ON THE TOPOLOGICAL THEORY OF FUNCTIONS

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1. Introduction. The present paper constitutes a continuation of the ideas and methods of M. Morse and M. Heins [1]. As in that work the subject treated is the theory of deformation classes of meromorphic functions and interior transformations. There the functions considered were defined over the open disc $|z| < 1$ and had only a finite number of zeros, poles and branch point antecedents. It is possible to transfer the results obtained to the situation where the domain of definition is any simply-connected domain of hyperbolic type or, alternatively, of parabolic type. We shall be concerned principally with restricted deformations of functions having these same domains of definition but which are allowed to possess infinitely many zeros, poles and branch point antecedents. The same invariants as in [1] serve to characterize the restricted deformation classes.

The proof of this result presents essential difficulties not met with in the case of finitely many zeros, poles and branch point antecedents. It is no longer possible to use the Lagrange interpolation formula to construct canonical meromorphic functions. In order to prove the existence of a deformation for the interior transformations to be treated it is necessary to give a uniform version of Mittag-Leffler's theorem. In this same connection, the ordinary form of the Tietze deformation theorem is inadequate and it is necessary to give a uniform version of it also.

I wish to thank Professor Morse for his helpful comments.

2. Fundamental definitions. We shall begin by recalling certain fundamental definitions and results.

An *interior transformation* $w = f(z)$ defined on an open set S of the complex z-sphere is a continuous map of S into the complex w-sphere such that given any point z_0 of S there exists a sense-preserving homeomorphism $\phi(z)$ from a neighbourhood N of z_0 to another neighbourhood N_1 of z_0 with z_0 fixed such that the function $f[\phi(z)] = F(z)$ is analytic on N except for a possible pole at z_0 and is not identically constant. The transformation f is said to have a *zero* or *pole* at z_0 according as F has a zero or pole at z_0 and the *order* in each case is taken to be the corresponding order for F. If z_0 is the antecedent of a branch point of the rth order of the inverse of F, z_0 is said to be an *antecedent of a branch point of the rth order* of the inverse of f. These definitions extend at once to a Riemann domain by applying them in the plane of a local uniformizing parameter. We shall be concerned with the case that S is a simply-connected domain, of hyperbolic or parabolic type, and shall consider mero-

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morphic functions and interior transformations which may have infinitely many zeros, poles and branch point antecedents. The totality of such points will be called the *characteristic set* (α) of the function. We shall assume that all zeros, poles and branch points involved are of order 1.

By a *deformation* of an interior transformation f we shall mean a family of interior transformations

$$w = F(z, t) \qquad (z \in S,\ 0 \leqslant t \leqslant 1)$$

depending continuously on the point z in S and the deformation parameter t together and having

$$F(z, 0) \equiv f(z) \qquad (z \in S).$$

We shall confine our attention to *restricted* deformations, namely those for which $F(z, t)$ has for every t the same zeros, poles and branch point antecedents as $f(z)$. Interior transformations which admit a (restricted) deformation one into the other will be said to belong to the same (restricted) deformation class.

The fundamental concept used in defining the invariants employed to characterize the deformation properties of interior transformations is that of the difference order $d(k)$ of a locally simple arc k. We consider arcs k which are represented by continuous and locally (1, 1) images

$$w(t) = u(t) + iv(t) \qquad (0 \leqslant t \leqslant 1)$$

of the interval [0, 1] and which intersect their end points $w(0) = a$ and $w(1) = b$ only when $t = 0$ and $t = 1$ respectively. We call such arc *locally simple* provided there exists a constant $e > 0$ such that any subarc of k whose diameter is less than e is simple. Such a constant e is called a *norm of local simplicity* of k. A set of locally simple curves which admit the same norm of local simplicity will be termed *uniformly locally simple.*

An *admissible deformation* of a locally simple arc k is defined by a family of arcs represented in the form

$$w(t) = H(t, \lambda) \qquad (0 \leqslant t \leqslant 1, 0 \leqslant \lambda \leqslant 1)$$

where H depends continuously on t and λ together and satisfies

$$H(0, \lambda) \equiv a,\ H(1, \lambda) \equiv b \qquad (0 \leqslant \lambda \leqslant 1).$$

The arc associated with the parameter λ will be called k^λ. Then the arcs k^λ are to be uniformly locally simple and are to intersect a and b as end points only.

Two arcs connected by such a deformation are said to belong to the same *deformation class.* This property is independent of the particular parametric representation of the arcs [2, Lemma 28.1].

