A Certain Linear Differential Equation.

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The Series

$$y = 1 + \frac{\Pi(\alpha)\Pi(\beta-m)}{\Pi(\alpha-n)\Pi(\beta)} \cdot \frac{x}{1!} + \frac{\Pi(\alpha)\Pi(\alpha+1)\Pi(\beta-m)\Pi(\beta-m+1)}{\Pi(\alpha-n)\Pi(\alpha-n+1)\Pi(\beta)\Pi(\beta+1)} \cdot \frac{x^2}{2!} + \dots \qquad (1)$$

if convergent, is a particular solution of the Differential Equation

$$\left[(\alpha)_n + n(\alpha)_{n-1}.xD + \frac{n.n-1}{2\,!}(\alpha)_{n-2}x^2D^2 + \quad \dots \quad \dots \right]y$$

$$-\frac{1}{x}\left[(\beta)_m.xD + m(\beta)_{m-1}x^2D^2 + \frac{m.m-1}{2!}(\beta)_{m-2}x^3D^3 + \quad \dots \right]y = 0 \qquad (2)$$

in which

$(\alpha)_n \equiv \dfrac{\Pi(\alpha)}{\Pi(\alpha-n)}$ and Π denotes Gauss's Π Function, D stands for $\dfrac{d}{dx}$.

The Differential Equation will contain a finite number or an infinite number of terms according as m and n are, or are not positive integers. When $m = n - 1$ and n is a positive integer, Equation (1) is identical with "(G)," Vol. XIII. p. 125, *Proceedings of Edinburgh Mathematical Society*.

Assume that

$$y = A_1 x^{m_1} + A_2 x^{m_2} + \quad \dots \quad \dots \quad + A_r x^{m_r} + \quad \dots \quad \dots \qquad (A)$$

is a possible form of solution of Equation (2). By differentiating r times in succession we obtain

$$\frac{d^r y}{dx^r} = A_1(m_1)_r x^{m_2 - r} + A_2(m_2)_r x^{m_2 - r} + \quad \dots \quad + A_r(m_r)_r x^{m_r - r} + \quad \dots$$

in which $(m_1)_r = \dfrac{\Pi(m_1)}{\Pi(m_1 - r)} = m_1.m_1 - 1.m_1 - 2 \quad \dots \quad m_1 - r + 1.$

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Substituting the values of the differential coefficients in the expression on the left side of equation (2) we have

$$(\alpha)_n\Big[A_1x^{m_1} + A_2x^{m_2} + \dots \dots + A_rx^{m_r} + \dots\Big]$$

$$+ n.(\alpha)_{n-1}\Big[A_1(m_1)_1x^{m_1} + A_2(m_2)_1x^{m_2} + \dots + A_r(m_r)_1x^{m_r} + \dots\Big]$$

$$+ \frac{n.n-1}{2!}(\alpha)_{n-2}\Big[A_1(m_1)_2x^{m_1} + A_2(m_2)_2x^{m_2} + \dots + A_2(m_r)_2x^{m_r} + \dots\Big]$$

$$\begin{matrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

$$+ \frac{n!}{r!\,n-r!}(\alpha)_{n-r}\Big[A_1(m_1)_rx^{m_1} + A_2(m_2)_rx^{m_2} + \dots + A_r(m_r)_rx^{m_r} + \dots\Big]$$

$$\begin{matrix} \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

$$- (\beta)_m\Big[A_1(m_1)_1x^{m_1-1} + A_2(m_2)_1x^{m_2-1} + \dots + A_r(m_r)_1x^{m_r-1} + \dots\Big]$$

$$- m(\beta)_{m-1}\Big[A_1(m_1)_2x^{m_1-1} + A_2(m_2)_2x^{m_2-1} + \dots + A_r(m_r)_2x^{m_r-1} + \dots\Big]$$

$$\begin{matrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

$$- \frac{m!}{r!\,m-r!}(\beta)_{m-r}\Big[A_1(m_1)_rx^{m_1-1} + A_2(m_2)_rx^{m_2-1} + \dots$$
$$\dots + A_r(m_r)_rx^{m_r-1} + \dots\Big]$$

$$\begin{matrix} \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

This expression must vanish identically and we see that a possible relation between the indices is

$$m_1 = m_2 - 1$$
$$m_2 = m_3 - 1$$
$$\dots\dots$$
$$\dots\dots$$
$$m_r = m_{r+1} - 1$$

