $a < x < b$, note that under $\mathbb{P}_x$, the walk $(X_t)_t$ has the same distribution as $(a + X_t)_t$ under $\mathbb{P}_{x-a}$. Thus, for $a < x < b$,

$$\mathbb{E}_x[T_{a,b}] = \mathbb{E}_{x-a}[T_{0,b-a}] = (b-x)(x-a).$$

2.5 L^p Maximal Inequality

The goal of this section is to prove the following theorem, which shows how to control the maximum of a martingale up to a certain time, using only the last value.

Theorem 2.5.1 (L^p maximal inequality) *Let $(M_t)_t$ be a martingale. Then, for any $1 < p < \infty$ and any t,*

$$\mathbb{E}[\max_{k\le t} |M_k|^p] \le (\tfrac{p}{p-1})^p \cdot \mathbb{E}\,|M_t|^p.$$

Proof Let $\mathcal{F}_t = \sigma(M_0, \dots, M_t)$ be the natural filtration. Let $N_t = \max_{k\le t} |M_k|$.

As a first step we show that for any $r > 0$,

$$r\cdot\mathbb{P}[N_t \ge r] \le \mathbb{E}\left[|M_t|\mathbf{1}_{\{N_t\ge r\}}\right].$$

(This is also known as *Doob's inequality*.) Indeed, fix $r > 0$ and $t > 0$. Let

$$T = \inf\{s \ge 0 : |M_s| \ge r\} \wedge t,$$

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which is a stopping time. Since $\{T=s\}\in\mathcal{F}_s$, we have that for all $s\le t$,

$$\mathbb{E}\left[|M_s|\mathbf{1}_{\{T=s\}}\right]=\mathbb{E}\left[|\,\mathbb{E}[M_t\mid\mathcal{F}_s]|\mathbf{1}_{\{T=s\}}\right]\le\mathbb{E}\left[|M_t|\mathbf{1}_{\{T=s\}}\right].$$

Summing over $s\le t$, using that $\mathbb{P}[T\le t]=1$, we have

$$\mathbb{E}\,|M_T|\le\mathbb{E}\,|M_t|.$$

Finally, note that $|M_T|\mathbf{1}_{\{N_t<r\}}=|M_t|\mathbf{1}_{\{N_t<r\}}$ by the definition of N_t and T, so that

$$\mathbb{E}\left[|M_t|\mathbf{1}_{\{N_t\ge r\}}\right]=\mathbb{E}\,|M_t|-\mathbb{E}\left[|M_t|\mathbf{1}_{\{N_t<r\}}\right]\ge\mathbb{E}\left[|M_T|\mathbf{1}_{\{N_t\ge r\}}\right]\ge r\cdot\mathbb{P}[N_t\ge r],$$

because $|M_T|\mathbf{1}_{\{N_t\ge r\}}\ge r\mathbf{1}_{\{N_t\ge r\}}$. This proves Doob's inequality above.

Fix some $R>0$ and denote $K_t=N_t\wedge R$. Note that $\{K_t\ge r\}=\{N_t\ge r\}$ for $r\le R$, and $\{K_t\ge r\}=\emptyset$ for $r>R$. Now, for $p>1$ we integrate Doob's inequality:

$$\begin{aligned}\mathbb{E}\,|K_t|^p&=\int_0^\infty pr^{p-1}\,\mathbb{P}[K_t\ge r]dr\le\int_0^R pr^{p-2}\,\mathbb{E}\left[|M_t|\mathbf{1}_{\{N_t\ge r\}}\right]dr\\&=\mathbb{E}\left[|M_t|\int_0^R pr^{p-2}\mathbf{1}_{\{r\le N_t\}}dr\right]=\tfrac{p}{p-1}\,\mathbb{E}\left[|M_t|\cdot|K_t|^{p-1}\right]\\&\le\tfrac{p}{p-1}\cdot\left(\mathbb{E}\,|M_t|^p\right)^{1/p}\cdot\left(\mathbb{E}\,|K_t|^p\right)^{(p-1)/p},\end{aligned}$$

where the last inequality is Hölder's inequality. Recalling that $K_t=N_t\wedge R$, taking $R\to\infty$, and using monotone convergence, we have

$$\left(\mathbb{E}\,|N_t|^p\right)^{1/p}\le\tfrac{p}{p-1}\cdot\left(\mathbb{E}\,|M_t|^p\right)^{1/p},$$

which is the required assertion. □

Exercise 2.27 Let $(X_t)_t$ be a lazy random walk on $\mathbb{Z}$; that is the μ-random walk for $\mu(1)=\mu(-1)=\frac{1}{2}(1-p)$ and $\mu(0)=p$ for some $p\in[0,1)$.

Let $M_t=\max_{k\le t}|X_k|$. Show that $\mathbb{E}\,|M_t|^2\le 4t$. ▷ solution ◁

Exercise 2.28 Let $(X_t)_t$ be a lazy random walk on $\mathbb{Z}$; that is the μ-random walk for $\mu(1)=\mu(-1)=\frac{1}{2}(1-p)$ and $\mu(0)=p$ for some $p\in[0,1)$. Let $M_t=\max_{k\le t}|X_k|$. Prove that there exists $C,c>0$ such that for all $t>0$ and all $m>0$,

$$c\exp\left(-C\tfrac{t}{m^2}\right)\le\mathbb{P}[M_t\le m]\le C\exp\left(-c(1-p)\tfrac{t}{m^2}\right).$$

▷ solution ◁

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2.6 Martingale Convergence

One amazing property of martingales is that they converge under appropriate conditions.

Definition 2.6.1 A **sub-martingale** is a process $(M_t)_t$ such that $\mathbb{E}\,|M_t| < \infty$ and $\mathbb{E}[M_{t+1} \mid M_0, \ldots, M_t] \geq M_t$ for all t.

A **super-martingale** is a process $(M_t)_t$ such that $\mathbb{E}\,|M_t| < \infty$ and $\mathbb{E}[M_{t+1} \mid M_0, \ldots, M_t] \leq M_t$ for all t.

A process $(H_t)_t$ is called **predictable** (with respect to $(M_t)_t$) if H_t is measurable with respect to $\sigma(M_0, \ldots, M_{t-1})$ for all t.

Of course any martingale is a sub-martingale and a super-martingale.

Exercise 2.29 Show that if $(M_t)_t$ is a sub-martingale then also $X_t := (M_t - a)\mathbf{1}_{\{M_t > a\}}$ is a sub-martingale. ▷ solution ◁

Exercise 2.30 Show that if $(M_t)_t$ is a sub-martingale (respectively, super-martingale) and $(H_t)_t$ is a bounded nonnegative predictable process, then the process

$$(H \cdot M)_t := \sum_{s=1}^{t} H_s (M_s - M_{s-1})$$

is a sub-martingale (respectively, super-martingale).

