Proceedings of the Edinburgh Mathematical Society (2013) **56**, 853–871
DOI:10.1017/S0013091513000606

MAXIMAL REGULARITY FOR DEGENERATE DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN PERIODIC VECTOR-VALUED FUNCTION SPACES

CARLOS LIZAMA[1] AND RODRIGO PONCE[2]

[1] *Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Casilla 307-Correo 2, Santiago, Chile* (carlos.lizama@usach.cl)
[2] *Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile* (rponce@inst-mat.utalca.cl)

(Received 4 October 2011)

Abstract Let A and M be closed linear operators defined on a complex Banach space X and let $a \in L^1(\mathbb{R}_+)$ be a scalar kernel. We use operator-valued Fourier multipliers techniques to obtain necessary and sufficient conditions to guarantee the existence and uniqueness of periodic solutions to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,\mathrm{d}s + f(t), \quad t > 0,$$

with initial condition $Mu(0) = Mu(2\pi)$, solely in terms of spectral properties of the data. Our results are obtained in the scales of periodic Besov, Triebel–Lizorkin and Lebesgue vector-valued function spaces.

Keywords: differential equations with delay; operator-valued Fourier multipliers; R-boundedness; UMD spaces; Besov vector-valued spaces; Lebesgue vector-valued spaces

2010 *Mathematics subject classification:* Primary 35K90; 45N05; 45D05
Secondary 34G10; 34K13

1. Introduction

In this paper, we study maximal regularity in Lebesgue, Besov and Triebel–Lizorkin vector-valued function spaces for the following class of differential equation with infinite delay:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,\mathrm{d}s + f(t), \quad 0 \leqslant t \leqslant 2\pi, \tag{1.1}$$

where $(A, D(A))$ and $(M, D(M))$ are (unbounded) closed linear operators defined on a Banach space X, with $D(A) \subseteq D(M)$, $a \in L^1(\mathbb{R}_+)$ a scalar-valued kernel and f an X-valued function defined on $[0, 2\pi]$.

The model (1.1) corresponds to problems related to viscoelastic materials, that is, materials whose stresses at any instant depend on the complete history of strains that the material has undergone (see [**21**]) or heat conduction with memory. For more details, see, for example, [**14**, **15**, **23**].

The recent linear theory of maximal regularity is not only important in its own right, but it is also the indispensable basis for the theory of nonlinear evolution equations (see, for example, [**1**, **13**, **23**] and references therein). In the case when $M = I$ (the identity in X) and $a \equiv 0$, (1.1) with periodic initial conditions has been studied by Arendt, Bu and Kim, and characterizations of the maximal regularity in Lebesgue, Besov and Triebel–Lizorkin vector-valued function spaces have been obtained using the resolvent set of A (see [**3**, **4**, **8**]).

On the other hand, characterizations of maximal regularity for (1.1) when $M = I$ and $a \in L^1(\mathbb{R})$ have been obtained by Keyantuo and Lizama [**18**] in Lebesgue and Besov vector-valued function spaces and by Bu and Fang [**7**] in Triebel–Lizorkin vector-valued spaces. We note that periodic solutions have also been studied by other authors, for example, by Burton and Zhang [**11**] using topological methods.

Characterizations of maximal regularity in these vector-valued function spaces for the degenerate abstract equation (1.1) with periodic initial conditions

$$Mu(0) = Mu(2\pi) \tag{1.2}$$

and $a \equiv 0$ have been studied recently by the authors [**22**]. The method used in [**22**] is based on the results given in [**3**, **4**, **8**], for operator-valued Fourier multipliers in Lebesgue, Besov and Triebel–Lizorkin vector-valued function spaces. It is worthwhile mentioning that this method enables us to improve and extend results on degenerate abstract equations obtained previously in the literature (cf. [**5**, **22**]).

In this paper, we apply the same method to obtain characterizations of maximal regularity for (1.1) in the above-mentioned vector-valued function spaces. The advantage of our approach is clear. We recover, as special cases, the results in [**3**, **4**, **7**, **8**, **18**, **22**]. In addition, we are also able to improve the results in [**22**], where the closedness of the operator $\mathrm{i}kM - A$ is assumed, for all $k \in \mathbb{Z}$, to prove boundedness of $(\mathrm{i}kM - A)^{-1}$. Indeed, we give a simple argument to show that this condition is not necessary under the presence of maximal regularity (see the proof of Theorem 3.4).

It is remarkable that in the characterizations that we obtain, no conditions on the commutativity of operators A and M, or on the existence of bounded inverses of A or M, are needed. Also, in the case of periodic Besov and Triebel–Lizorkin function spaces, no geometrical assumption on the underlying Banach space X is needed.