Therefore $m_{r+1} = m_r + 1$ and $m_{r+1} = m_1 + r.$

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The coefficients of all the powers of x must vanish separately, the coefficient of x^{m_1-1} is

$$-A_1\left[(m_1)_1(\beta)_m+m(m_1)_2(\beta)_{m-1}+\ \dots\ +\frac{m!}{r!\,\overline{m-r}!}(m_1)_{r+1}(\beta)_{m-r}+\quad\dots\right]$$

which may be written

$$-A_1m_1\left[(\beta)_m+m(\beta)_{m-1}(m_1-1)_1+\dots+\frac{m!}{r!\,\overline{m-r}!}(\beta)_{m-r}(m_1-1)_r+\quad\dots\right] \tag{3}$$

This series consists of a finite number of terms, only when m is a positive integer; it is convergent however for all values of m provided that $\beta+m_1>0$; and Expression (3) reduces to $-A_1m_1(\beta+m_1-1)_m$ (*Proc. Lon. Math. Soc.*, Vol. XXVI. p. 285). Now A_1 is not zero and we see that a possible value of m_1 is zero, other values of m_1 are the roots in m_1 of

$$(\beta+m_1-1)_m=0$$

that is of $\underset{\kappa=\infty}{\mathrm{L}}\dfrac{(\beta+m_1-m)(\beta+m_1-m+1)\ \dots\ (\beta+m_1-m+\kappa)}{(\beta+m_1)(\beta+m_1+1)\quad\dots\quad(\beta+m_1+\kappa)}.\kappa^m=0$

$\therefore$ the other values of m_1 are

$$m-\beta,\ m-\beta-1,\ m-\beta-2,\ \text{etc.},\ \dots\ m-\beta-\kappa.$$

The coefficient of x^{m_r} is

$$A_r\left[(\alpha)_n+n(\alpha)_{n-1}(m_r)_1+\quad\dots\quad\dots+\frac{n!}{r!\,\overline{n-r}!}(\alpha)_{n-r}(m_r)_r+\quad\dots\right]$$

$$-A_{r+1}\left[(\beta)_m(m_{r+1})_1+m(\beta)_{m+1}(m_{r+1})_2+\dots+\frac{m!}{r!\,\overline{m-r}!}(\beta)_{m-r}(m_{r+1})_r+\dots\right] \tag{4}$$

which may be written

$$A_r(\alpha+m_r)_n-A_{r+1}m_{r+1}(\beta+m_{r+1}-1)_m\quad\dots\quad\dots\quad\dots\quad\dots \tag{5}$$

This expression must vanish identically, therefore

$$A_{r+1}=\frac{A_r}{m_{r+1}}\cdot\frac{(\alpha+m_r)_n}{(\beta+m_{r+1}-1)_m}=\frac{A_r}{m_{r+1}}\,\frac{(\alpha+m_r)_n}{(\beta+m_r)_m} \tag{6}$$

The two series in (4) are respectively convergent if

$$\alpha+m_r+1\quad>0$$
$$\beta+m_r+1\quad>0$$

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Expression (6) which shows the relation between successive coefficients of series (A) is only valid subject to the conditions

$$\left.\begin{array}{ll} \alpha+m_r+1 & >0 \\ \beta+m_r+1 & >0 \end{array}\right\} \quad \ldots \quad \ldots \quad (7)$$

Now $m_r = m_1 + r - 1$ and the possible values of m_1 are zero and $m - \beta - s$, where s is zero or any positive integer.