Show that when $(M_t)_t$ is a martingale and $(H_t)_t$ is bounded and predictable but not necessarily nonnegative, then $(H \cdot M)_t$ is a martingale. ▷ solution ◁

Exercise 2.31 Show that if $(M_t)_t$ is a sub-martingale and T is a stopping time then $(M_{T \wedge t})_t$ is a sub-martingale. ▷ solution ◁

Lemma 2.6.2 (Upcrossing Lemma) *Let $(M_t)_t$ be a sub-martingale. Fix $a < b \in \mathbb{R}$ and let U_t be the number of upcrossings of the interval (a, b) up to time t; more precisely, define: $N_0 = -1$ and inductively*

$$N_{2k-1} = \inf\{t > N_{2k-2} : M_t \leq a\} \quad \textit{and} \quad N_{2k} = \inf\{t > N_{2k-1} : M_t \geq b\}.$$

Set $U_t = \sup\{k : N_{2k} \leq t\}$.

Then

$$(b - a) \cdot \mathbb{E}[U_t] \leq \mathbb{E}\left[(M_t - a)\mathbf{1}_{\{M_t > a\}}\right] - \mathbb{E}\left[(M_0 - a)\mathbf{1}_{\{M_0 > a\}}\right].$$

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Proof Define $X_t = a + (M_t - a)\mathbf{1}_{\{M_t > a\}}$. Set $H_t = \mathbf{1}_{\{\exists\, k:\, N_{2k-1} < t \le N_{2k}\}}$. Note that H_t is $\sigma(X_0, \ldots, X_{t-1})$-measurable, since $H_t = 1$ if and only if $N_{2k-1} \le t-1$ and $N_{2k} > t-1$.

Now, one verifies that

$$\sum_{s=1}^{t} H_s \cdot (X_s - X_{s-1}) \ge \sum_{k=1}^{U_t} \sum_{s=N_{2k-1}+1}^{N_{2k}} (X_s - X_{s-1}) = \sum_{k=1}^{U_t} (M_{N_{2k}} - a) \ge (b-a)U_t.$$

Note that $(X_t)_t$ is a sub-martingale by Exercise 2.29. By Exercise 2.30, since $H_s \in [0,1]$, $A_t := \sum_{s=1}^t H_s \cdot (X_s - X_{s-1})$ and $B_t := \sum_{s=1}^t (1 - H_s) \cdot (X_s - X_{s-1})$ are also sub-martingales. Specifically, $\mathbb{E}[B_t] \ge \mathbb{E}[B_0] = 0$. We have that

$$(b-a)\,\mathbb{E}[U_t] \le \mathbb{E}[A_t] \le \mathbb{E}[A_t + B_t] = \mathbb{E}[X_t - X_0].$$

This is the required form. □

Theorem 2.6.3 (Martingale convergence theorem) *Let $(M_t)_t$ be a sub-martingale such that $\sup_t \mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] < \infty$. Then there exists a random variable M_∞ such that $M_t \to M_\infty$ a.s. and $\mathbb{E}\,|M_\infty| < \infty$.*

Proof Since $(M_t - a)\mathbf{1}_{\{M_t > a\}} \le M_t \mathbf{1}_{\{M_t > 0\}} + |a|$, we have by the upcrossing lemma that $(b-a)\,\mathbb{E}[U_t] \le \mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] + |a|$, where U_t is the number of upcrossings of the interval (a,b). Let $U = U_{(a,b)} = \lim_{t\to\infty} U_t$ be the total number of upcrossings of (a,b). By Fatou's lemma,

$$\mathbb{E}[U] \le \liminf_{t\to\infty} \mathbb{E}[U_t] \le \frac{|a| + \sup_t \mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right]}{b-a} < \infty.$$

Specifically, $U < \infty$ a.s. Since this holds for all $a < b \in \mathbb{R}$, taking a union bound over all $a < b \in \mathbb{Q}$, we have that

$$\mathbb{P}[\exists\, a < b \in \mathbb{Q} : \liminf_{t\to\infty} M_t \le a < b \le \limsup_{t\to\infty} M_t] \le \sum_{a<b\in\mathbb{Q}} \mathbb{P}\left[U_{(a,b)} = \infty\right] = 0.$$

But then, a.s. we have that $\limsup M_t \le \liminf M_t$, which implies that an a.s. limit $M_t \to M_\infty$ exists.

By Fatou's lemma again,

$$\mathbb{E}\left[M_\infty \mathbf{1}_{\{M_\infty > 0\}}\right] \le \liminf_{t\to\infty} \mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] \le \sup_t \mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] < \infty.$$

Another application of Fatou's lemma gives

$$\begin{aligned}
\mathbb{E}\left[-M_\infty \mathbf{1}_{\{M_\infty < 0\}}\right] &\le \liminf_{t\to\infty} \left(\mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] - \mathbb{E}[M_t]\right) \\
&\le \liminf_{t\to\infty} \left(\mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] - \mathbb{E}[M_0]\right) \\
&\le \sup_t \mathbb{E}\left[M_t \mathbf{1}_{\{M_t > 0\}}\right] - \mathbb{E}[M_0] < \infty.
\end{aligned}$$

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Thus

$$\mathbb{E}|M_\infty| = \mathbb{E}\left[M_\infty \mathbf{1}_{\{M_\infty > 0\}}\right] - \mathbb{E}\left[M_\infty \mathbf{1}_{\{M_\infty < 0\}}\right] < \infty. \qquad \square$$

Exercise 2.32 Show that if $(M_t)_t$ is a super-martingale and $M_t \geq 0$ for all t a.s., then $M_t \to M_\infty$ a.s., for some integrable random variable M_∞. ▷ solution ◁

Exercise 2.33 Show that if $(M_t)_t$ is a uniformly integrable martingale then $M_t \to M_\infty$ a.s. and in L^1 for some integrable M_∞. ▷ solution ◁

Exercise 2.34 Let $(\mathcal{F}_t)_t$ be a filtration. Let X be an integrable random variable.

Show that $\mathbb{E}[X \mid \mathcal{F}_t] \to \mathbb{E}[X \mid \mathcal{F}_\infty]$ a.s. and in L^1, where $\mathcal{F}_\infty = \sigma\left(\bigcup_t \mathcal{F}_t\right)$.

▷ solution ◁

Exercise 2.35 (Backward martingale convergence theorem) Let $(\sigma_n)_n$ be a non-increasing sequence of σ-algebras. Let X be an integrable random variable.

Show that $\mathbb{E}[X \mid \sigma_n] \to \mathbb{E}[X \mid \sigma_\infty]$ a.s. and in L^1, where $\sigma_\infty = \bigcap_n \sigma_n$.

(Hint: use the upcrossing lemma.) ▷ solution ◁

2.7 Bounded Harmonic Functions

We will now use the martingale convergence theorem to study the space of the bounded harmonic functions.

Theorem 2.7.1 *Let G be a finitely generated group, and let μ be an adapted probability measure on G. Then,* $\dim \mathsf{BHF}(G,\mu) \in \{1, \infty\}$. *That is, there are either infinitely many linearly independent bounded harmonic functions or the only bounded harmonic functions are the constants.*

Proof Let h be a bounded harmonic function. Let $(X_t)_t$ be the μ-random walk on G. Then, $(h(X_t))_t$ is a bounded martingale. Thus, $h(X_t) \to L$ a.s. for some integrable random variable L. Hence, $h(X_{t+k}) - h(X_t) \to 0$ a.s. for any k.

