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THE COEFFICIENT RING OF A PRIMITIVE GROUP RING

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All rings are associative with unity. A ring R is prime if $xRy \neq 0$ whenever x and y are nonzero. A ring R is (left) primitive if there exists a faithful irreducible left R-module.

If the group ring $R[G]$ is primitive, what can we say about R? First, since every primitive ring is prime, we know that R is prime, by the following

THEOREM 1 (Connell [**1**, 675]). *The group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroup.*

We cannot, however, conclude that R is primitive. This was shown recently by Formanek with the following

THEOREM 2 (Formanek [**3**]). *If R is a domain (not necessarily commutative) and $G = A * B$ is a non-trivial free product of groups (except $G = Z_2 * Z_2$), and $|G| \geqq |R|$, then $R[G]$ is primitive.*

Of interest here is the cardinality condition, $|G| \geqq |R|$. If R is a field, then the cardinality condition is unnecessary. Formanek showed, however, that the condition is necessary for certain commutative domains.

In this paper we generalize Theorem 2 by showing that R need not be a domain. On the other hand, we give an example of a prime semiprimitive ring R such that $R[G]$ is not primitive, where G is any group.

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1. Strongly prime rings.

Definition. A ring R is said to be (left) *strongly prime* (SP) if for all $0 \neq r \in R$, there exists a finite set $S(r) \subset R$ such that for all $0 \neq t \in R$ we have $tS(r)r \neq \{0\}$.

The set $S(r)$ is a (left) *insulator* of r. If there is an integer n such that every nonzero element has an n-element insulator, then R is said to be *bounded strongly prime.*

Every SP ring is prime and every prime ring may be embedded in an SP ring. Left SP does not imply right SP. These and other properties are discussed in [**6**] and [**7**].

THEOREM 3. *Domains, simple rings, and prime left Goldie rings are all SP.*

Proof. That a domain is SP is obvious. If R is simple and $0 \neq r \in R$, then

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$1 = s_1rt_1 + \ldots + s_nrt_n$, for some $s_i, t_i \in R$. We then let $S(r) = \{s_1, \ldots, s_n\}$. Suppose R is a prime left Goldie ring. By Goldie's Theorem and the Faith-Utumi Theorem, there exists a positive integer n, a division ring D, a left order C of D, and a complete set of matrix units $\{e_{ij}\}$, such that

$$\sum Ce_{ij} \subset R \subset \sum De_{ij}.$$

Let c be a nonzero element of C and let $r = \sum r_{ij}e_{ij} \in R$ and $t = \sum t_{ij}e_{ij} \in R$ be nonzero elements, with $r_{uv} \neq 0$ and $t_{xy} \neq 0$. Then $t_{xy}cr_{ux} \neq 0$ occurs as a component in $tce_{yu}r$. We then let $S(r) = \{ce_{ij}\}$.

In the above cases the rings are SP on both sides.

THEOREM 4 [**5**; **6**]. *If R is bounded SP, but not every element has a one-element insulator, then R is Goldie, (hence two-sided SP).*

2. Generalization of Formanek's theorem.

Definition. Let R be a ring and let $\{J_i\}$ be the set of nonzero two-sided ideals of R. R is said to be (left) *weakly primitive* if it has a proper left ideal M with the following property: for every J_i there exists a finite set $S(J_i) \subset J_i$ such that $\{r \in R | r \cdot S(J_i) \subset M\} \subset M$.

We now come to the main theorem of this paper.

THEOREM 5. *For any ring R, the following are equivalent:*

(1) *R is weakly primitive;*

(2) *if X is a set of indeterminates with $|X| \geqq |R|$, then the free algebra $R[X]$ is left primitive;*

(3) *if $G = A * B$ is a free product of groups A and B, $|A| = \infty$, $|B| > 1$, with $|G| \geqq |R|$, then the group ring $R[G]$ is left primitive;*

(4) *there exists a monoid G such that the monoid ring $R[G]$ is left primitive.*

Before proving this theorem, we state a

LEMMA. *A ring R is (left) primitive if and only if R has a proper left ideal M comaximal with every nonzero two-sided ideal of R, i.e. if $(0) \neq J$ is an ideal, then $M + J = R$.*

Proof. (1) $\Rightarrow$ (3). We may assume that $|A| \geqq |B|$. Thus $|A| = |G| \geqq |R|$, hence $|R[G]| = |A|$.