The plan of the paper is the following: after some preliminaries in § 2, assuming that X have the unconditional martingale difference property (UMD), we characterize in § 3 the uniqueness and existence of a strong L^p-solution for the problem (1.1), (1.2) solely in terms of a property of R-boundedness for the sequence of operators $\mathrm{i}kM(\mathrm{i}kM - (1 + \tilde{a}(\mathrm{i}k))A)^{-1}$. Here the tilde denotes the Laplace transform of $a(t)$. In § 4, we obtain a characterization in the context of Besov spaces. We notice that, as a particular case of this characterization, the following simple condition to guarantee the existence and uniqueness of solutions in Hölder spaces $C^s((0, 2\pi); X)$, $0 < s < 1$, in general Banach spaces X, is obtained.

Theorem 1.1. *Let $s > 0$ and $A\colon D(A) \subseteq X \to X$, $M\colon D(M) \subseteq X \to X$ be closed linear operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$, the sequence $\{\tilde{a}(\mathrm{i}k)\}_{k\in\mathbb{Z}}$ is 2-regular and $(\mathrm{i}kM-(1+\tilde{a}(\mathrm{i}k))A)$ are closed operators for all $k \in \mathbb{Z}$. Then, the following assertions are equivalent:*

(i) *for every $f \in C^s((0,2\pi);X)$ there is a unique strong C^s-solution of (1.1) such that $Mu(0) = Mu(2\pi)$;*

(ii) *$(\mathrm{i}kM - (1+\tilde{a}(\mathrm{i}k))A)^{-1}$ exists for all $k \in \mathbb{Z}$ and*

$$\sup_{k\in\mathbb{Z}} \|\mathrm{i}kM(\mathrm{i}kM - (1+\tilde{a}(\mathrm{i}k))A)^{-1}\| < \infty.$$

In § 5 we give the corresponding characterization in case of the scale of Triebel–Lizorkin vector-valued spaces. The difference with the scale of Besov vector-valued spaces is only that we need more regularity of the sequence $\tilde{a}(\mathrm{i}k)$. In § 6, we apply our results to two concrete examples.

2. Preliminaries

Given $1 \leqslant p < \infty$, we denote by $L^p_{2\pi}(\mathbb{R}, X)$ the space of all 2π-periodic Bochner measurable X-valued functions f, such that the restriction of f to $[0, 2\pi]$ is p-integrable. For a function $f \in L^1_{2\pi}(\mathbb{R}, X)$ we denote by $\hat{f}(k)$ the kth Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt} f(t)\,\mathrm{d}t$$

for all $k \in \mathbb{Z}$. We remark that the Fourier coefficients determine the function f, that is, $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ if and only if $f(t) = 0$ almost everywhere (a.e.). Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote its domain by $D(A)$ and its resolvent set by $\rho(A)$. We denote by $[D(A)]$ the domain of A equipped with the graph norm.

We begin with some preliminaries about operator-valued Fourier multipliers. More information can be found in [**4**] in the periodic case; for the non-periodic case, see, for example, [**25**].

Definition 2.1 (Arendt and Bu [3]). For $1 \leqslant p < \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p-multiplier if, for each $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exists $u \in L^p_{2\pi}(\mathbb{R}, Y)$ such that

$$\hat{u}(k) = M_k\hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

It follows from the uniqueness theorem of the Fourier series that u is uniquely determined by f.

For $j \in \mathbb{N}$, denote by r_j the jth Rademacher function on $[0, 1]$, i.e. $r_j(t) = \mathrm{sgn}(\sin(2^j\pi t))$, and for $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \to r_j(t)x$.

Definition 2.2. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded if there is a constant $C_p > 0$, $p \in [1, \infty)$, such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$, $j = 1, \dots, N$ the inequality

$$\left\| \sum_{j=1}^{N} r_j \otimes T_j x_j \right\|_{L^p((0,1);Y)} \leqslant C_p \left\| \sum_{j=1}^{N} r_j \otimes x_j \right\|_{L^p((0,1);X)} \tag{2.1}$$

is valid.

If (2.1) holds for some $p \in [1, \infty)$, then it holds for all $p \in [1, \infty)$. The smallest C_p in (2.1) is called the R-bound of $\mathcal{T}$; we denote it by $R_p(\mathcal{T})$.

We note that large classes of classical operators are R-bounded (see [**16**] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this paper.

Remark 2.3. Several properties of R-bounded families can be found in [**13**]. For the reader's convenience, we summarize here some results from [**13**, § 3].

(a) If $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded, then it is uniformly bounded, with

$$\sup\{\|T\| \colon T \in \mathcal{T}\} \leqslant R_p(\mathcal{T}).$$

(b) The definition of R-boundedness is independent of $p \in [1, \infty)$.

(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

(d) Let X, Y be Banach spaces and let $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X, Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S \colon T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leqslant R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and let $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST \colon T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leqslant R_p(\mathcal{S})R_p(\mathcal{T})$.

(f) Let X, Y be Banach spaces and let $\mathcal{T} \subset \mathcal{B}(X, Y)$ be R-bounded. If $\{\alpha_k\}_{k \in \mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T \colon T \in \mathcal{T}, k \in \mathbb{Z}\}$ is R-bounded.