Fix $x \in G$. Let $k > 0$ be such that $\mathbb{P}[X_k = x] = \alpha > 0$. Exercise 2.36 proves that $\mathbb{P}[X_{t+k} = X_t x \mid X_t] = \alpha$ a.s. for all t (this is known as the *Markov property*, which will be discussed in Section 3.1, Exercise 3.1). Thus, for any $\varepsilon > 0$,

$$\mathbb{P}[|h(X_t x) - h(X_t)| > \varepsilon] \leq \alpha^{-1}\,\mathbb{P}[|h(X_{t+k}) - h(X_t)| > \varepsilon] \to 0.$$

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Now assume that $\dim \mathsf{BHF}(G,\mu) < \infty$. Then there exists a ball $B = B(1,r)$ such that for all $f, f' \in \mathsf{BHF}(G,\mu)$, if $f|_B = f'|_B$ then $f = f'$. Define a norm on $\mathsf{BHF}(G,\mu)$ by $||f||_B = \max_{x\in B} |f(x)|$. Since all norms on finite-dimensional spaces are equivalent, there exists a constant $K > 0$ such that $||f||_B \le ||f||_\infty \le K \cdot ||f||_B$ for all $f \in \mathsf{BHF}(G,\mu)$.

Now, since $||y.h||_\infty = ||h||_\infty$, for any t we have a.s.

$$\inf_{c\in\mathbb{C}} ||h-c||_\infty = \inf_{c\in\mathbb{C}} \left\|X_t^{-1}.h - c\right\|_\infty \le K \cdot \inf_{c\in\mathbb{C}} \left\|X_t^{-1}.h - c\right\|_B$$
$$\le K \cdot \inf_{c\in\mathbb{C}} \max_{x\in B} |h(X_t x) - c| \le K \cdot \max_{x\in B} |h(X_t x) - h(X_t)|.$$

Since this last term converges to 0 in probability, $\inf_{c\in\mathbb{C}} ||h-c||_\infty = 0$ must hold. Thus, h is constant. □

Exercise 2.36 Let $(X_t)_t$ be the μ-random walk for an adapted probability measure μ on a group G. Show that

$$\mathbb{P}[X_{t+k} = X_t x \mid X_t] = \mathbb{P}[X_k = x] \text{ a.s.}$$

for all t, k. ▷ solution ◁

Exercise 2.37 Check that $||f||_B$ in the proof of Theorem 2.7.1 is indeed a norm, for the specific B chosen. (In general, it is only a semi-norm.)

The following is a major open problem in the theory of bounded harmonic functions. It basically states that the property of having only constant bounded harmonic functions should not change if we restrict to "nice" random walk measures. We will return to this conjecture in Chapter 6.

Conjecture 2.7.2 Let G be a finitely generated group. Then for any two $\mu, \nu \in \mathsf{SA}(G,2)$, we have $\dim \mathsf{BHF}(G,\mu) = \dim \mathsf{BHF}(G,\nu)$.

Exercise 2.38 Let G be a finitely generated group and $\mu \in \mathsf{SA}(G,1)$. Recall the space of Lipschitz harmonic functions $\mathsf{LHF}(G,\mu)$.

Show that, if there exists a nonconstant positive $h \in \mathsf{LHF}(G,\mu)$, then $\dim \mathsf{LHF}(G,\mu) = \infty$. (Hint: consider LHF modulo the constant functions.)

▷ solution ◁

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2.8 Solutions to Exercises

Solution to Exercise 2.1 :(
Let $A = \{Y > 0\}$ and $B = \{Y \le 0\}$. Note that $A, B \in \mathcal{G}$. Thus, $\mathbb{E}[|Y|\mathbf{1}_A] = \mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A] \le \mathbb{E}[|X|\mathbf{1}_A]$, and similarly $\mathbb{E}[|Y|\mathbf{1}_B] = -\mathbb{E}[Y\mathbf{1}_B] = -\mathbb{E}[X\mathbf{1}_B] \le \mathbb{E}[|X|\mathbf{1}_B]$. Thus, $\mathbb{E}|Y| \le \mathbb{E}|X| < \infty$. So Y is integrable. Similarly for Z.

Now, let $A = \left\{Y - Z > \frac{1}{n}\right\}$. Then since $A \in \mathcal{G}$, we have

$$0 = \mathbb{E}[(Y - Z)\mathbf{1}_A] \ge \mathbb{P}[A] \cdot \tfrac{1}{n},$$

implying that $\mathbb{P}\left[Y > Z + \frac{1}{n}\right] = 0$. A union bound over n implies that $\mathbb{P}[Y > Z] = 0$. Reversing the roles of Y, Z, we have that $\mathbb{P}[Z > Y] = 0$ as well, culminating in $\mathbb{P}[Y \ne Z] = 0$. :) ✓

Solution to Exercise 2.2 :(
When X is $\mathcal{G}$-measurable, since for any $A \in \mathcal{G}$ we have $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A]$ trivially, we have that $\mathbb{E}[X \mid \mathcal{G}] = X$ a.s.

If X is independent of $\mathcal{G}$ then for any $A \in \mathcal{G}$ we have $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[X] \cdot \mathbb{E}[\mathbf{1}_A] = \mathbb{E}[\mathbb{E}[X] \cdot \mathbf{1}_A]$. Since a constant random variable is measurable with respect to any σ-algebra (and specifically $\mathbb{E}[X]$ is $\mathcal{G}$-measurable), we have the second assertion.

The third assertion is a direct consequence, since a constant is always independent of $\mathcal{G}$. :) ✓

Solution to Exercise 2.3 :(
Since $\Omega \in \mathcal{G}$ we have

$$\mathbb{E}[X] = \mathbb{E}[X\mathbf{1}_\Omega] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]\mathbf{1}_\Omega] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]].$$:) ✓

Solution to Exercise 2.4 :(
Since $B \in \mathcal{G}$,

$$\mathbb{E}[\mathbf{1}_B\, \mathbb{P}[A \mid \mathcal{G}]] = \mathbb{E}[\mathbf{1}_A\mathbf{1}_B] = \mathbb{P}[B \mid A] \cdot \mathbb{P}[A].$$:) ✓

Solution to Exercise 2.5 :(
Linearity: let $Z = a\,\mathbb{E}[X \mid \mathcal{G}] + \mathbb{E}[Y \mid \mathcal{G}]$. So Z is $\mathcal{G}$-measurable. Also, for any $A \in \mathcal{G}$,

$$\mathbb{E}[(aX + Y)\mathbf{1}_A] = a\,\mathbb{E}[X\mathbf{1}_A] + \mathbb{E}[Y\mathbf{1}_A] = a\,\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]\mathbf{1}_A] + \mathbb{E}[\mathbb{E}[Y \mid \mathcal{G}]\mathbf{1}_A] = \mathbb{E}[Z\mathbf{1}_A].$$

Monotonicity: By linearity it suffices to show that if $X \ge 0$ a.s. then $\mathbb{E}[X \mid \mathcal{G}] \ge 0$ a.s. Indeed, for $X \ge 0$ a.s., let $Y = \mathbb{E}[X \mid \mathcal{G}]$. Then we may consider $\{Y < 0\} \in \mathcal{G}$, and we have that $0 \le \mathbb{E}\left[X\mathbf{1}_{\{Y<0\}}\right] = \mathbb{E}\left[Y\mathbf{1}_{\{Y<0\}}\right] \le 0$, so $Y\mathbf{1}_{\{Y<0\}} = 0$ a.s. So if we set $Z = Y\mathbf{1}_{\{Y\ge 0\}}$ we have that $Y = Z \ge 0$ a.s.