We proceed to the definition of the *difference order* of a locally simple arc joining two finite points a and b ($a \neq b$).

A subarc of k will be defined by an interval (σ, τ) for t. If this subarc is simple the vector $w(\tau) - w(\sigma)$ has a well defined direction and this direction will vary continuously with σ and τ so long as the arc given by (σ, τ) remains simple and $\sigma < \tau$. Such a variation is termed an admissible chord variation and if we suppose that the angle

$$\frac{1}{2\pi} \arg [w(\tau) - w(\sigma)] \tag{1}$$

has been chosen so as to change continuously in the course of the variation, then the algebraic increment of the angle depends only on the initial and final simple subarcs.

Let k_a and k_b be respectively proper simple subarcs of k of which the initial point of k_a is a and the terminal point of k_b is b. Let the chord subtending k_a vary admissibly into the chord subtending k_b. Let $P(k, k_a\ k_b)$ represent the accompanying algebraic increment of (1). Let the algebraic increment of

$$\frac{1}{2\pi} \arg [w(t) - a] \tag{2}$$

as t increases from its terminal value t_a on k_a to 1 be denoted by $Q_a(k, k_a)$ and let the algebraic increment of

$$\frac{1}{2\pi} \arg [w(t) - b] \tag{3}$$

as t increases from 0 to its initial value t_b on k_b be denoted by $Q_b(k, k_b)$. Then the difference order $d(k)$ of k is defined by

$$d(k) = P(k, k_a, k_b) - Q_a(k, k_a) - Q_b(k, k_b). \tag{4}$$

The difference order $d(k)$ as defined by (4) where $a \neq b$ is an integer which is independent of the choice of k_a and k_b among proper simple subarcs of k with end points as prescribed. It is further independent of any admissible deformation of k [1, Lemma 3.3].

The case a finite, $b = \infty$. In this case continuity, local simplicity and related concepts must be interpreted appropriately in terms of the w-sphere. For a locally simple arc k joining the finite point a to the point at infinity, there will exist a value τ with $0 < \tau < 1$ such that the subarc corresponding to values of t with $\tau \leqslant t < 1$ is simple in the finite w-plane. On this arc $w(t)$ becomes infinite as t tends to 1. Let $k(s)$ denote the subarc of k defined by $(0, s)$ with $0 < s < 1$. Then $d(k(s))$ is well defined and is clearly independent of s provided $s > \tau$ so that we may set

$$d(k) = d(k(s)) \qquad (\tau < s < 1).$$

This quantity possesses invariance under admissible deformations as before.

The case $a = b$ (finite). Let $P(k, k_a, k_b)$ be as above and let $Q_a(k, k_a, k_b)$ equal the algebraic increment of

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$$\frac{1}{2\pi} \arg [w(t) - a]$$

as t varies monotonically from its terminal value t_a on k_a to its initial value t_b on k_b. Then the difference order $d(k)$ of k is defined by

$$d(k) = P(k, k_a, k_b) - Q_a(k, k_a, k_b).$$

This quantity has the following properties. The value of $d(k)$ when $a = b$ is equal to $\frac{1}{2}$ modulo 1. It is independent of the choice of k_a and k_b among proper simple subarcs of k with end points as prescribed and is invariant under admissible deformations of k [1, Lemma 3.5].

3. The invariants. We shall for the present confine our attention to the special hyperbolic case where S is the open disc $|z| < 1$. We will consider an interior transformation $f(z)$ defined on S and possessing zeros and poles which we denote by a_j and branch point antecedents which we denote by b_j. In general the index will run, in the first case, through all non-negative integers and, in the second, through all positive integers. Exceptionally either sequence might terminate, but since this would result only in a considerable simplification of the situation we shall not mention this possibility explicitly. Evidently the points of the characteristic set can have no limit point in the interior of the unit circle. We shall assume that there is at least one zero which we take to be a_0. If there were poles but no zeros we would consider the reciprocal function. For each a_j with $j > 0$ we consider a simple curve h_j joining a_0 to a_j on S and meeting no other point of the characteristic set. Two such curves h_j will be said to be of the same topological type if they can be deformed isotopically one into the other on S without intersecting the characteristic set of f elsewhere than in the end points of h_j. Let h_j^f denote the image of h_j under f. The curve h_j^f will join $f(a_0)$ to $f(a_j)$ in the w-plane and be locally simple. An isotopic deformation of h_j through curves of the same topological type will induce an admissible deformation in the curves h_j^f. Thus the difference order $d(h_j^f)$ is invariant under such deformations. It will, however, in general be different for curves of different topological types.