Proposition 2.4 (Arendt and Bu [3]). *Let X be a Banach space and let $\{M_k\}_{k \in \mathbb{Z}}$ be an L^p-multiplier, where $1 \leqslant p < \infty$. Then, the set $\{M_k \colon k \in \mathbb{Z}\}$ is R-bounded.*

A Banach space X is said to be UMD if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwarz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV\left(\frac{1}{t}\right) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [**6**] and Burkholder [**9**]. Some examples of UMD spaces include the Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$, and Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD space. Moreover, a UMD space is reflexive and therefore, $L^1(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ (in the infinite-dimensional case) and $C^s([0, 2\pi]; X)$ are not UMD. More information on UMD spaces can be found in [**6**, **9**, **10**].

We recall the following theorem, due to Arendt and Bu [**3**, Theorem 1.3], in the context of UMD spaces.

Theorem 2.5 (Arendt and Bu [3]). *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X, Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

We shall need the following lemmas.

Lemma 2.6 (Arendt and Bu [3]). *Let $f, g \in L^p_{2\pi}(\mathbb{R}; X)$, where $1 \leqslant p < \infty$ and A is a closed operator on a Banach space X. Then, the following assertions are equivalent:*

(i) *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e.;*

(ii) *$\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$.*

Lemma 2.7 (Lizama and Ponce [22]). *Let M be a closed operator, let $u \in L^p_{2\pi}(\mathbb{R}; [D(M)])$ and $u' \in L^p_{2\pi}(\mathbb{R}; X)$ for $1 \leqslant p < \infty$. Then, the following assertions are equivalent:*

(i)
$$\int_0^{2\pi} (Mu)'(t)\,\mathrm{d}t = 0$$
and there exist $x \in X$ such that
$$Mu(t) = x + \int_0^t (Mu)'(s)\,\mathrm{d}s$$
a.e. on $[0, 2\pi]$;

(ii) *$\widehat{(Mu)'}(k) = \mathrm{i}kM\hat{u}(k)$ for all $k \in \mathbb{Z}$.*

Let a be a complex valued function. We define the set

$$\rho_{M,a}(A) = \{\lambda \in \mathbb{C} \colon (\lambda M - (1 + a(\lambda))A) \colon D(A) \cap D(B) \to X \\ \text{is invertible and } (\lambda M - (1 + a(\lambda))A)^{-1} \in \mathcal{B}(X)\},$$

and denote by $\sigma_{M,a}(A)$ the complementary set $\mathbb{C} \setminus \rho_{M,a}(A)$. If $M = I$ is the identity operator on X and $a \equiv 0$, we simply denote the set $\rho_{M,a}(A)$ by $\rho(A)$, and as usual we call this set the resolvent set of A. Denote by $\tilde{a}(\lambda)$ the Laplace transform of a. In what follows, we always assume that $\tilde{a}(\mathrm{i}k)$ exists for all $k \in \mathbb{Z}$.

Henceforth, we use the following notation:

$$a_k := \tilde{a}(\mathrm{i}k),$$

and we assume that $a_k \neq -1$ for all $k \in \mathbb{Z}$.

Remark 2.8. Note that by the Riemann–Lebesgue lemma we have that the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{1/(\alpha + a_k)\}_{k\in\mathbb{Z}}$, $\alpha \neq a_k (k \in \mathbb{Z})$, are bounded.

Finally, from [**20**] we recall the concept of n-regularity for $n = 1, 2, 3$. The general notion of n-regularity is the discrete analogue for the notion of n-regularity related to Volterra integral equations (see [**23**, Chapter I, § 3.2]).

Definition 2.9. A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C} \setminus \{0\}$ is said to be

(i) 1-regular if the sequence
$$\left\{k\frac{(c_{k+1} - c_k)}{c_k}\right\}_{k\in\mathbb{Z}}$$
is bounded,

(ii) 2-regular if it is 1-regular and the sequence
$$\left\{k^2\frac{(c_{k+1} - 2c_k + c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$$
is bounded,

(iii) 3-regular if it is 2-regular and the sequence
$$\left\{k^3\frac{(c_{k+2} - 3c_{k+1} + 3c_k - c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$$
is bounded.

Note that if $\{c_k\}_{k\in\mathbb{Z}}$ is 1-regular, then $\lim_{|k|\to\infty} c_{k+1}/c_k = 1$. For more details of n-regularity sequences, see [**20**].

3. Maximal regularity on vector-valued Lebesgue spaces

To characterize the maximal regularity, we begin with the study of the relation between multipliers and R-boundedness of a sequences of operators. Consider the problem

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}Mu(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,\mathrm{d}s + f(t), \quad 0 \leqslant t \leqslant 2\pi, \\ &Mu(0) = Mu(2\pi), \end{aligned}\right\} \tag{3.1}$$

where $(A, D(A))$ and $(M, D(M))$ are closed linear operators on X, $D(A) \subseteq D(M)$, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel and $f \in L^p_{2\pi}(\mathbb{R}, X)$, $p \geqslant 1$.

https://doi.org/10.1017/S0013091513000606 Published online by Cambridge University Press

From [**22**], we recall that, for a given closed operator M, and $1 \leqslant p < \infty$, the set $H^{1,p}_{\mathrm{per},M}(\mathbb{R};[D(M)])$ is defined by

$$H^{1,p}_{\mathrm{per},M}(\mathbb{R};[D(M)]) = \{u \in L^p_{2\pi}(\mathbb{R};[D(M)]) \colon \exists v \in L^p_{2\pi}(\mathbb{R};X), \text{ such that } \hat{v}(k) = \mathrm{i}kM\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

When $M = I$, we denote this by $H^{1,p}_{\mathrm{per}}(\mathbb{R};X)$ (see [**3**]).