Monotone convergence: Write $Y_n = \mathbb{E}[X_n \mid \mathcal{G}]$, $Y = \mathbb{E}[X \mid \mathcal{G}]$. By monotonicity above, $Y_n \le Y_{n+1} \le Y$ a.s. Let $Z = \lim_n Y_n$, which exists a.s. because the sequence is monotone. Z is $\mathcal{G}$-measurable as a limit of $\mathcal{G}$-measurable random variables. Also, for any $A \in \mathcal{G}$ we have $X_n\mathbf{1}_A \nearrow X\mathbf{1}_A$ and $Y_n\mathbf{1}_A \nearrow Z\mathbf{1}_A$ a.s. So by monotone convergence, $\mathbb{E}[Z\mathbf{1}_A] \nwarrow \mathbb{E}[Y_n\mathbf{1}_A] = \mathbb{E}[X_n\mathbf{1}_A] \nearrow \mathbb{E}[X\mathbf{1}_A]$. Hence $Z = \mathbb{E}[X \mid \mathcal{G}]$ a.s. :) ✓

Solution to Exercise 2.6 :(
Note that $Y\,\mathbb{E}[X \mid \mathcal{G}]$ is a $\mathcal{G}$-measurable random variable, as a product of two such random variables. So we need to show that for any $A \in \mathcal{G}$ we have

$$\mathbb{E}[XY\mathbf{1}_A \mid \mathcal{G}] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]Y\mathbf{1}_A] \qquad \text{a.s.}$$

If $Y = \mathbf{1}_B$ then this is immediate. If Y is a simple random variable this follows by linearity. For a nonnegative Y we may approximate by simple random variables and use the monotone convergence theorem. For general Y, we may write $Y = Y^+ - Y^-$ where $Y^\pm$ are nonnegative, and use linearity. :) ✓

Solution to Exercise 2.7 :(
Start with the assumption that $X \ge 0$.

Let $Y = \sum_n \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{\mathbb{P}[A_n]} \cdot \mathbf{1}_{A_n}$. It is immediate to verify that Y is $\mathcal{G}$-measurable.

Also, $\mathbb{E}[Y] = \mathbb{E}[X]$, so $Q(B) := \mathbb{E}[X]^{-1} \cdot \mathbb{E}[Y\mathbf{1}_B]$ defines a probability measure on $(\Omega, \mathcal{F})$. Similarly, $P(B) := \mathbb{E}[X]^{-1} \cdot \mathbb{E}[X\mathbf{1}_B]$ defines a probability measure on $(\Omega, \mathcal{F})$. The system $\{\emptyset, A_n : n \in \mathbb{N}\}$ is a π-system. For any n we have

$$\mathbb{E}[X] \cdot P(A_n) = \mathbb{E}[X\mathbf{1}_{A_n}] = \mathbb{E}[Y\mathbf{1}_{A_n}] = \mathbb{E}[X] \cdot Q(A_n).$$

Since P, Q are equal on a π-system generating $\mathcal{G}$, they must be equal on all of $\mathcal{G}$ by Dynkin's lemma. That is, for any $B \in \mathcal{G}$ we have

$$\mathbb{E}[X\mathbf{1}_{A_n}\mathbf{1}_B] = \mathbb{E}[X]P(A_n \cap B) = \mathbb{E}[X]Q(A_n \cap B) = \mathbb{E}[Y\mathbf{1}_{A_n}\mathbf{1}_B],$$

which implies, since $A_n \in \mathcal{G}$, that a.s.

$$\mathbb{E}[X \mid \mathcal{G}] \cdot \mathbf{1}_{A_n} = \mathbb{E}[X\mathbf{1}_{A_n} \mid \mathcal{G}] = Y\mathbf{1}_{A_n} = \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{\mathbb{P}[A_n]} \cdot \mathbf{1}_{A_n}.$$

For general integrable X, decompose $X = X^+ - X^-$. :) ✓

Solution to Exercise 2.11 :(
Since $|M_t| \leq \sum_{k=1}^t |U_k|$, we have that M_t is integrable and $\mathcal{F}_t$-measurable.

Note that $M_{t+1} = M_t + U_{t+1}$ and note that U_{t+1} is independent of $\mathcal{F}_t$ and of $\mathcal{F}_t'$. Thus,

$$\mathbb{E}[M_{t+1} \mid \mathcal{F}_t] = M_t + \mathbb{E}[U_{t+1}] = M_t,$$
$$\mathbb{E}[M_{t+1} \mid \mathcal{F}_t'] = M_t + \mathbb{E}[U_{t+1}] = M_t.$$

:) ✓

Solution to Exercise 2.12 :(
Since $M_0, \ldots, M_t$ are all measurable with respect to $\mathcal{F}_t$, we have that $\mathcal{F}_t' \subset \mathcal{F}_t$ for all t. Thus, by the tower property,

$$\mathbb{E}[M_{t+1} \mid \mathcal{F}_t'] = \mathbb{E}[\mathbb{E}[M_{t+1} \mid \mathcal{F}_t] \mid \mathcal{F}_t'] = \mathbb{E}[M_t \mid \mathcal{F}_t'] = M_t$$

a.s. :) ✓

Solution to Exercise 2.13 :(
Note that in both cases M_t is measurable with respect to $\sigma(X_t) \subset \sigma(X_0, \ldots, X_t)$.

In the first case, denoting by $e_1, \ldots, e_d$ the standard basis of $\mathbb{Z}^d$, then

$$\mathbb{E}[M_{t+1} \mid X_0, \ldots, X_t] = \frac{1}{2d}\sum_{j=1}^d \left\langle X_t + e_j, v\right\rangle + \left\langle X_t - e_j, v\right\rangle = \frac{1}{2d}\sum_{j=1}^d 2\langle X_t, v\rangle = M_t.$$

In the second case,

$$\mathbb{E}\left[||X_{t+1}||^2 \mid X_0, \ldots, X_t\right] = \frac{1}{2d}\sum_{j=1}^d ||X_t + e_j||^2 + ||X_t - e_j||^2 = \frac{1}{2d}\sum_{j=1}^d 2\left(||X_t||^2 + 1\right) = ||X_t||^2 + 1.$$

Thus,

$$\mathbb{E}[M_{t+1} \mid X_0, \ldots, X_t] = ||X_t||^2 + 1 - (t+1) = M_t.$$

:) ✓

Solution to Exercise 2.14 :(
Because $\{T \leq t\} \in \sigma(M_0, \ldots, M_t)$ and $\{T > t\} = \{T \leq t\}^c \in \sigma(M_0, \ldots, M_t)$, we get that $M_{T\wedge t} = M_t\mathbf{1}_{\{T>t\}} + \sum_{j=0}^t M_j\mathbf{1}_{\{T=j\}}$ is measurable with respect to $\sigma(M_0, \ldots, M_t)$. Also,

$$\mathbb{E}\,|M_{T\wedge t}| = \mathbb{E}\sum_{j=0}^{t-1}\mathbf{1}_{\{T>j\}} \cdot (|M_{j+1}| - |M_j|) + \mathbb{E}\,|M_0| < \infty.$$