By a slight generalization of the Weierstrass product representation for integral functions as it is usually given we can construct one function regular in S which has simple zeros at the points a_j and no other zeros in S and a second function regular in S which has simple zeros at the points b_j and no other zeros in S. There is a good deal of arbitrariness in the choice of these functions and we shall later want to make our choice subject to certain conditions. However we shall always use two definite functions $A(z,\alpha)$ and $B(z,\alpha)$, say, kept fixed for all time for a given characteristic set (α). We shall not go further into the above generalization of the product representation at this point since we shall later have to prove a stronger result which includes this.

Now let us set

$$C_j(z, \alpha) = \frac{(z - a_0)(z - a_j)B(z, \alpha)}{A(z, \alpha)} \qquad (j > 0)$$

where $C_j(z, \alpha)$ is assumed to take its limit values at a_0 and a_j, namely

$$C_j(a_0, \alpha) = \frac{(a_0 - a_j)B(a_0, \alpha)}{A'(a_0, \alpha)}, \; C_j(a_j, \alpha) = \frac{(a_j - a_0)B(a_j, \alpha)}{A'(a_j, \alpha)} \qquad (j > 0).$$

Corresponding to a variation of z along h_j we set

$$V(h_j) = \frac{1}{2\pi} [\arg C_j(z, \alpha)]_{z=a_0}^{z=a_j}.$$

Regardless of the arguments used

$$V(h_j) \equiv \frac{1}{2\pi} [\arg C_j(a_j, \alpha) - \arg C_j(a_0, \alpha)] \qquad (\text{mod } 1).$$

In [1] there is proved the following theorem which we quote: [1, Theorem 5.2].

THEOREM A. *The value of the difference*

$$d(h_j^f) - V(h_i)$$

is independent of h_j among simple curves which join a_0 to a_j without intersecting the other points of the characteristic set (α).

The proof of this theorem in no wise uses the assumption there in force that there are only finitely many points in the characteristic set but merely that between two choices for h_j lie only finitely many points of this set. This condition is certainly satisfied here and in all cases with which we shall deal. This enables us to make the following definition.

We set

$$J_j(f, \alpha) = d(h_j^f) - V(h_j) \qquad (j > 0)$$

for any simple arc h_j which joins a_0 to a_j without intersecting $(\alpha) - (a_0, a_j)$. The numbers J_j are independent of the choice of h_j among admissible arcs h_j and of restricted deformations of f. We should remark that these quantities can be defined analogously for S the finite z-plane and their definition extends at once to any domain conformally equivalent to either of the preceding.

Our objective is to characterize the deformation properties of meromorphic functions and interior transformations in terms of the numbers J_j which we call the *invariants* of the function in question. This we will do by means of the following theorem.

THEOREM B. *A necessary and sufficient condition that the meromorphic functions or interior transformations f_1 and f_2, defined on S and possessing the same characteristic set, belong to the same restricted deformation class is that*

$$J_j(f_1, \alpha) = J_j(f_2, \alpha) \qquad (j > 0).$$

In the case of meromorphic functions the deformation can be carried out through meromorphic functions alone.

The necessity is obvious from what we have observed above. We shall next treat the sufficiency in the case of meromorphic functions.

4. Meromorphic functions. Let us suppose that the function f is meromorphic on S and possesses the characteristic set (α). Then the function ϕ defined by the equation

$$\frac{f'(z)}{f(z)} = \phi(z)\frac{B(z, \alpha)}{A(z, \alpha)} \tag{5}$$

is regular on S apart from removable singularities and never zero. We term ϕ the *residual function* of f.

We have immediately at our disposal the following theorem [1, Theorem 10.1].

THEOREM C. *The algebraic increment of the argument of the residual function ϕ of f as z traverses a simple regular arc h_i leading from a_0 to a_j on S is equal to $2\pi J_j(f, \alpha)$.*

On multiplying the two sides of (5) by $z - a_j$ and letting z tend to a_j as limit, we find that

$$\phi(a_j) = e_j\,\frac{A'(a_j, \alpha)}{B(a_j, \alpha)} \qquad (j \geqslant 0) \tag{6}$$

where $e_j = 1$ if a_j is a zero and $e_j = -1$ if a_j is a pole. Regarding this latter quantity as a function determined by the characteristic set we define

$$e_j\,\frac{A'(a_j, \alpha)}{B(a_j, \alpha)} = g_j(\alpha) \qquad (j \geqslant 0).$$

The following lemma is proved precisely as in [1] provided we include as a condition of admissibility that the points of the characteristic set have no limit point in the interior of S [1, Lemma 10.2].