Lemma 3.1. *Let X be a UMD space. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, $\{I/(1+a_k)\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. By Remarks 2.8 and 2.3 (f), $\{I/(1+a_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Moreover,

$$k\left(\frac{1}{1+a_{k+1}} - \frac{1}{1+a_k}\right) = -k\left(\frac{a_{k+1}-a_k}{a_k}\right)a_k\frac{1}{1+a_{k+1}}\frac{1}{1+a_k}.$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, we conclude the proof of the lemma by using Remark 2.8 and Theorem 2.5. □

The following proposition is an extension of [**22**, Proposition 3.2].

Proposition 3.2. *Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Let $A\colon D(A) \subseteq X \to X$ and $M\colon D(M) \subseteq X \to X$ be linear closed operators defined on a UMD space X. Suppose that $D(A) \subseteq D(M)$. Then, the following assertions are equivalent:*

(i) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{\mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;*

(ii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{\mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. Define $N_k := (\mathrm{i}kM-(1+a_k)A)^{-1}$ and $M_k := \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}$, $k \in \mathbb{Z}$. By the closed graph theorem we can show that if $\mathrm{i}k \in \rho_{M,\tilde{a}}(A)$, then M_k are bounded operators for each $k \in \mathbb{Z}$. By Proposition 2.4 it follows that (i) implies (ii). Conversely, by Theorem 2.5, it is sufficient to prove that the set $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact, we note the following:

$$\begin{aligned} k[M_{k+1}-M_k] &= k[\mathrm{i}(k+1)MN_{k+1} - \mathrm{i}kMN_k] \\ &= \mathrm{i}kMN_{k+1}[(k+1)[\mathrm{i}kM-(1+a_k)A] - k[\mathrm{i}(k+1)M-(1+a_{k+1})A]]N_k \\ &= \mathrm{i}kMN_{k+1}\left[k\frac{(a_{k+1}-a_k)}{1+a_k}\right](1+a_k)AN_k - M_k(1+a_k)AN_k. \end{aligned} \tag{3.2}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, the sequence

$$\left\{k\left(\frac{a_{k+1}-a_k}{1+a_k}\right)\right\}_{k\in\mathbb{Z}}$$

is bounded. The identity $(1+a_k)AN_k = M_k - I$ implies that $\{(1+a_k)AN_k\}_{k\in\mathbb{Z}}$ is R-bounded. We conclude the proof by using Remark 2.3. □

https://doi.org/10.1017/S0013091513000606 Published online by Cambridge University Press

Next, we introduce the following definition of the solution of (3.1).

Definition 3.3. We say that a function $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R};[D(M)]) \cap L^p_{2\pi}(\mathbb{R};[D(A)])$ is a strong L^p-solution of (3.1) if $u(t) \in D(A)$ and (3.1) holds for almost every $t \in [0, 2\pi]$.

Let $a \in L^1(\mathbb{R}_+)$ and let A be a closed operator defined on X. For $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$ define

$$F(t) := (a \mathbin{\dot{*}} Au)(t) = \int_{-\infty}^{t} a(t-s)Au(s)\,\mathrm{d}s.$$

An easy computation shows that $\hat{F}(k) = A\tilde{a}(\mathrm{i}k)\hat{u}(k)$, $k \in \mathbb{Z}$ (where the hat denotes Fourier transform). It is remarkable that, in the case when $a \equiv 0$, the next theorem improves the main result in [**22**].

Theorem 3.4. *Let X be a UMD space. Let $A\colon D(A) \subseteq X \to X$ and $M\colon D(M) \subseteq X \to X$ be linear closed operators. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, the following assertions are equivalent:*

(i) *for every $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exists a unique strong L^p-solution of (3.1);*

(ii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and $\{\mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;*

(iii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and $\{\mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. (i) $\Rightarrow$ (ii) Let $k \in \mathbb{Z}$ and let $y \in X$. Define $f(t) = \mathrm{e}^{\mathrm{i}kt}y$. By hypothesis, there exists $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R};[D(M)]) \cap L^p_{2\pi}(\mathbb{R};[D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a \mathbin{\dot{*}} Au)(t) + f(t)$. Taking the Fourier transforms of both sides, we have $\hat{u}(k) \in D(A)$ and

$$\begin{aligned} \mathrm{i}kM\hat{u}(k) &= (1+a_k)A\hat{u}(k) + \hat{f}(k) \\ &= (1+a_k)A\hat{u}(k) + y. \end{aligned}$$

Thus, $(\mathrm{i}kM - (1+a_k)A)\hat{u}(k) = y$ for all $k \in \mathbb{Z}$ and we conclude that $(\mathrm{i}kM - (1+a_k)A)$ is surjective. Let $x \in D(A)$. If $(\mathrm{i}kM - (1+a_k)A)x = 0$, then $u(t) = \mathrm{e}^{\mathrm{i}kt}x$ defines a periodic solution of (3.1). Hence, $u \equiv 0$ by the assumption of uniqueness, and thus $x = 0$. Therefore, $(\mathrm{i}kM - (1+a_k)A)$ is bijective.