It now suffices to show that

$$\mathbb{E}\left[M_{T\wedge(t+1)} \mid M_0, \ldots, M_t\right] = M_{T\wedge t}.$$

Indeed, since $\{T = t\} = \{T \leq t\} \setminus \{T \leq t-1\} \in \sigma(M_0, \ldots, M_t)$, we have that

$$\mathbb{E}[M_{T\wedge(t+1)} \mid M_0, \ldots, M_t] = \mathbb{E}\left[M_{t+1}\mathbf{1}_{\{T>t\}} \mid M_0, \ldots, M_t\right] + \sum_{j=0}^t \mathbb{E}\left[M_j\mathbf{1}_{\{T=j\}} \mid M_0, \ldots, M_t\right]$$
$$= \mathbb{E}[M_{t+1} \mid M_0, \ldots, M_t] \cdot \mathbf{1}_{\{T>t\}} + \sum_{j=0}^t M_j\mathbf{1}_{\{T=j\}} = M_{T\wedge t}.$$

:) ✓

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Solution to Exercise 2.15 :(
For any t we have that $\{T \wedge T' \leq t\} = \{T \leq t\} \cup \{T' \leq t\} \in \mathcal{F}_t$. :) ✓

Solution to Exercise 2.16 :(
If f is μ-harmonic then for any $x \in G$,

$$\mathbb{E}\left[|f(X_t)|\mathbf{1}_{\{X_{t-1}=x\}}\right] = \sum_y \mu(y)|f(xy)| \cdot \mathbb{P}[X_{t-1} = x] < \infty,$$

so that $f(X_t)$ is integrable for every t. Also,

$$\mathbb{E}[f(X_{t+1}) \mid \mathcal{F}_t] = \mathbb{E}\left[X_t^{-1}.f(U_{t+1}) \mid \mathcal{F}_t\right] = \sum_y \mu(y)X_t^{-1}.f(y) = \sum_y \mu(y)f(X_t y) = f(X_t),$$

which shows that $(f(X_t))_t$ is a martingale.

Now assume that f is such that $(f(X_t))_t$ is a martingale. Since μ is adapted, for any $x \in G$, there exists $t > 0$ such that $\mathbb{P}[X_t = x] > 0$. So,

$$f(x) = \mathbb{E}[f(X_{t+1}) \mid X_t = x] = \sum_y \mu(y)f(xy),$$

which implies that f is harmonic at x. As before, the above sum converges absolutely because $f(X_{t+1})$ is integrable. :) ✓

Solution to Exercise 2.17 :(
Set $Y_K = |X|\mathbf{1}_{\{|X|>K\}}$. So $Y_K \to 0$ a.s. Since $0 \leq Y_K \leq |X|$ for all K, by dominated convergence we have that

$$\lim_{K\to\infty} \mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|>K\}}\right] = \lim_{K\to\infty} \mathbb{E}[Y_K] = 0. \qquad :)\ ✓$$

Solution to Exercise 2.18 :(
Uniform integrability implies that there exists K such that for all α we have $\mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|>K\}}\right] < 1$. Since $\mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|\leq K\}}\right] \leq K$, we arrive at

$$\sup_\alpha \mathbb{E}\,|X_\alpha| \leq \sup_\alpha \left(\mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|>K\}}\right] + \mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|\leq K\}}\right]\right) \leq 1 + K. \qquad :)\ ✓$$

Solution to Exercise 2.19 :(
Choose $p = 1 + \varepsilon$ and $q = \frac{1+\varepsilon}{\varepsilon}$. Hölder's inequality gives for any α,

$$\begin{aligned}\mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|>K\}}\right] &\leq (\mathbb{E}[|X_\alpha|^p])^{1/p} \cdot (\mathbb{P}[|X_\alpha| > K])^{1/q} \\ &\leq (\mathbb{E}[|X_\alpha|^p])^{1/p} \cdot (\mathbb{E}[|X_\alpha|^{1+\varepsilon}])^{1/q} \cdot K^{-(1+\varepsilon)/q} \\ &= \mathbb{E}[|X_\alpha|^{1+\varepsilon}] \cdot K^{-\varepsilon}.\end{aligned}$$

Thus,

$$\lim_{K\to\infty} \sup_\alpha \mathbb{E}\left[|X_\alpha|\mathbf{1}_{\{|X_\alpha|>K\}}\right] \leq \sup_\alpha \mathbb{E}\left[|X_\alpha|^{1+\varepsilon}\right] \cdot \lim_{K\to\infty} K^{-\varepsilon} = 0. \qquad :)\ ✓$$

Solution to Exercise 2.20 :(
Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $A \in \mathcal{F}$ and $\mathbb{P}[A] < \delta$ then $\mathbb{E}[|X|\mathbf{1}_A] < \varepsilon$. (Otherwise we could find $(A_n)_n \subset \mathcal{F}$ such that $\mathbb{P}[A_n] < n^{-1}$ and $\mathbb{E}[|X|\mathbf{1}_{A_n}] \geq \varepsilon$. But since X is integrable, $\mathbb{E}[|X|\mathbf{1}_{A_n}] \to 0$ by the dominated convergence theorem, a contradiction.)

Take $K > \delta^{-1}\,\mathbb{E}\,|X|$. Then, since $\{\mathbb{E}[|X| \mid \mathcal{F}_\alpha] > K\} \in \mathcal{F}_\alpha$,

$$\begin{aligned}\mathbb{E}\left[|\mathbb{E}[X \mid \mathcal{F}_\alpha]|\mathbf{1}_{\{|\mathbb{E}[X|\mathcal{F}_\alpha]|>K\}}\right] &\leq \mathbb{E}\left[\mathbb{E}[|X| \mid \mathcal{F}_\alpha]\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{F}_\alpha]>K\}}\right] \\ &= \mathbb{E}\left[\mathbb{E}[|X|\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{F}_\alpha]>K\}} \mid \mathcal{F}_\alpha]\right] \\ &= \mathbb{E}\left[|X|\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{F}_\alpha]>K\}}\right].\end{aligned}$$

Using $A = \{\mathbb{E}[|X| \mid \mathcal{F}_\alpha] > K\}$, we have that

$$\mathbb{P}[A] \leq \mathbb{E}[\mathbb{E}[|X| \mid \mathcal{F}_\alpha]] \cdot K^{-1} = \mathbb{E}\,|X| \cdot K^{-1} < \delta,$$

so $\mathbb{E}[|X|\mathbf{1}_A] < \varepsilon$. This was uniform over α, so we conclude that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $K > \delta^{-1}\,\mathbb{E}\,|X|$ then

$$\sup_\alpha \mathbb{E}\left[|\,\mathbb{E}[X \mid \mathcal{F}_\alpha]|\mathbf{1}_{\{|\mathbb{E}[X|\mathcal{F}_\alpha]|>K\}}\right] < \varepsilon.$$

This is exactly uniform integrability. :) ✓

Solution to Exercise 2.21 :(
By Fatou's lemma,

$$\mathbb{E}\,|X| = \mathbb{E}[\lim_n |X_n|] \leq \liminf_{n\to\infty} \mathbb{E}\,|X_n| \leq \sup_n \mathbb{E}\,|X_n| < \infty.$$:) ✓

Solution to Exercise 2.22 :(
Note that if $|M_t| \leq m$ a.s., then obviously $(M_t)_t$ is uniformly integrable.