An element $g = a_1b_1a_2b_2 \ldots a_nb_na_{n+1}$ of G, in reduced form, is said to have length $2n + 1$, denoted by $l(g)$. $l(g)$ is similarly defined for words q beginning with b and or ending with b, etc. A product of two elements of G, g and g', is said to be pure if $l(g \cdot g') = l(g) + l(g')$. For completeness, we define $l(1) = 0$.

From each nonzero ideal J of $R[G]$, choose $\alpha = \alpha(J) \in J$, $(\alpha \neq 0)$, such that α has minimal support. Suppose

(1) $\alpha = \sum r(\alpha, g)g$,

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where $r(\alpha, g) \in R$ and $g \in G$. Let $g(\alpha)$ be an element of maximal length in the support of α. Thus $g(\alpha)$ has coefficient $r(\alpha, g(\alpha))$. Consider the R-ideal $I(\alpha) = \langle r(\alpha, g(\alpha))\rangle$, and suppose that $S(I(\alpha)) = \{s_1, s_2, \ldots, s_n\}$. (Here we are assuming that R is weakly primitive, with a left ideal M satisfying the conditions of the definition.) For each $s_j \in S(I(\alpha))$, let $\alpha(s_j)$ be an element of $R\alpha R$ with the coefficient of $g(\alpha)$ being s_j. Thus, using our notation, $\alpha(s_j) = \sum r(\alpha(s_j), g)g$, and the support of $\alpha(s_j)$ is the same as the support of α.

Fix $b \in B - \{1\}$, and let W: $(R[G] - \{0\}) \times N \rightarrow A - \{1\}$, be α bijection. Given α, chosen above, let $T(\alpha, j)$ equal

$$\begin{aligned} &bW(\alpha, j_1)b\alpha(s_j)bW(\alpha, j_1)b + W(\alpha, j_2)bW(\alpha, j_2)b\alpha(s_j)bW(\alpha, j_2)bW(\alpha, j_2) \\ &\quad + bW(\alpha,j_3)b\alpha(s_j)W(\alpha,j_3)bW(\alpha,j_3) + W(\alpha,j_4)bW(\alpha,j_4)b\alpha(s_j)W(\alpha,j_4)b \\ &\quad + bW(\alpha,j_5)\alpha(s_j)bW(\alpha,j_5)bW(\alpha,j_5) + W(\alpha,j_6)bW(\alpha,j_6)\alpha(s_j)bW(\alpha,j_6)b \\ &\quad + bW(\alpha,j_7)\alpha(s_j)W(\alpha,j_7)bW(\alpha,j_7)b \\ &\quad + W(\alpha, j_8)bW(\alpha, j_8)\alpha(s_j)W(\alpha, j_8)bW(\alpha, j_8), \end{aligned}$$

where $W(\alpha, j_k)$ is chosen such that it is not equal to any factor of the reduced form of any element in the support of α, $(k = 1, 2, \ldots, 8)$, and $j_1 < j_2 < \ldots < j_8 < (j + 1)_1$. Finally, let $H(\alpha) = \sum_j T(\alpha, j)$, a finite sum.

Note that if $\beta \in R[G]$, and $r \in R$ occurs as the coefficient of some term in the expansion of $\beta \cdot T(\alpha, j)$, then r is the coefficient of a pure product in this expansion.

Let M' be the left ideal of $R[G]$ generated by the set $\{H(\alpha) + 1\}$, where we have chosen some α from each ideal $(\neq (0))$ in $R[G]$.

In order to prove that $R[G]$ is primitive, it is sufficient to show that M' is proper, for M' is obviously comaximal with every nonzero two-sided ideal of $R[G]$.

If M' is not proper, then there exists $\beta_1, \beta_2, \ldots, \beta_m \in R[G]$ such that

$$(2)\quad \sum_{i=1}^{m} \beta_i[H(\alpha_i) + 1] = 1.$$

Since $1 \notin M$ (by the definition of weakly primitive), then we must have either:

(3) $r(\beta_i, y) \cdot r(\alpha_i(s_j), z) \notin M$, for some $i = 1, 2, \ldots, m$, and some $s_j \in S(I(\alpha_j))$,

or

(3′) $r(\beta_i, y) \notin M$, for some $i = 1, 2, \ldots, m$.