Now, we must prove that $(\mathrm{i}kM - (1+a_k)A)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$. Suppose that $(\mathrm{i}kM - (1+a_k)A)^{-1}$ has no bounded inverse. Then, for each $k \in \mathbb{Z}$ there exists a sequence $(y_{n,k})_{n\in\mathbb{Z}} \subset X$ such that $\|y_{n,k}\| \leqslant 1$ and

$$\|(\mathrm{i}kM - (1+a_k)A)^{-1}y_{n,k}\| \geqslant n^2 \quad \text{for all } n \in \mathbb{Z}.$$

Thus, we obtain that the sequence $x_k := y_{k,k}$ satisfies

$$\|(\mathrm{i}kM - (1+a_k)A)^{-1}x_k\| \geqslant k^2 \quad \text{for all } k \in \mathbb{Z}.$$

Let

$$f(t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{\mathrm{i}kt}\frac{x_k}{k^2}.$$

Observe that $f \in L^p_{2\pi}(\mathbb{R}, X)$ and so, by hypothesis, there exists a unique strong solution $u \in L^p_{2\pi}(\mathbb{R}, X)$ of (3.1). One can check that

$$u(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (\mathrm{i}kM - (1 + a_k)A)^{-1} \mathrm{e}^{\mathrm{i}kt} \frac{x_k}{k^2}.$$

Since

$$\left\| (\mathrm{i}kM - (1 + a_k)A)^{-1} \mathrm{e}^{\mathrm{i}kt} \frac{x_k}{k^2} \right\| \geqslant 1 \quad \text{for all } k \in \mathbb{Z} \setminus \{0\},$$

we obtain $u \notin L^p_{2\pi}(\mathbb{R}, X)$: a contradiction. Thus, we conclude that $(\mathrm{i}kM - (1 + a_k)A)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$, and therefore $\mathrm{i}k \in \rho_{M,\tilde{a}}(A)$ for all $k \in \mathbb{Z}$.

Using the closed graph theorem, we have that there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that

$$\|(Mu)'\|_{L^p} + \|a \mathbin{\tilde{*}} Au\|_{L^p} + \|Au\|_{L^p} \leqslant C\|f\|_{L^p}.$$

Note that for $f(t) = \mathrm{e}^{\mathrm{i}tk} y$, $y \in X$, the solution u of (3.1) is given by

$$u(t) = (\mathrm{i}kM - (1 + a_k)A)^{-1} \mathrm{e}^{\mathrm{i}kt} y.$$

Hence,

$$\|\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1} y\| \leqslant C\|y\|.$$

We obtain that, for $k \in \mathbb{Z}$, $\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}$ is a bounded operator. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$. By hypothesis, there exists $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a \mathbin{\tilde{*}} Au)(t) + f(t)$. Taking the Fourier transform of both sides, and using that $(\mathrm{i}kM - (1 + a_k)A)$ is invertible, we have $\hat{u}(k) \in D(A)$ and $\hat{u}(k) = (\mathrm{i}kM - (1 + a_k)A)^{-1} \hat{f}(k)$. Now, since $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}; [D(M)])$ and by the definition of $H^{1,p}_{\mathrm{per},M}(\mathbb{R}; [D(M)])$, there exists $v \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{v}(k) = \mathrm{i}kM\hat{u}(k)$ for all $k \in \mathbb{Z}$. Therefore, we have

$$\hat{v}(k) = \mathrm{i}kM\hat{u}(k) = \mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1} \hat{f}(k).$$

(ii) $\Rightarrow$ (i) Define $M_k = \mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}$ and $N_k = (\mathrm{i}kM - (1 + a_k)A)^{-1}$. Suppose that $\{\mathrm{i}k\}_{k \in \mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and that $\{M_k\}_{k \in \mathbb{Z}}$ is an L^p-multiplier. For $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists $u \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{u}(k) = \mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1} \hat{f}(k)$ for all $k \in \mathbb{Z}$. The identity $I = M_k - (1 + a_k)AN_k$ implies that

$$\begin{aligned} \hat{u}(k) &= \mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1} \hat{f}(k) \\ &= (I + (1 + a_k)AN_k)\hat{f}(k). \end{aligned}$$