Also, since $T < \infty$ a.s., we have that $|M_{T\wedge t}| \to |M_T|$ a.s. as $t \to \infty$. Thus, $|M_T| \leq m$ a.s., implying that $\mathbb{E}[|M_T|] < \infty$.

By the optional stopping theorem we have that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. :) ✓

Solution to Exercise 2.23 :(
This is similar to the proof of Theorem 2.3.3.

Define

$$\varphi_K(x) = \begin{cases} K & \text{if } x > K, \\ x & \text{if } |x| \leq K, \\ -K & \text{if } x < -K. \end{cases}$$

Note that $|\varphi_K(x) - x| \leq |x|\mathbf{1}_{\{|x|>K\}}$. Since $\varphi_K(X_n) \to \varphi_K(X)$ a.s., and $|\varphi_K(X_n)| \leq K$ for all n, by dominated convergence we have that $\varphi_K(X_n) \to \varphi_K(X)$ in L^1. Thus,

$$\mathbb{E}\,|X_n - X| \leq \mathbb{E}\,|\varphi_K(X_n) - \varphi_K(X)| + \mathbb{E}[|X_n|\mathbf{1}_{\{|X_n|>K\}}] + \mathbb{E}[|X|\mathbf{1}_{\{|X|>K\}}],$$

implying that

$$\limsup_{n\to\infty} \mathbb{E}\,|X_n - X| \leq \sup_t \mathbb{E}\left[|X_t|\mathbf{1}_{\{|X_t|>K\}}\right] + \mathbb{E}\left[|X|\mathbf{1}_{\{|X|>K\}}\right].$$

Since $\mathbb{E}\,|X| < \infty$ (as the a.s. limit of a uniformly integrable sequence), this goes to 0 as $K \to \infty$. :) ✓

Solution to Exercise 2.24 :(
Set $X_t := |M_t| - |M_{T\wedge t}|$. Using Jensen's inequality we have that a.s.

$$\mathbb{E}[|M_{t+1}| \mid M_t, \ldots, M_0] \geq |\mathbb{E}[M_{t+1} \mid M_t, \ldots, M_0]| = |M_t|.$$

(That is, $(|M_t|)_t$ is a *sub-martingale*.) Note that

$$|M_{T\wedge(t+1)}| - |M_{T\wedge t}| = (|M_{t+1}| - |M_t|)\mathbf{1}_{\{T>t\}},$$

so

$$X_{t+1} - X_t = (|M_{t+1}| - |M_t|)\mathbf{1}_{\{T\leq t\}}.$$

Thus,

$$\mathbb{E}[X_{t+1} - X_t \mid M_0, \ldots, M_t] = \mathbf{1}_{\{T\leq t\}} \cdot \mathbb{E}[|M_{t+1}| - |M_t| \mid M_0, \ldots, M_t] \geq 0.$$

(That is, $(X_t)_t$ is also a *sub-martingale*.) Taking expectations we get that $\mathbb{E}[X_t] \geq \mathbb{E}[X_{t-1}] \geq \cdots \geq \mathbb{E}[X_0] = \mathbb{E}[M_0 - M_{T\wedge 0}] = 0$. Hence, $\mathbb{E}\,|M_t| \geq \mathbb{E}\,|M_{T\wedge t}|$, as required. :) ✓

Solution to Exercise 2.25 :(
Since $\sup_t \mathbb{E}\,|M_{T\wedge t}| \leq \sup_t \mathbb{E}\,|M_t| < \infty$, we have that $M_{T\wedge t} \to M_\infty$ a.s., for some integrable M_∞. But $M_{T\wedge t} \to M_T$ a.s. as well, which implies that $M_T = M_\infty$ a.s., so M_T is integrable. :) ✓

Solution to Exercise 2.26 :(
Let $K = b - a$. Compute, for any $a < x < b$,

$$\mathbb{P}[X_{t+K} \notin (a,b) \mid X_t = x,\ T_{a,b} > t] \geq \mathbb{P}[\forall\, 0 \leq j < K,\ U_{t+j+1} = 1 \mid X_t = x,\ T_{a,b} > t] = 2^{-K},$$

using the fact that $(U_{t+j+1})_{j=0}^{\infty}$ are all independent of $\mathcal{F}_t$, and that $\{T_{a,b} > t\} = \{T_{a,b} \leq t\}^c \in \mathcal{F}_t$.

Since $T_{a,b} > t + K$ implies that $T_{a,b} > t$ and that $X_{t+K} \in (a,b)$, we may bound

$$\begin{aligned}\mathbb{P}[T_{a,b} > t+K] &= \sum_{x=a+1}^{b-1} \mathbb{P}[T_{a,b} > t+K \mid X_t = x,\ T_{a,b} > t] \cdot \mathbb{P}[X_t = x,\ T_{a,b} > t] \\ &\leq \left(1 - 2^{-K}\right) \cdot \sum_{x=a+1}^{b-1} \mathbb{P}[X_t = x,\ T_{a,b} > t] = \left(1 - 2^{-K}\right) \cdot \mathbb{P}[T_{a,b} > t].\end{aligned}$$

Inductively we obtain that

$$\mathbb{P}[T_{a,b} > Kn] \leq \left(1 - 2^{-K}\right)^n . \qquad :) \checkmark$$

Solution to Exercise 2.27 :(
This is just the L^p maximal inequality, with $p = 2$, together with the fact that $\mathbb{E}\,|X_t|^2 = (1-p)t \leq t$, which stems from the OST applied to the martingale $\left(|X_t|^2 - (1-p)t\right)_t$. :) ✓

Solution to Exercise 2.28 :(
We start with the upper bound. Consider $\left(|X_t|^2 - (1-p)t\right)_t$. This is easily seen to be a martingale. Started at $|x| \leq m$ and up to the stopping time

$$T = T_{m+1} \wedge T_{-m-1} = \inf\{t \geq 0 \ :\ |X_t| = m+1\},$$

this is a bounded martingale, so by the OST (Theorem 2.3.3),

$$|x|^2 = \mathbb{E}_x\left[|X_T|^2 - (1-p)T\right] = (m+1)^2 - (1-p)\,\mathbb{E}_x[T].$$

Hence, by Markov's inequality, uniformly over $|x| \leq m$, we have $\mathbb{P}_x\left[T > \frac{2}{1-p}(m+1)^2\right] \leq \frac{1}{2}$.