If (3′) occurs, then $r(\beta_i, y) \cdot r(\alpha_i(s_j), g(\alpha_i)) \notin M$, for some j, as the set $\{r(\alpha_i(s_j), g(\alpha_i))\}_{j=1}^{n}$ equals the set $S(J_i)$. Hence we may assume that (3) holds. We choose y and z in (3) so that $l(y) + l(z)$ is maximal in all the products in the expansion of (2), for which (3) holds. We now have $r = r(\beta_i, y) \cdot r(\alpha_i(s_j), z)$ as the coefficient, in the expansion of (2), of a group element x, with $l(x) = l(y) + l(z) + 6$.

In order to arrive at a contradiction, we first make two observations for arbitrary products occurring in the expansion of (2).

First, if r' is the coefficient of a group element x' in the expansion of $\beta_t \cdot T(\alpha_{i'}, j')$, then r' is the coefficient of x'' (in the same expansion), where x'' ends in $W(\alpha_{i'}, j_k')b$ or $W(\alpha_{i'}, j_k')$, and $l(x'') \geqq l(x')$. Therefore, by the maximality of $l(x)$, with respect to property (3), $x' = x$, with $(i, j) \neq (i', j')$, implies that $r' \in M$.

Second, if $r(\beta_i, y') \cdot r(\alpha_i(s_j), z') \notin M$ is the coefficient, in the expansion of $\beta_t \cdot T(\alpha_i, j)$, of a group element x', and $l(x')$ is maximal with this coefficient, then $l(g') = l(y') + l(z') + 6$. Thus we may suppose that

$$x = y - z - W(\alpha_i, j_k),$$

or

$$x = y - z - W(\alpha_i, j_k) \cdot b,$$

and

$$x' = y' - z' - W(\alpha_i, j_{k'}),$$

or

$$x' = y' - z' - W(\alpha_i, j_{k'}) \cdot b,$$

where both x and x' are in reduced form; hence $x = x'$ implies $k = k'$. Let us suppose, therefore, that (for example)

$$x = ybW(\alpha_i, j_1)bzbW(\alpha_i, j_1)b, \quad \text{and}$$

$$x' = y'bW(\alpha_i, j_1)bz'bW(\alpha_i, j_1)b,$$

where both are in reduced form, and neither z nor z' contains $W(\alpha_i, j_1)$ as a factor in their reduced form. Then, if $x = x'$,we must have $y = y'$ and $z = z'$.

By the second observation we see that x occurs as an element in the support of $\beta_i T(\alpha_i, j)$ and has coefficient $r \notin M$. (Here we use the maximality of $l(y) + l(z)$, hence of $l(x)$.) Now rx must cancel with a sum of other terms in the expansion of (2), therefore, by the maximality of $l(x)$ with respect to (3), along with the first observation, we see that $r'x$, $(r' \notin M)$, does not occur in the expansion of $\beta_{i'} \cdot T(\alpha_{i'}, j')$, for $(i, j) \neq (i', j')$. We conclude that x is in the support of $\beta_{i'}$ for some i', and $r(\beta_{i'}, x) \notin M$. Then for some j',

(4) $\quad r(\beta_{i'}, x) \cdot r(\alpha_{i'}(s_{j'}), g(\alpha_{i'})) \notin M.$

However, $l(y) + l(z) < l(x) \leqq l(x) + l(g(\alpha_{i'}))$, thus (4) contradicts the fact that y and z were chosen such that $l(y) + l(z)$ was maximal with respect to (3). This completes the proof.

(1) $\Rightarrow$ (2) The proof is similar to the above proof (and, in fact, is easier).

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) The proof of these is trivial.

(4) $\Rightarrow$ (1) Suppose $R[G]$ is left primitive; hence, there is a proper left ideal M' of $R[G]$ which is comaximal with every nonzero two-sided ideal of $R[G]$. Let $(0) \neq J$ be an ideal of R, and put $J' = J[G]$. We then have $\alpha(J) \in J'$ such that $\alpha(J) - 1 \in M'$. Let $S(J)$ be the set of coefficients of the support of $\alpha(J)$. If $rS(J) \subset M = M' \cap R$, (where $r \in R$), then $r\alpha(J) \in M'$; since

$\alpha(J) - 1 \in M'$, we must have $r \in M$. This completes the proof of the theorem. By a similar argument, $R[G]$ $WP \Rightarrow R$ WP.