So, we obtain $\widehat{(u - f)}(k) = (1 + a_k)AN_k\hat{f}(k)$. By Lemma 3.1, $\{I/(1 + a_k)\}_{k \in \mathbb{Z}}$ is an L^p-multiplier. Thus, for $u - f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists $v \in L^p_{2\pi}(\mathbb{R}, X)$ such that

$$\hat{v}(k) = \frac{1}{1 + a_k} \widehat{(u - f)}(k) = AN_k\hat{f}(k).$$

Since $0 \in \rho_{M,\tilde{a}}(A)$, we obtain that $A^{-1} \in \mathcal{B}(X)$, and therefore $w := A^{-1}v \in L^p_{2\pi}(\mathbb{R}, X)$ and $\hat{w}(k) = N_k \hat{f}(k)$. Hence, $\mathrm{i}kM\hat{w}(k) - (1 + a_k A)\hat{w}(k) = \hat{f}(k)$. Now, observe that for all $k \in \mathbb{Z}$ we have

$$\hat{u}(k) = \mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}\hat{f}(k) = \mathrm{i}kM\hat{w}(k).$$

Thus $w \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$. Moreover, $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$. Since A and M are closed operators and

$$\widehat{(Mw)'}(k) = \mathrm{i}kM\hat{w}(k) = (1 + a_k)A\hat{w}(k) + \hat{f}(k) \quad \text{for all } k \in \mathbb{Z},$$

one has $(Mw)'(t) = Aw(t) + (a * Au)(t) + f(t)$ a.e by Lemmas 2.6 and 2.7. Thus, we conclude that w is a strong L^p-solution of (3.1).

Finally, to see the uniqueness, let $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}, [D(A)])$ such that $(Mu)'(t) = Au(t) + (a * Au)(t)$. Taking the Fourier transforms of both sides, we have $\hat{u}(k) \in D(A)$ and $(\mathrm{i}kM - (1 + a_k)A)\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Since $(\mathrm{i}kM - (1 + a_k)A)$ is invertible for all $k \in \mathbb{Z}$, we obtain $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$, and thus $u \equiv 0$.

(ii) $\Leftrightarrow$ (iii) See Proposition 3.2. □

In the Hilbert case, we obtain a simple condition for existence and uniqueness of solutions of (3.1). Since in Hilbert spaces the concepts of R-boundedness and boundedness are equivalent [**13**], the proof of the next corollary follows from Theorem 3.4.

Corollary 3.5. *Let H be a Hilbert space and let $A\colon D(A) \subset H \to H$ and $M\colon D(M) \subset H \to H$ be closed linear operators satisfying $D(A) \subseteq D(M)$. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, for $1 < p < \infty$, the following assertions are equivalent:*

(i) *for every $f \in L^p_{2\pi}(\mathbb{R}, H)$, there exists a unique strong L^p-solution of (3.1);*

(ii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} \|\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}\| < \infty$.*

Note that the solution $u(\cdot)$ given in Theorem 3.4 satisfies the following maximal regularity property.

Corollary 3.6. *In the context of Theorem 3.4, if condition (iii) is fulfilled, we have $(Mu)', Au, a * Au \in L^p_{2\pi}(\mathbb{R}, X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that*

$$\|(Mu)'\|_{L^p} + \|Au\|_{L^p} + \|a * Au\|_{L^p} \leqslant C\|f\|_{L^p}. \tag{3.3}$$

Remark 3.7. Fejer's theorem (see [**3**, Proposition 1.1]) can be used to construct the solution u given in Theorem 3.4. More precisely, if $M_k = \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}$ satisfies condition (ii) or (iii) in Theorem 3.4, then for $f \in L^p_{2\pi}(\mathbb{R}, X)$ the solution $u \in L^p_{2\pi}(\mathbb{R}, X)$ of (3.1) is given by

$$u(\cdot) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes M_k \hat{f}(k),$$

where $e_k(t) := \mathrm{e}^{\mathrm{i}kt}$, $t \in \mathbb{R}$, and the convergence holds in $L^p_{2\pi}(\mathbb{R}, X)$.

https://doi.org/10.1017/S0013091513000606 Published online by Cambridge University Press

4. Maximal regularity on Hölder and Besov spaces

In this section, we formulate analogous theorems to those in the above section, in the context of Hölder and Besov spaces.

Besov spaces form one class of function spaces that are of special interest. The relatively complicated definition is recompensed by useful applications to differential equations (see [**2**] for a concrete model).

We consider solutions to (3.1) in $B^s_{p,q}((0,2\pi);X)$, the vector-valued periodic Besov spaces for $1 \leqslant p \leqslant \infty$, $s > 0$. It is remarkable that in this case there are no geometrical conditions on the Banach space X. For the definition and main properties of these spaces we refer to [**4**,**19**]. For the scalar case, see [**12**,**24**]. Contrary to the L^p case, the multiplier theorems established for vector-valued Besov spaces are valid for arbitrary Banach spaces X (see [**1**,**4**,**17**]). Note also that, for $0 < s < 1$, $B^s_{\infty,\infty}$ is the usual Hölder space C^s. We now summarize some useful properties of $B^s_{p,q}((0,2\pi);X)$. See [**4**, § 2] for a proof.