Let $U_t = X_t - X_{t-1}$ for all $t \geq 1$, so that $(U_t)_{t\geq 1}$ are i.i.d. $-\mu$. Since $(U_{s+k})_{k\geq 1}$ are independent of $\mathcal{F}_s$, for any $|x| \leq m$ and any $t > s$ we have that

$$\begin{aligned}\mathbb{P}_x[T > t \mid \mathcal{F}_s] &= \mathbf{1}_{\{T>s\}} \cdot \mathbb{P}_x\left[\forall\, 1 \leq k \leq t-s,\ \left|X_s + \sum_{j=1}^{k} U_{s+j}\right| \leq m\right] \\ &= \mathbf{1}_{\{T>s\}} \cdot \mathbb{P}_{X_s}[T > t-s] \leq \mathbf{1}_{\{T>s\}} \cdot \sup_{|y|\leq m} \mathbb{P}_y[T > t-s].\end{aligned}$$

Thus, for $|x| \leq m$ and any $t > \frac{2}{1-p}(m+1)^2$,

$$\begin{aligned}\mathbb{P}_x[M_t \leq m] &= \mathbb{P}_x[T > t] = \mathbb{P}_x\left[T > t,\ T > \tfrac{2}{1-p}(m+1)^2\right] \\ &\leq \mathbb{P}_x\left[T > \tfrac{2}{1-p}(m+1)^2\right] \cdot \sup_{|y|\leq m} \mathbb{P}_y\left[T > t - \tfrac{2}{1-p}(m+1)^2\right] \leq \cdots \leq 2^{-\lfloor \frac{(1-p)t}{2(m+1)^2} \rfloor},\end{aligned}$$

which gives the desired upper bound.

Now for the lower bound. For $0 \leq x \leq m$ we have

$$\mathbb{P}_x[X_t < -m] \leq \mathbb{P}_x[|X_t - X_0| \geq m+1] \leq \mathbb{P}[|X_t| \geq m+1] \leq \frac{\mathbb{E}\,|X_t|^2}{(m+1)^2} \leq \frac{t}{(m+1)^2}.$$

Similarly, for $-m \leq x \leq 0$,

$$\mathbb{P}_x[X_t > m] \leq \frac{t}{(m+1)^2}.$$

Under $\mathbb{P}_0$, both X_t and $-X_t$ have the same distribution. Shifting by x, for some $|x| \leq m$, we get that $\mathbb{P}_x[X_t \leq x] = \mathbb{P}[X_t \leq 0] \geq \frac{1}{2}$, and similarly $\mathbb{P}_x[X_t \geq x] \geq \frac{1}{2}$. Recall that by the L^p maximal inequality we know that

$$\mathbb{E}_x\, |M_t|^2 \leq 4\,\mathbb{E}_x\, |X_t|^2 = 4\left(x^2 + (1-p)t\right) \leq 4\left(x^2 + t\right),$$

since $\left(|X_t|^2 - (1-p)t\right)_t$ is a martingale.

Putting all this together we have for any $0 \leq x \leq m$,

$$\begin{aligned}\mathbb{P}_x[M_t \leq km,\ |X_t| \leq m] &\geq \mathbb{P}_x[X_t \leq x] - \mathbb{P}_x[X_t < -m] - \mathbb{P}_x[M_t > km] \\ &\geq \frac{1}{2} - \frac{t}{(m+1)^2} - \frac{\mathbb{E}_x\left[|M_t|^2\right]}{k^2m^2} \geq \frac{1}{2} - \frac{t}{(m+1)^2} - \frac{4\left(x^2+t\right)}{k^2m^2},\end{aligned}$$

and similarly for $-m \leq x \leq 0$ we have

$$\mathbb{P}_x[M_t \leq km,\ |X_t| \leq m] \geq \frac{1}{2} - \frac{t}{(m+1)^2} - \frac{4\left(x^2+t\right)}{k^2m^2}.$$

Choosing $k = 8$ and $\ell = \left\lfloor \frac{1}{8}m^2 \right\rfloor$, we arrive at the conclusion that for all $|x| \leq m$ we have

$$\mathbb{P}_x[M_\ell \leq 8m,\ |X_\ell| \leq m] > \tfrac{1}{4}.$$

Similarly to the proof of the upper bound, for any $|x| \leq m$ and any $t > s$ we have

$$\begin{aligned}\mathbb{P}_x[M_t \leq 8m,\ |X_s| \leq m \mid \mathcal{F}_s] &= \mathbf{1}_{\{M_s \leq 8m,\ |X_s| \leq m\}} \cdot \mathbb{P}_x\left[\forall\, 1 \leq k \leq t-s,\ \left|X_s + \sum_{j=1}^{k} U_{s+j}\right| \leq 8m\right] \\ &= \mathbf{1}_{\{M_s \leq 8m,\ |X_s| \leq m\}} \cdot \mathbb{P}_{X_s}[|M_{t-s}| \leq 8m] \\ &\geq \mathbf{1}_{\{M_s \leq 8m,\ |X_s| \leq m\}} \cdot \inf_{|y| \leq m} \mathbb{P}_y[|M_{t-s}| \leq 8m].\end{aligned}$$

We obtain that for any $|x| \leq m$ and $t > \ell$,

$$\begin{aligned}\mathbb{P}_x[M_t \leq 8m] &\geq \mathbb{P}_x[M_t \leq 8m,\ |X_\ell| \leq m] \\ &\geq \mathbb{P}_x[M_\ell \leq 8m,\ |X_\ell| \leq m] \cdot \inf_{|y| \leq m} \mathbb{P}_y[M_{t-\ell} \leq 8m] \geq \cdots \geq 4^{-\lfloor t/\ell \rfloor},\end{aligned}$$

which completes the proof of the lower bound. :) ✓

Solution to Exercise 2.29 :(

The function $\varphi(x) = x\mathbf{1}_{\{x>0\}}$ is convex and nondecreasing, so by Jensen's inequality $\mathbb{E}[\varphi(M_{t+1} - a) \mid M_0, \ldots, M_t] \geq \varphi(\mathbb{E}[M_{t+1} - a \mid M_0, \ldots, M_t]) \geq \varphi(M_t - a)$. :) ✓

Solution to Exercise 2.30 :(

We write a solution only for the sub-martingale case, since all are very similar.

$$\mathbb{E}[(H \cdot M)_{t+1} - (H \cdot M)_t \mid M_0, \ldots, M_t] = H_{t+1}\, \mathbb{E}[(M_{t+1} - M_t) \mid M_0, \ldots, M_t] \geq 0.$$

We have used the fact that H_{t+1} is $\sigma(M_0, \ldots, M_t)$-measurable. :) ✓

Solution to Exercise 2.31 :(

Let $H_t = \mathbf{1}_{\{T \geq t\}}$, which is a bounded predictable process. Then, $M_{T\wedge t} = M_0 + \sum_{s=1}^{t} H_s(M_s - M_{s-1})$, which is a sub-martingale by Exercise 2.30. :) ✓

Solution to Exercise 2.32 :(

The process $X_t := -M_t$ is a sub-martingale and $\sup_t \mathbb{E}\left[X_t \mathbf{1}_{\{X_t>0\}}\right] \leq 0 < \infty$. So X_t converges a.s., which implies the a.s. convergence of M_t. :) ✓

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Solution to Exercise 2.33 :(
Since $(M_t)_t$ is uniformly integrable,

$$\sup_t \mathbb{E}\left[M_t \mathbf{1}_{\{M_t>0\}}\right] \leq \sup_t \mathbb{E}|M_t| < \infty,$$

so by martingale convergence $M_t \to M_\infty$ a.s., for some integrable M_∞. By uniform integrability again, $M_t \to M_\infty$ in L^1 as well. :) ✓

