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CERTAIN CONDITIONS UNDER WHICH NEAR-RINGS ARE RINGS

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In 1969, Ligh proved that distributively generated (d-g) Boolean near-rings are rings, and hinted that some of the more complicated polynomial identities implying commutativity in rings may turn d-g near-rings into rings. In the present paper we investigate the following conditions: (1) $xy = (xy)^{n(x,y)}$; (2) $xy = (yx)^{n(x,y)}$; (3) $xy = y^{m(x,y)}x^{n(x,y)}$; (4) $xy = xy^{n(x,y)}x$; (5) $xy = x^{n(x,y)}y^{m(x,y)}$; and finally prove that under appropriate additional hypotheses a d-g near-ring must be a commutative ring. Indeed the theorem proved here is a wide generalisation of many recently established results.

1. Introduction

A ring (near-ring) R is called periodic if for each x in R there exist distinct positive integers m, n such that $x^m = x^n$. A long standing result of Herstein [7] states that a periodic ring is commutative if its nilpotent elements are central. In order to get the analogue of this result in near-rings, Bell [4] established that if R is a distributively generated (d-g) near-ring with its nilpotent elements lying in the centre, then the set N of nilpotent elements of R forms an ideal, and if R/N is periodic then R must be commutative. Now we consider the following properties and notice that a ring (near-ring) satisfying any one of the them is necessarily periodic.

(1) For each x, y in R there exists a positive integer $n = n(x,y) > 1$ such that $xy = (xy)^n$.

(2) For each x, y in R there exists a positive integer $n = n(x,y) > 1$ such that $xy = (yx)^n$.

(3) For each x, y in R there exist positive integers $m = m(x,y)$, $n = n(x,y)$, at least one of them greater than 1, such that $xy = y^m x^n$.

(4) For each x, y in R there exists an integer $n = n(x,y) > 1$ such that $xy = xy^n x$.

(5) For each x, y in R there exist integers $m = m(x,y) > 1$, $n = n(x,y) > 1$ such that $xy = x^n y^m$.

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Recently commutativity of rings under most of the above mentioned conditions has been investigated (see [**2**], [**3**] and [**10**]).

In a paper Ligh [**9**] has remarked that some conditions implying commutativity in rings might turn a class of near-rings into rings. The purpose of the present paper is to examine whether some of our conditions may imply that certain near-rings are rings.

Besides providing a simpler and more attractive proof of a result due to Bell [**5**], our theorem generalises the results proved in [**1**], [**2**] and [**10**].

2. Notations and preliminaries

Throughout the paper R is a left near-ring with multiplicative centre Z, and N denotes the set of nilpotent elements of R. An element x of R is said to be *distributive* if $(y+z)x = yx + zx$ and if all the elements are distributive then the near-ring R is said to be distributive. A near-ring R is *distributively generated* (d-g) if it contains a multiplicative subsemigroup of distributive elements which generates the additive group R^+, and a near-ring R will be called *strongly distributively generated* (s-d-g) if it contains a set of distributive elements whose squares generate R^+.

An *ideal* of a near-ring R is defined to be a normal subgroup I of R^+ such that

(i) $RI \subseteq I$ and

(ii) $(x+i)y - xy \in I$ for all x, y in R and $i \in I$.

In a (d-g) near-ring (ii) may be replaced by (ii)′ $IR \subseteq I$. A near-ring R is called *zero-symmetric* if $0x = 0$ for all x in R and *zero-commutative* if $xy = 0$ implies $yx = 0$ for all x, y in R (recall that left distributivity yields $x0 = 0$).

3. Results

We begin with the following known results which will be used extensively. The proofs of (I) and (II) are straightforward whereas those of (III), (IV) and (V) can be found in [**6**].

(I) If R is a zero-commutative near-ring, then $xy = 0$ implies $xry = 0$ for all r in R.

(II) A d-g near-ring is always zero-symmetric.

(III) A d-g near-ring R is distributive if and only if R^2 is additively commutative.

(IV) A d-g near-ring R with unity 1 is a ring if R is distributive or if R^+ is commutative.

(V) If N is a two-sided ideal in a d-g near-ring R, then the elements of the quotient group $R^+ - N$ form a d-g near-ring, which will be represented by R/N.

We pause to observe that a d-g near-ring R satisfying any of our conditions (1)-(4) is necessarily zero-commutative. For example, let R satisfy (1) and for a pair of elements x, y in R, $xy = 0$. Then by virtue of (II), $yx = (yx)^{n_1 = n(y,x)} = yxyxyx \ldots yx = y(xy)^{n_1 - 1}x = (y0)x = 0x = 0$.

Now we prove the following:

LEMMA. *Let R be a d-g near-ring satisfying any one of the conditions (1)–(5). Then $N \subseteq Z$.*

PROOF: We look at each of the conditions in turn:

(1) Since R is zero-commutative, it follows that if $a \in N$ and x is an arbitrary element of R then ax is nilpotent. Thus the nilpotent elements of R annihilate R on both sides. Hence a is central.

(2) Argue as above.

(3) Let $a \in N$ and $x \in R$; then there exist integers $m_1 = m(a,x) \geqslant 1$, $n_1 = n(a,x) > 1$ such that $ax = x^{m_1}a^{n_1}$. Now choose $m_2 = m(x^{m_1}, a^{n_1}) \geqslant 1$, $n_2 = n(x^{m_1}, a^{n_1}) > 1$ such that $x^{m_1}a^{n_1} = a^{n_1 n_2}x^{m_1 m_2}$, and hence $ax = a^{n_1 n_2}x^{m_1 m_2}$. It is now clear that for arbitrary t we have integers m_1, $m_2 \ldots m_t \geqslant 1$ and n_1, $n_2 \ldots n_t > 1$ such that $ax = a^{n_1 n_2 \cdots n_t}x^{m_1 m_2 \cdots m_t}$. But since $a \in N$, $a^{n_1 n_2 \cdots n_t} = 0$ for sufficiently large t. Thus $ax = 0$ for a in N and x in R. Since R is zero-commutative, the nilpotent elements of R annihilate R on both sides and therefore a is central.

(4) and (5) may be proved on the same lines as (3). □

Now in view of [4], Lemma 1 our lemma at once yields that N is a two-sided ideal which in turn, together with the main theorem of [4], proves the following:

THEOREM. *Let R be a d-g near-ring satisfying any one of the conditions (1)–(5). Then R is commutative.*

COROLLARY 1. *Let R be a d-g near-ring satisfying any one of the conditions (1)–(5). If $R^2 = R$, then R is a commutative ring.*

PROOF: In view of our theorem a d-g near-ring satisfying any one of the conditions (1)–(5) is commutative. Thus for any x, y, z in R, we have $(y+z)x = x(y+z) = xy + xz = yx + zx$. This implies that R is distributive and hence by (III) R^2 is additively commutative. Now $R^2 = R$ yields that R is also additively commutative. Hence R is a commutative ring. □

COROLLARY 2. *Let R be a d-g near-ring with unity satisfying any one of the conditions (1)–(5). Then R is a commutative ring.*

PROOF: Application of (IV) together with our theorem yields the result. □

Corollary 3. *Let R be a s-d-g near-ring satisfying any one of the conditions (1)–(5). Then R is a commutative ring.*

Proof: By the theorem, R is a commutative s-d-g near-ring in which every element is distributive and by (III) R^2 is additively commutative. Hence the additive group R^+ of the s-d-g near-ring is also commutative and R is a commutative ring. ∎

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