LEMMA 1. *Corresponding to an arbitrary admissible characteristic set (α) and to a function ψ which is non-null and regular on S and satisfies (6) in terms of (α), there exists a function F which is meromorphic on S, possesses the characteristic set (α) and for which the residual function is ψ.*

We will employ the convention of denoting, for a complex quantity X, by Arg X that value of the argument for which

$$0 \leqslant \operatorname{Arg} X < 2\pi$$

and similarly

$$\text{Log } X = \log |X| + i \text{ Arg } X.$$

Now we may express the invariants of any interior transformation by

$$J_j(f, \alpha) = \frac{1}{2\pi}\{\text{Arg } g_j(\alpha) - \text{Arg } g_0(\alpha)\} + I_j(f, \alpha)$$

where the numbers $I_j(f, \alpha)$ are integers invariant under restricted deformations of f and uniquely determined by f and (α). This is a consequence of the facts that the first term differs from $-V(h_j)$ by half of an odd integer or by an integer according as a_j is a zero or a pole and that in these same cases $d(h_j^f)$ is respectively half of an odd integer or an integer.

The following theorem shows that all possible values of the integers I_j are realized for meromorphic functions and hence more generally for interior transformations.

THEOREM D. *Corresponding to any admissible characteristic set* (α) *and arbitrary sequence* $\{r_j\}$ *of integers, there exists a function* $F(z, \alpha, r)$ *which is meromorphic in* z *on* S, *whose characteristic set is* (α) *and whose invariants* $I_j(F, \alpha) = r_j$ $(j > 0)$.

Indeed, let us set

$$c_j = c_j(\alpha, r) = \text{Log } g_j(\alpha) + 2\pi r_j i \qquad (j \geqslant 0)$$

where we take $r_0 = 0$. Let us further set

$$A'_j = A'(a_j, \alpha).$$

Since all zeros of $A(z, \alpha)$ are simple, none of these quantities are zero. Thus we can construct a function $Q(z, \alpha, c)$ which has simple poles at the points $a_j (j \geqslant 0)$, the corresponding residues being c_j/A'_j and which is regular elsewhere in S. This is done by a slight variation of the proof of Mittag-Leffler's theorem as it is usually given [6]. Since, in any case, we shall later need a much stronger result which will imply this fact, we shall not go into this variation here.

Now the function

$$P(z) = P(z, \alpha, c) = A(z, \alpha)\, Q(z, \alpha, c)$$

is regular for $z \in S$ and has the values

$$P(a_j) = c_j \qquad (j \geqslant 0).$$

The function

$$\psi(z) = \exp P(z)$$

gives the residual function corresponding to a function $F(z, \alpha, r)$ of the desired type. Indeed,

$$\psi(a_j) = \exp P(a_j) = \exp c_j = g_j(\alpha) = e_j \frac{A'(a_j, \alpha)}{B(a_j, \alpha)} \qquad (j \geqslant 0),$$

and $\psi(z)$ is regular and non-zero on S. Hence it is a residual function by Lemma 1. Let $F(z, \alpha, r)$ be the corresponding meromorphic function. Since ψ is non-zero on S there exists a single-valued continuous branch of arg ψ over S and, by Theorem C, for any such

$$2\pi J_j(\mathrm{F}, \alpha) = \arg \psi(a_j) - \arg \psi(a_0).$$

Then by the definition of $\psi(z)$

$$\begin{aligned} 2\pi J_j(F, \alpha) &= \Im(P(a_j)) - \Im(P(a_0)) \\ &= \Im(c_j) - \Im(c_0) \\ &= (\operatorname{Arg} g_j(\alpha) + 2\pi r_j) - \operatorname{Arg} g_0(\alpha). \end{aligned}$$

Hence

$$I_j(F, \alpha) = r_j \qquad (j > 0)$$

as required.

There is a certain arbitrary character in the definition of $F(z, \alpha, r)$ since in the definition of $A(z, \alpha)$, $B(z, \alpha)$ and $Q(z, \alpha, c)$ were involved certain arbitrary choices. However, to a certain extent $F(z, \alpha, r)$ can be used as a canonical function associated with the chosen sets (α) and (r), since, when these are given, we may make a perfectly definite choice of the functions A, B and Q.

We shall now give the proof of the part of Theorem B which relates to meromorphic functions. It is the same as the corresponding proof in [1] but we include it for the sake of completeness.