Solution to Exercise 2.34 :(
Let $M_t = \mathbb{E}[X \mid \mathcal{F}_t]$. Since $(M_t)_t$ is a uniformly integrable martingale, it converges a.s. and in L^1 to some integrable M_∞. Now, for any event A, we have that $M_t\mathbf{1}_A \to M_\infty\mathbf{1}_A$ a.s. and in L^1 as well. Thus, if $A \in \mathcal{F}_n$ for some n,

$$\mathbb{E}[M_\infty \mathbf{1}_A] = \lim_{t\to\infty} \mathbb{E}[M_t\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_A].$$

Consider the probability measures:

$$\mu(A) := \frac{\mathbb{E}[((M_\infty)^+ + X^-)\mathbf{1}_A]}{\mathbb{E}[(M_\infty)^+ + X^-]} \qquad \text{and} \qquad \nu(A) := \frac{\mathbb{E}[((M_\infty)^- + X^+)\mathbf{1}_A]}{\mathbb{E}[(M_\infty)^- + X^+]}.$$

Since M_∞, X are integrable, these are indeed probability measures, and μ, ν agree on the π-system $\bigcup_t \mathcal{F}_t$. Thus, by Dynkin's lemma (also known as the $\pi - \lambda$ theorem) μ, ν must agree on all of $\mathcal{F}_\infty$. Hence, $M_\infty = \mathbb{E}[X \mid \mathcal{F}_\infty]$ a.s. :) ✓

Solution to Exercise 2.35 :(
Set $X_n = \mathbb{E}[X \mid \sigma_n]$.

Fix n and consider $M_t := X_{n-t}$ for $t \leq n$ and $M_t = X_0$ for $t \geq n$. Then $(M_t)_t$ is a martingale. If U_n is the number of upcrossings of the interval (a, b) by $M_0, \ldots, M_n$, then $(b-a)\,\mathbb{E}[U_n] \leq \mathbb{E}\left[(M_n - a)\mathbf{1}_{\{M_n>a\}}\right] = \mathbb{E}\left[(X_0 - a)\mathbf{1}_{\{X_0>a\}}\right]$.

Now, let U_∞ be the number of upcrossings of the interval (a, b) by $(X_n)_n$. Then $U_n \nearrow U_\infty$, so by monotone convergence, $\mathbb{E}[U_\infty] < \infty$. Exactly as in the proof of the martingale convergence theorem, this holding for all $a < b \in \mathbb{Q}$ implies that $X_n \to X_\infty$ a.s. for some integrable X_∞. Since $(X_n)_n$ is uniformly integrable, we get that $X_n \to X_\infty$ in L^1 as well.

Set $Y = \limsup_{n\to\infty} X_n$. So $Y = X_\infty$ a.s. Note that for any n, all $(X_t)_{t\geq n}$ are σ_n-measurable, so we have that $Y = \limsup_{n\leq t\to\infty} X_t$ is also σ_n-measurable. This implies that Y is measurable with respect to $\sigma_\infty = \bigcap_n \sigma_n$.

For any $A \in \sigma_\infty$, we have that $X_n\mathbf{1}_A \to X_\infty\mathbf{1}_A$ in L^1, so

$$\mathbb{E}[Y\mathbf{1}_A] = \mathbb{E}[X_\infty\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[X \mid \sigma_n]\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[X\mathbf{1}_A].$$

Thus, $\mathbb{E}[X \mid \sigma_\infty] = Y = X_\infty$ a.s. :) ✓

Solution to Exercise 2.36 :(
Write $X_t = U_1 \cdots U_t$ for $(U_t)_{t\geq 1}$ i.i.d.-μ elements.

Set $Y = X_t^{-1}X_{t+k}$, and note that Y is independent of $\mathcal{F}_t$, and specifically that Y is independent of X_t. Thus,

$$\mathbb{P}[X_{t+k} = X_t x \mid X_t] = \mathbb{P}[Y = x \mid X_t] = \mathbb{P}[Y = x] \quad \text{a.s.}$$

Since $(U_t)_{t\geq 1}$ all have the same distribution, we find that $Y = U_{t+1}\cdots U_{t+k}$ has the same distribution as $X_k = U_1 \cdots U_k$. :) ✓

Solution to Exercise 2.38 :(
Let $V = \mathsf{LHF}(G, \mu)/\mathbb{C}$ (modulo the constant functions). Fix some finite symmetric generating set S of G. Assume that $\dim \mathsf{LHF}(G, \mu) < \infty$, so $\dim V < \infty$.

Recall the Lipschitz semi-norm $||\nabla_S f||_\infty := \sup_{x\in G, s\in S} |f(xs) - f(x)|$. We have that $||\nabla_S f||_\infty = 0$ if and only if f is constant. Also, $||\nabla_S(f + c)||_\infty = ||\nabla_S f||_\infty$ for any constant c. Thus, $||\nabla_S \cdot ||_\infty$ induces a norm on V.

Another semi-norm on G is given by $||f||_B := \max_{x\in B} |f(x) - f(1)|$, where B is some finite subset. Note that $||f + c||_B = ||f||_B$ for any constant c. Because $\dim \mathsf{LHF}(G, \mu) < \infty$, if $B = B(1, r)$ for r large enough then $|| \cdot ||_B$ is a semi-norm on $\mathsf{LHF}(G, \mu)$ such that $||f||_B = 0$ if and only if f is constant. Thus, $|| \cdot ||_B$ induces a norm on V as well.

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Since V is finite dimensional, all norms on it are equivalent. Thus, there exists a constant $K > 0$ such that $||\nabla_S v||_\infty \le K \cdot ||v||_B$ for any $v \in V$. Since these semi-norms are invariant to adding constants, this implies that $||\nabla_S f||_\infty \le K \cdot ||f||_B$ for all $f \in \mathsf{LHF}(G, \mu)$.

Now, let $h \in \mathsf{LHF}(G, \mu)$ be a positive harmonic function. Then, $(h(X_t))_t$ is a positive martingale, implying that it converges a.s. Thus, for any fixed k, we have $h(X_{t+k}) - h(X_t) \to 0$ a.s.

Fix $x \in G$ and let k be such that $\mathbb{P}[X_k = x] = \alpha > 0$. By Exercise 2.36, $\mathbb{P}[X_{t+k} = X_t x \mid X_t] = \alpha$, independently of t. We have that a.s. convergence implies convergence in probability, so for any $\varepsilon > 0$,

$$\mathbb{P}[|h(X_t x) - h(X_t)| > \varepsilon] \le \alpha^{-1}\, \mathbb{P}[|h(X_{t+k}) - h(X_t)| > \varepsilon] \to 0.$$

So $h(X_t x) - h(X_t) \to 0$ in probability, for any $x \in G$. Since B is a finite ball this implies that $\max_{x \in B} |h(X_t x) - h(X_t)| \to 0$ in probability.

Now we also use the fact that $||\nabla_S (x.h)||_\infty = ||\nabla_S h||_\infty$. Thus, for all t, we have a.s. that

$$||\nabla_S h||_\infty = \left\| \nabla_S \left(X_t^{-1}.h \right) \right\|_\infty \le K \cdot \left\| X_t^{-1}.h \right\|_B = K \cdot \max_{x \in B} |h(X_t x) - h(X_t)|.$$

Since this converges to 0 in probability, we have $||\nabla_S h||_\infty = 0$ and h is constant. :) ✓

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