When $m_1 = 0$

The conditions (7) become $\alpha + r > 0$

$$\beta + r > 0$$

and since the least value of r is 1

$$\alpha + 1 > 0$$

$$\beta + 1 > 0$$

Subject to these conditions

$$y = A_1\left[1 + \frac{(\alpha)_n}{(\beta)_m}\cdot\frac{x}{1!} + \frac{(\alpha)_n(\alpha+1)_n}{(\beta)_m(\beta+1)_m}\cdot\frac{x^2}{2!} + \quad \ldots \quad \ldots \right.$$

$$\left. \ldots \quad + \frac{(\alpha)_n(\alpha+1)_n \;\ldots\; (\alpha+r)_n}{(\beta)_m(\beta+1)_m \;\ldots\; (\beta+r)_m}\cdot\frac{x^{r+1}}{\overline{r+1}!} + \quad \ldots \right] (8)$$

is a solution of the Differential Equation (2) provided the series on the Right side of (8) is convergent.

When $m_1 = m - \beta - s$, the conditions become

$$\alpha + m - \beta - s + r > 0$$

$$\beta + m - \beta - s + r > 0$$

Now the least value of r is 1, therefore

$$\left.\begin{array}{l} s < \alpha - \beta + m + 1 \\ s < \qquad m + 1 \end{array}\right\} \quad \ldots \quad \ldots \quad (9)$$

and subject to these conditions

$$y = A_s x^{m-\beta-s}\left[1 + \frac{(\alpha+m-\beta-s)_n}{(m-s)_m}\cdot\frac{x}{(m-\beta-s+1)} + \quad \ldots \right.$$

$$+ \frac{(\alpha+m-\beta-s)_n \ldots (\alpha+m-\beta-s+r)_n}{(m-s)_m \;\ldots\; (m-s+r)_m}\cdot\frac{x^{r+1}}{(m-\beta-s+1)\ldots(m-\beta-s+r+1)} + \ldots$$

$$\left. \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \right] \quad \ldots \quad (10)$$

s being zero or any positive integer subject to the conditions (9) is a solution of the Differential Equation (2).

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If $\alpha = n$ and $\beta = m$.

The Series (8) becomes

$$y = A_1\left[1 + \frac{(n)_n}{(m)_m}\cdot\frac{x}{1!} + \frac{(n)_n(n+1)_n}{(m)_m(m+1)_m}\cdot\frac{x^2}{2!} + \quad \dots \quad \dots \quad \dots\right]$$

$$= A_1\left[1 + \frac{\Pi(n)}{\Pi(m)}\cdot\frac{x}{1!} + \frac{\Pi(n).\Pi(n+1)}{\Pi(m)\Pi(m+1)}\cdot\frac{x^2}{2!} + \dots \quad \dots \quad \dots\right] \quad (11)$$

The Series (10) becomes

$$y = A_s x^{-s}\left[1 + \frac{(n-s)_n}{(m-s)_m}\cdot\frac{x}{1-s} + \frac{(n-s)_n(n-s+1)_n}{(m-s)_m(m-s+1)_m}\cdot\frac{x^2}{\overline{1-s}.\overline{2-s}} + \dots\right] \quad (12)$$

s being zero or any positive integer subject to the conditions

$$s < m + 1$$

$$s < n + 1$$

These series are solutions of the Differential Equations

$$\Pi(n)\left[1 + \frac{n}{(1!)^2}xD + \frac{n.\overline{n-1}}{(2!)^2}x^2D^2 + \quad \dots \quad \dots\right]y$$

$$-\frac{1}{x}\Pi(m)\left[xD + \frac{m}{(1!)^2}x^2D^2 + \frac{m.\overline{m-1}}{(2!)^2}x^3D^3 + \quad \dots \quad \dots\right]y = 0 \quad (13)$$

a particular case of (2) when $\alpha = n$, $\beta = m$.

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