Let, then, f_1 and f_2 be two functions meromorphic on S with the same characteristic set (α) for which

$$J_j(f_1, \alpha) = J_j(f_2, \alpha) \qquad (j > 0).$$

Let ϕ_1 and ϕ_2 be the corresponding residual functions and let us set

$$\psi(z, t) = \exp\{(1 - t) \log \phi_1(z) + t \log \phi_2(z)\} \qquad (0 \leqslant t \leqslant 1)$$

where $\log \phi_1$ and $\log \phi_2$ are continuous branches with

$$\log \phi_1(a_0) = \log \phi_2(a_0). \tag{7}$$

The latter can be arranged since

$$\phi_1(a_0) = \phi_2(a_0).$$

Clearly $\psi(z, t)$ is non-zero and regular on S. Equality of the invariants for f_1 and f_2 implies that

$$\arg \phi_1(z) \Big|_{a_0}^{a_j} = \arg \phi_2(z) \Big|_{a_0}^{a_j} \qquad (j > 0)$$

or again

$$\log \phi_1(z) \Big|_{a_0}^{a_j} = \log \phi_2(z) \Big|_{a_0}^{a_j} \qquad (j > 0).$$

Then from (7) we deduce

$$\log \phi_1(a_j) = \log \phi_2(a_j) \qquad (j > 0).$$

Thus

$$\psi(a_j, t) = \exp \log \phi_1(a_j) = \phi_1(a_j) \qquad (j > 0).$$

By Lemma 1 this implies that $\psi(z, t)$ is the residual function of a meromorphic function with characteristic set (α)

$$f(z, t) = \exp \left(\int \psi(z, t) \frac{B(z, \alpha)}{A(z, \alpha)} dz \right) \qquad (0 \leqslant t \leqslant 1).$$

This means that

$$f(z, 0) = C_1 f_1(z) \qquad (C_1 \neq 0),$$

$$f(z, 1) = C_2 f_2(z) \qquad (C_2 \neq 0),$$

with C_1, C_2 constants. The function

$$\frac{f(z, t)}{C_1^{1-t} C_2^{t}} \qquad (0 \leqslant t \leqslant 1)$$

gives the required meromorphic deformation of f_1 into f_2.

5. Interior transformations. We now proceed to the classification of interior transformations under restricted deformation. It is here that the infinitude of the characteristic set first makes itself felt in an essential manner since there are no longer canonical meromorphic functions depending in a trivial manner on the characteristic set.

We make here a remark which will be useful later also: if we can prove the classification theorem after a preliminary homeomorphism of S it follows immediately for the original situation.

Indeed, equality of the invariants of two interior transformations with the same characteristic set is preserved by such an operation since in their definition, if we use arcs corresponding under the homeomorphism for the evaluation, the term $d(h_j^f)$ is unaltered and the term $-V(h_j)$ is changed by an amount independent of the particular interior transformation. Further, if we can perform the deformation after the homeomorphism we can transform it back to the original situation and conversely.

Now we may assume that all points of the characteristic set (α) lie in the semi-circle where $\Re z \geqslant 0$. Indeed, if we draw every circle with centre $z = 0$ and passing through a point of the characteristic set, then on these circles we can deform the identity mapping isotopically into a homeomorphism of the circle for which the characteristic points have their images in $\Re z \geqslant 0$ and then extend this homeomorphism over the intervening circular rings [5].

Now we can prove:

LEMMA 2. *Any interior transformation f^* of S admits a restricted deformation into a function f for which a sense-preserving homeomorphism ζ of S exists such that $f\zeta$ is meromorphic on S.*

Indeed, by virtue of what we may call the "uniformization theorem" [4] there exists a homeomorphism σ from a simply-connected domain R of the complex plane to S such that $f^*\sigma$ is meromorphic on R. If R is of hyperbolic type it can be mapped conformally onto S and σ is carried into a homeomorphism ζ as desired. If R is of parabolic type we can perform a preliminary restricted deformation of f^* and reduce this case to the preceding one. Indeed let θ^t be an isotopic deformation of S onto $S' = \{z \mid |z| < 1, \Re z > -1 + \delta, 0 < \delta < \frac{1}{3}\}$, leaving fixed the points of S with $\Re z > -1 + 2\delta$. Then θ^1 will be a homeomorphism from S to S' and the composite function $f^*\theta^t$ constitutes a restricted deformation of f^*. The Riemann image under this new function, as a proper subdomain of the Riemann image under f^*, is of hyperbolic type and thus comes under the case first treated.