COROLLARY 1. *If R is left strongly prime, then R is the coefficient ring of a primitive group ring.*

Proof. A ring is left strongly prime if and only if, for every nonzero two-sided ideal J, we have a finite set $S(J) \subset J$ such that $\text{ann}_l S(J) = (0)$. Thus (0) satisfies the conditions of the definition.

COROLLARY 2. *Every prime ring is a subring of the coefficient ring of a primitive group ring.*

Proof. Every prime ring is a subring of a strongly prime ring [**7**].

COROLLARY 3. *If a regular ring R is a coefficient ring of a primitive group ring, then R is primitive.*

Proof. Suppose $(0) \neq J$ is an ideal of R, and let $S(J) = \{s_1, s_2, \ldots, s_k\}$. Since R is regular, $s_1R + s_2R + \ldots + s_kR = eR$, for some idempotent e. Therefore, $e \in J$, and $(1 - e)S(J) = \{0\}$, hence $1 - e \in M$, and so M is a proper left ideal of R comaximal with every nonzero two-sided ideal of R.

Thus the question of whether every prime regular ring is the coefficient ring of a primitive group ring is equivalent to a question of Kaplansky: Is every prime regular ring primitive? Partial solutions to this problem have been given in [**2**] and [**4**].

COROLLARY 4. *If R is bounded left strongly prime, and the right zero-divisors of R form a right ideal, then R is right weakly primitive.*

Proof. We may assume that R is not right SP, hence every $r \in R$, $(r \neq 0)$ has a one-element left insulator $s(r)$. Let M be the proper left ideal consisting of the right zero-divisors; we will show that M satisfies the conditions of the definition. If $J(\neq (0))$ is an ideal of R, choose $0 \neq t \in J$, and let $S(J) = s(t)t$. If $S(J)r \in M$, then $S(J)r$ is a right zero-divisor, whence $uS(J)r = 0$, for some $u \neq 0$. But $uS(J) \neq 0$, hence r is a right zero-divisor, i.e. $r \in M$.

3. Example. We give an example of a prime semiprimitive ring R such that $R[G]$ is not primitive, where G is any group.

Let $F = Z_2[X, Y_i]$, $i = 1, 2, \ldots$, be the free Z_2-algebra in noncommuting variables. For a given monomial

$$m = X^{i_1} Y_{j_1} X^{i_2} \ldots Y_{j_n} X^{i_{n+1}}, \; i \geqq 0, j \geqq 1,$$

repetitions allowed, define

$$\begin{aligned} c(m) &= (\max j_k) \text{ (times the number of times } Y_{\max jk} \text{ appears)}, \\ &\quad \text{if } m \text{ has a } Y \text{ term}, \\ &= 0, \text{ otherwise}, \\ d(m) &= \textstyle\sum i_k = \text{degree of } X \text{ in } m. \end{aligned}$$

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Let I be the ideal generated by all monomials m such that $d(m) > c(m) \geqq 1$. Set $R = F/I$. R was given in [**10**] as an example of a semiprimitive ring with nonzero singular ideal.

THEOREM 5 (Osofsky [**10**]). *R is a prime semiprimitive ring.*

THEOREM 6. *Let G be any group. Then $R[G]$ is not primitive.*

Proof. Assume that $R[G]$ is primitive, and let M be a proper left ideal of $R[G]$ comaximal with every nonzero two-sided ideal. By hypothesis, there exists $a \in (X)$ such that $a - 1 \in M$. Choose h so that if Y_j occurs in any monomial in a, then $h > j$. Then there exists $b \in (Y_h)$ such that $b - 1 \in M$. Let n be a positive integer and consider $a^n b$. By our choice of h, $(\max j_k)$ (the number of times $Y_{\max j_k}$ appears) is independent of n, in any monomial in $a^n b$. However, in such a monomial, X occurs at least n times, and so for sufficiently large n, $a^n b = 0$. Then

$$-1 = a^n(b - 1) + \left[\sum_{i=0}^{n-1} a^i\right](a - 1) \in M,$$

contradicting the fact that M is proper.

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