(i) $B^s_{p,q}((0,2\pi);X)$ is a Banach space.

(ii) If $s > 0$, then $B^s_{p,q}((0,2\pi);X) \hookrightarrow L^p((0,2\pi);X)$, and the natural injection from $B^s_{p,q}((0,2\pi);X)$ into $L^p((0,2\pi);X)$ is a continuous linear operator.

(iii) Let $s > 0$. Then $f \in B^{s+1}_{p,q}((0,2\pi);X)$ if and only if f is differentiable a.e. and $f' \in B^s_{p,q}((0,2\pi);X)$.

We begin with the definition of operator-valued Fourier multipliers in the context of periodic Besov spaces.

Definition 4.1. Let $1 \leqslant p,q \leqslant \infty$, $s > 0$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}((0,2\pi);X)$ there exists a function $g \in B^s_{p,q}((0,2\pi);Y)$ such that

$$\hat{g}(k) = M_k \hat{f}(k), \quad k \in \mathbb{Z}.$$

Definition 4.2 (Keyantuo *et al.* [20]). We say that $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ satisfies the Marcinkiewicz condition of order 2 if

$$\sup_{k\in\mathbb{Z}} \|M_k\| < \infty, \qquad \sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty, \tag{4.1}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+1} - 2M_k + M_{k-1})\| < \infty. \tag{4.2}$$

We recall the following operator-valued Fourier multiplier theorem on Besov spaces.

Theorem 4.3 (Arendt and Bu [4]). *Let X, Y be Banach spaces and suppose $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$ satisfies the Marcinkiewicz condition of order 2. Then, for $1 \leqslant p,q \leqslant \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*

The following proposition is analogous to Proposition 3.2.

Proposition 4.4. *Let $A\colon D(A) \subseteq X \to X$, $M\colon D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, the following assertions are equivalent:*

(i) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and $\{\mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier for $1 \leqslant p, q \leqslant \infty$;*

(ii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and $\sup_{k\in\mathbb{Z}} \|\mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}\| < \infty$.*

Proof. (i) $\Rightarrow$ (ii) This follows the same lines as the proof in [**18**, Proposition 3.4].

(ii) $\Rightarrow$ (i) For $k \in \mathbb{Z}$, define $M_k = \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}$ and $N_k = (\mathrm{i}kM-(1+a_k)A)^{-1}$. From the identity (3.2) we obtain

$$\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty, \tag{4.3}$$

proving (4.1). To verify (4.2), note that

$$\begin{aligned}
&k^2[M_{k+1} - 2M_k + M_{k-1}] \\
&\quad = \mathrm{i}k^2M[(k+1)N_{k+1} - 2kN_k + (k-1)N_{k-1}] \\
&\quad = \mathrm{i}k^2MN_{k+1}[(k+1)N_k^{-1} - 2kN_{k+1}^{-1} + (k-1)N_{k+1}^{-1}N_{k-1}N_k^{-1}]N_k \\
&\quad = \mathrm{i}k^2MN_{k+1}[(k+1)N_k^{-1} - 2k[(\mathrm{i}kM - (1+a_k)A) - (a_{k+1} - a_k)A + \mathrm{i}M] \\
&\qquad\qquad + (k-1)[(\mathrm{i}(k-1)M - (1+a_{k-1})A) \\
&\qquad\qquad\qquad + 2\mathrm{i}M - (a_{k+1} - a_{k-1})A]N_{k-1}N_k^{-1}]N_k \\
&\quad = \mathrm{i}k^2MN_{k+1}[(k+1)N_k^{-1} - 2kN_k^{-1} + 2k(a_{k+1} - a_k)A - 2\mathrm{i}kM \\
&\qquad\qquad + [(k-1)N_{k-1}^{-1} + 2\mathrm{i}(k-1)M - (k-1)(a_{k+1} - a_{k-1})A]N_{k-1}N_k^{-1}]N_k \\
&\quad = \mathrm{i}k^2MN_{k+1}[(k+1)I - 2kI + 2k(a_{k+1} - a_k)AN_k - 2\mathrm{i}kMN_k \\
&\qquad\qquad + [(k-1)I + 2\mathrm{i}(k-1)MN_{k-1} - (k-1)(a_{k+1} - a_{k-1})AN_{k-1}]] \\
&\quad = \mathrm{i}k^2MN_{k+1}[2k(a_{k+1} - a_k)AN_k - 2(M_k - M_{k-1}) - (k-1)(a_{k+1} - a_{k-1})AN_{k-1}] \\
&\quad = \mathrm{i}kMN_{k+1}[2k^2(a_{k+1} - a_k)AN_k - 2k(M_k - M_{k-1}) \\
&\qquad\qquad\qquad - k(k-1)(a_{k+1} - a_{k-1})AN_{k-1}] \\
&\quad = \mathrm{i}kMN_{k+1}\biggl[2k^2\frac{(a_{k+1} - a_k)}{1+a_k}(M_k - I) - 2k(M_k - M_{k-1}) \\
&\qquad\qquad\qquad - k(k-1)\frac{(a_{k+1} - a_{k-1})}{1+a_{k-1}}(M_{k-1} - I)\biggr].
\end{aligned}$$