It is at this stage enough to show that an interior transformation of the type of the function f of the lemma admits a restricted deformation into a meromorphic function.

Let f be an interior transformation for which there exists a sense-preserving homeomorphism ζ of S onto itself such that $f\zeta$ is meromorphic on S. Let η be the inverse homeomorphism of ζ on S. We will denote by $C(r)$ the circle $|z| = r$ and by $L(r)$ its image under η. Further $m(r)$ will denote the minimum of $|z|$ on $L(r)$. Since, as $r \to 1$, $L(r)$ passes outside of every compact subset of S, we have

$$\lim_{r \to 1} m(r) = 1.$$

We will need the following lemma, the present simple proof of which is due to a suggestion by Professor Morse.

LEMMA 3. *There exists an isotopic deformation η^t $(0 \leqslant t \leqslant 1)$ of S which deforms the identity mapping of S into the homeomorphism η and which satisfies the following condition: if $L^t(r)$ and $m^t(r)$ are the quantities for η^t analogous to $L(r)$ and $m(r)$ for η then*

$$\lim_{r \to 1} m^t(r) = 1$$

uniformly in t.

Indeed let us consider the space $\tilde{S}$ obtained from S by Alexandroff's compactification. That is, we adjoin to S a point P and define a neighbourhood of P to consist of P together with the complement of a compact set in S. The space $\tilde{S}$ is topologically equivalent to a 2-sphere and the homeomorphism η extends at once to a sense-preserving homeomorphism $\tilde{\eta}$ of $\tilde{S}$ having P as fixed point. It is well known that $\tilde{\eta}$ can be generated by an isotopic deformation from the identity $\tilde{\eta}^t$ $(0 \leqslant t \leqslant 1)$ on $\tilde{S}$ where P is fixed under each $\tilde{\eta}^t$ [3]. Given any neighbourhood W of P, since $\tilde{\eta}^t(z)$ is continuous in z and t together, for each t', $0 \leqslant t' \leqslant 1$, we can find a neighbourhood $U^{t'}$ of P and an interval $I^{t'}$ open with respect to $[0, 1]$ containing t' such that $\tilde{\eta}^t(z)$ maps the direct product $U^{t'} \times I^{t'}$ into W. We can cover the interval $[0, 1]$ by a finite number of such intervals and let $U_1, \ldots, U_k$ be the corresponding neighbourhoods of P. The intersection of these neighbourhoods is mapped into W by $\tilde{\eta}^t$, regarded as a function of z only, for all t. It is clear that by restricting $\tilde{\eta}^t$ to the complement of P, that is, passing back to S we obtain the desired isotopic deformation η^t.

The argument will now proceed in a number of steps. We will make use of our notations defined previously. For a deformation η^t of the type in Lemma 3 we will denote by $a_n{}^t$, $b_n{}^t$ the respective images of a_n, b_n under η^t and by $(\mathbf{a}^t)$ the corresponding characteristic set.

(i) *The product*

$$\prod_{n=0}^{\infty} \left((1 - \frac{z}{a_n{}^t} \right) \exp \left\{ \sum_{j=1}^{n} \frac{1}{j} \left(\frac{z}{a_n{}^t} \right)^j \right\} = A(z, \mathbf{a}^t) \tag{8}$$

is a continuous function of z and t together.

In order to prove continuity for $z = z_0$ it is enough to choose R with $|z_0| < R < 1$ and allow only variations of z within the circle $|z| < R$. The general factor in the product (8) can be written

$$\exp \left\{ - \sum_{n+1}^{\infty} \frac{1}{j} \left(\frac{z}{a_n{}^t} \right)^j \right\} = \exp v_n(z),$$

say. By the condition on η^t we have $a_j{}^t = \theta_j{}^t a_j$ where $|\theta_j{}^t| \to 1$ uniformly in t as $j \to \infty$. Now

$$|v_n(z)| \leqslant \sum_{n+1}^{\infty} \left| \frac{z}{a_n{}^t} \right|^j = \left| \frac{z}{a_n{}^t} \right|^{n+1} \bigg/ \left(1 - \left| \frac{z}{a_n{}^t} \right| \right)$$

$$\leqslant \theta \left| \frac{z}{a_n{}^t} \right|^{n+1}$$

provided n is large enough, namely so that $|a_n{}^t| \geqslant R/(1 - \epsilon)$ for all t, $1 > \epsilon > 0$, $\theta = \theta(\epsilon)$ independent of n. This shows that the infinite product (8) converges uniformly and absolutely in $|z| \leqslant R$, $0 \leqslant t \leqslant 1$. In particular, the product

