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Solid convergence structure equals pseudotopology

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A new brief proof of the fact stated in the title is presented.

Pseudotopological spaces were introduced by Choquet in [1], and solid convergence spaces by Schroder in [4]. Schroder remarked, without proof, that every solid convergence structure is a pseudotopology.

The statement of the title of this note was first proved in [5]. It is shown in [5] that every concrete category A has a quasitopos hull over sets, and that this quasitopos hull is the category of sheaves for a canonical Grothendieck topology of A. Day and Kelly [3] had given two characterizations of topological quotient maps $f : X \to Y$ such that every pullback of f, in the category *Top* of topological spaces, is a quotient map. These characterizations can easily be expanded to describe the canonical topology of *Top*. One of the characterizations shows that the quasitopos hull of *Top* is the category of solid convergence spaces; see [2]. The other characterization shows that the quasitopos hull of *Top* is the category of pseudotopological spaces.

Obviously, a less involved and more elementary proof of the fact stated as the title of this note seems desirable. We shall give such a proof.

Let X be a convergence space, F a filter on X, and x a point of X. A *cover* γ of x assigns, by definition, to every filter φ converging to x a set $\gamma(\varphi)$ in φ. We compare the following two statements.

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(A) Every ultrafilter on X finer than F converges to x .

(B) For every cover γ of x , there is a finite list $\varphi_1, \ldots, \varphi_n$ of filters converging to x such that $\gamma(\varphi_1) \cup \ldots \cup \gamma(\varphi_n)$ is in F .

If F converges to x , then the pair (F, x) satisfies (A) and (B). X is called a *pseudotopological space* if F always converges to x when (A) is satisfied, and X is called a *solid convergence space* if F always converges to x when (B) is satisfied.

PROPOSITION. (A) *and* (B) *are logically equivalent.*

Proof. Assume first that the conclusion of (B) is false for some cover of x . The difference sets

$$A\setminus(\gamma(\varphi_1) \cup \ldots \cup \gamma(\varphi_n)) ,$$

with A in F and $\varphi_1, \ldots, \varphi_n$ converging to x , then are non-empty and form a filter basis. The filter G on X with this basis is finer than F . No filter ψ finer than G can converge to x , for otherwise both $\gamma(\psi)$ and $X\setminus\gamma(\psi)$ would be in ψ , a contradiction.

Assume now that some ultrafilter ψ finer than F does not converge to x . A set $\gamma(\varphi)$ in φ , but not in ψ , can be chosen for every filter φ converging to x ; this defines a cover of x . If $\gamma(\varphi_1) \cup \ldots \cup \gamma(\varphi_n)$ is in F , and hence in ψ , for a finite list of filters $\varphi_1, \ldots, \varphi_n$ converging to x , then one of the sets $\gamma(\varphi_i)$ is in ψ , contrary to the construction of γ .

REMARK. (A) and (B) remain equivalent if (B) is sharpened by restricting covers of x to ultrafilters converging to x .

References

[1] G. Choquet, "Convergences", *Ann. Univ. Grenoble Sect. Sci. Math. Phys. (N.S.)* 23 (1948), 57-112.

[2] Brian Day, "Limit spaces and closed span categories", *Category seminar*, 65-75 (Proc. Sydney Category Theory Seminar 1972/73. Lecture Notes in Mathematics, 420. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[3] B.J. Day and G.M. Kelly, "On topological quotient maps preserved by pullbacks or products", *Proc. Cambridge Philos. Soc.* 67 (1970), 553-558.

[4] M. Schroder, "Solid convergence spaces", *Bull. Austral. Math. Soc.* 8 (1973), 443-459.

[5] Oswald Wyler, "Are there topoi in topology?", *Categorical topology*, 699-719 (Proc. Conf., Mannheim, 1975. Lecture Notes in Mathematics, 540. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

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