$$\prod_{N+1}^{\infty} \left(1 - \frac{z}{a_n{}^t} \right) \exp \left\{ \sum_{j=1}^{n} \frac{1}{j} \left(\frac{z}{a_n{}^t} \right)^j \right\}$$

is arbitrarily close to 1 for N large enough, uniformly in $|z| \leqslant R, 0 \leqslant t \leqslant 1$. The terms

$$\prod_{n=0}^{N} \left(1 - \frac{z}{a_n{}^t} \right) \exp \left\{ \sum_{j=1}^{n} \frac{1}{j} \left(\frac{z}{a_n{}^t} \right)^j \right\}$$

depend continuously on z and t together for $|z| \leqslant R$, $0 \leqslant t \leqslant 1$, proving the result. We note that in case $a_n{}^t = 0$ for some t (which can occur only for finitely many n) the corresponding primary factor must be redefined in a suitable manner.

(ii) *The product*

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{b_n{}^t}\right)\exp\left\{\sum_{i=1}^{n}\frac{1}{j}\left(\frac{z}{b_n{}^t}\right)^j\right\} = B(z, \mathbf{a}^t)$$

is a continuous function of z and t together.

This is immediate by (i).

(iii) *The quantity* $A'(a_j{}^t, \mathbf{a}^t)$ *is a continuous, non-zero function of t for each* j $(j \geqslant 0)$.

Indeed

$$A'(z, \mathbf{a}^t) = -\frac{1}{a_j{}^t}\, Q_j(z, t) + \left(1-\frac{z}{a_j{}^t}\right) Q'_j(z, t)$$

where

$$Q_j(z, t) = \exp\left\{\sum_{k=1}^{j}\frac{1}{k}\left(\frac{z}{a_j{}^t}\right)^k\right\}\prod_{n\neq j}\left(1-\frac{z}{a_n{}^t}\right)\exp\left\{\sum_{k=1}^{n}\frac{1}{k}\left(\frac{z}{a_n{}^t}\right)^k\right\}.$$

Further $Q_j(z, t)$ is a continuous function of z and t together and

$$A'(a_j{}^t, \mathbf{a}^t) = -\frac{1}{a_j{}^t}\; Q(a_j{}^t, t)$$

which depends continuously on t. Once again if $a_j{}^t = 0$ for some t the corresponding primary factor must be suitably redefined.

(iv) *The quantity*

$$g_j(\mathbf{a}^t) = e_j\,\frac{A'(a_j{}^t, \mathbf{a}^t)}{B\,(a_j{}^t, \mathbf{a}^t)}$$

is a continuous non-zero function of t for each j $(j \geqslant 0)$.

Indeed $B(a_j{}^t, \mathbf{a}^t) \neq 0$, $0 \leqslant t \leqslant 1$, and the result follows by (iii).

(v) Let us define

$$c_j{}^t = \log g_j(\mathbf{a}^t) + 2\pi r_j i \qquad (j \geqslant 0),$$

where some definite determination of the logarithm is chosen at a fixed point for each j and the r_j are integers. The function

$$c_j{}^t/A'_j{}^t, \text{ with } A'_j{}^t = A'(a_j{}^t, \mathbf{a}^t)$$

is a continuous function of t, $0 \leqslant t \leqslant 1$, and hence bounded in absolute value

$$|c_j{}^t/A'_j{}^t| \leqslant K_j \qquad (0 \leqslant t \leqslant 1).$$

There exists an increasing sequence of integers $\{j_n\}$ such that

$$\sum_{n=0}^{\infty}{}_n K\theta^{j_n}$$

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converges for each θ, $0 \leqslant \theta < 1$. Indeed it is only necessary to arrange that $\lim_{n\to\infty} (K_n)^{1/j_n} = 1$. *Then the function, to play the role of $Q(z, \alpha, c)$ in Theorem D,*

$$Q(z, a^t, c^t) = \sum_{n=0}^{\infty} \frac{c_n{}^t}{A'_n{}^t}\left\{\frac{1}{z - a_n{}^t} + \frac{1}{a_n{}^t}\left[1 + \frac{z}{a_n{}^t} + \ldots + \left(\frac{z}{a_n{}^t}\right)^{j_n-1}\right]\right\}$$

is a continuous function of z and t together (Q taken on the Riemann sphere). As before, to prove continuity of Q for a given value of z, say z_0, it is enough to choose R with $|z_0| < R < 1$ and allow only variations of z within the circle $|z| < R$. In this case

$$\left|\frac{1}{z - a_n{}^t} + \frac{1}{a_n{}^t}\left[1 + \frac{z}{a_n{}^t} + \ldots + \left(\frac{z}{a_n{}^t}\right)^{j_n-1}\right]\right|$$
$$\leqslant \left|\frac{z}{a_n{}^t}\right|^{j_n} \left|\frac{1}{a_n{}^t}\right| \frac{1}{1 - \left|\frac{z}{a_n{}^t}\right|}$$
$$\leqslant \theta \left|\frac{z}{a_n{}^t}\right|^{j_n}$$

provided n is large enough, namely so that $|a_n{}^t| \geqslant R/(1 - \epsilon)$ for all t, $1 > \epsilon > 0$, $\theta = \theta(\epsilon)$ independent of n. Thus the above series converges uniformly and absolutely for $|z| \leqslant R$, $0 \leqslant t \leqslant 1$. In particular, the sum

$$\sum_{u=N+1}^{\infty} \frac{c_n{}^t}{A'_n{}^t}\left\{\frac{1}{z - a_n{}^t} + \frac{1}{a_n{}^t}\left[1 + \frac{z}{a_n{}^t} + \ldots + \left(\frac{z}{a_n{}^t}\right)^{j_n-1}\right]\right\}$$

is arbitrarily small in absolute value uniformly in $|z| \leqslant R$, $0 \leqslant t \leqslant 1$, for N large enough. The terms

$$\sum_{n=0}^{N} \frac{c_n{}^t}{A'_n{}^t}\left\{\frac{1}{z - a_n{}^t} + \frac{1}{a_n{}^t}\left[1 + \frac{z}{a_n{}^t} + \ldots + \left(\frac{z}{a_n{}^t}\right)^{j_n-1}\right]\right\}$$

depend continuously on z and t together for $|z| \leqslant R$, $0 \leqslant t \leqslant 1$, proving the result.

Now let, as in the proof of Theorem D,

$$P(z, a^t, c^t) = A(z, a^t)\, Q(z, a^t, c^t)$$

and

$$\psi^t(z) = \exp\{P(z, a^t, c^t)\}.$$

Then

$$F^t(z) = \exp\left(\int \frac{\psi^t(z) B(z, a^t)}{A(z, a^t)}\, dz\right)$$

depends continuously on z and t together, $z \in S$, $0 \leqslant t \leqslant 1$. This is evident from the above considerations.

We are now in a position to prove the part of Theorem B which deals with interior transformations. Let f be an interior transformation on S, ζ a homeomorphism of S such that $f\zeta$ is meromorphic on S and η the inverse of ζ on S.

Let $\eta^t (0 \leqslant t \leqslant 1)$ be an isotopic deformation from the identity generating η and of the type of Lemma 3. We will denote the meromorphic function $f\zeta$ by λ: it has the characteristic set (α^1). The meromorphic function F^1 also has this characteristic set and the determination of the logarithms and the integers r_j in the definition of $c_j{}^t$ can be chosen so that F^1 will have the same invariants as λ after the manner of Theorem D. From this point on we retain this choice of the determination. Then by the first part of the theorem there exists a restricted deformation of meromorphic type of λ into F^1 which we denote by λ^t $(0 \leqslant t \leqslant 1)$. The composite function

$$\lambda^t \eta \qquad (0 \leqslant t \leqslant 1)$$

will define a restricted deformation of $\lambda\eta \equiv f$ into $F^1\eta$. The function $F^t\eta^t$ always has the characteristic set (α) and as t decreases from 1 to 0 it defines a restricted deformation of $F^1\eta$ into F^0. This last is a meromorphic function. Combining these two deformations gives the required restricted deformation of f into a meromorphic function. By the way in which it was obtained, the latter naturally has the same invariants as f. This completes the proof of the theorem.

6. General domains. We will now consider how the results of the preceding three sections extend to domains other than the open disc. For this the remark made at the beginning of §5 is essential. Any simply-connected domain of hyperbolic type admits a conformal map on the unit disc and thus the results carry over as regards both meromorphic functions and interior transformations. In the finite z-plane the deformation properties of meromorphic functions are obtained simply by transferring word for word the proofs previously given. In the case of interior transformations we can return to the situation of the unit disc by a preliminary homeomorphism. As above the results extend at once to any simply-connected domain of parabolic type.

We may remark also that the extension to the case where zeros, poles and branch point antecedents of higher orders are allowed requires only formal modifications.

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