Using the identities

$$2k^2(a_{k+1} - a_k) = k^2(a_{k+1} - 2a_k + a_{k-1}) + k^2(a_{k+1} - a_{k-1})$$

and

$$k(k-1)(a_{k+1} - a_{k-1}) = k^2(a_{k+1} - a_{k-1}) - k(a_{k+1} - a_{k-1}),$$

https://doi.org/10.1017/S0013091513000606 Published online by Cambridge University Press

we obtain

$$
\begin{aligned}
2k^2 & \frac{(a_{k+1}-a_k)}{1+a_k}[M_k - I] - k(k-1)\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I] \\
&= k^2\frac{(a_{k+1}-2a_k+a_{k-1})}{1+a_k}[M_k - I] \\
&\quad + k^2\frac{(a_{k+1}-a_{k-1})}{1+a_k}\left[\frac{(M_k - M_{k-1}) + (a_k - a_{k-1})I + a_{k-1}M_k - a_k M_{k-1}}{1+a_{k-1}}\right] \\
&\quad + k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I] \\
&= k^2\frac{(a_{k+1}-2a_k+a_{k-1})}{1+a_k}[M_k - I] \\
&\quad + k\frac{(a_{k+1}-a_{k-1})}{1+a_k}\left[\frac{1}{1+a_{k-1}}k(M_k - M_{k-1}) + k\frac{(a_k - a_{k-1})}{1+a_{k-1}}I \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{k}{1+a_{k-1}}[a_{k-1}M_k - a_k M_{k-1}]\right] \\
&\quad + k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I].
\end{aligned}
$$

Since the identities

$$
\begin{aligned}
\frac{k}{1+a_{k-1}}[a_{k-1}M_k - a_k M_{k-1}] &= \frac{a_{k-1}}{1+a_{k-1}}k[M_k - M_{k-1}] + k\frac{(a_{k-1}-a_k)}{1+a_{k-1}}M_{k-1}, \\
k\frac{(a_{k+1}-a_{k-1})}{1+a_k} &= k\frac{(a_{k+1}-a_k)}{1+a_k} + \frac{k}{k-1}(k-1)\frac{(a_k - a_{k-1})}{a_{k-1}}a_{k-1}\frac{1}{1+a_k}, \\
k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}} &= k\frac{(a_{k+1}-a_k)}{1+a_{k-1}} + k\frac{(a_k - a_{k-1})}{1+a_{k-1}},
\end{aligned}
$$

and

$$
k[M_k - M_{k-1}] = \frac{k}{k-1}(k-1)[M_k - M_{k-1}]
$$

are valid, and from the fact that $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ is bounded and $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, we conclude from the above identities and Remark 2.8 that

$$
\sup_{k\in\mathbb{Z}} \|k^2(M_{k+1} - 2M_k + M_{k-1})\| < \infty. \tag{4.4}
$$

Thus, $\{M_k\}_{k\in\mathbb{Z}}$, satisfies the Marcinkiewicz condition of order 2 and therefore, by Theorem 4.3, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. $\square$

Lemma 4.5. *Let X be a Banach space. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, $\{I/(1+a_k)\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier for $1 \leqslant p, q \leqslant \infty$.*

Proof. Define $m_k := 1/(1+a_k)$, $k \in \mathbb{Z}$. By Remark 2.8, the sequence $\{m_k\}_{k\in\mathbb{Z}}$ is bounded. Moreover, $\{m_k\}_{k\in\mathbb{Z}}$ satisfies the identities

$$
k[m_{k+1} - m_k] = -k\frac{a_{k+1}-a_k}{1+a_k}\frac{1}{1+a_{k+1}}
$$

and

$$\begin{aligned} k^2[m_{k+1} - 2m_k + m_{k-1}] = &-\frac{1}{(1+a_{k+1})(1+a_k)(1+a_{k-1})} k^2(a_{k+1} - 2a_k + a_{k-1}) \\ &+ \frac{2}{(1+a_{k+1})(1+a_k)(1+a_{k-1})} k(a_k - a_{k-1})k(a_{k+1} - a_k) \\ &- \frac{a_k}{(1+a_{k+1})(1+a_k)(1+a_{k-1})} k^2(a_{k+1} - 2a_k + a_{k-1}). \end{aligned}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, we conclude that $\{m_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2, and therefore $\{m_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. □

Definition 4.6. Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. A function $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ is said to be a strong $B^s_{p,q}$-solution of (3.1) if $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ and (3.1) holds for almost every $t \in (0, 2\pi)$.

The next theorem is the main result of this section and is analogous to Theorem 3.4 in the context of Besov spaces. We note that there are no special conditions in the space X.

Theorem 4.7. *Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. Let $A\colon D(A) \subseteq X \to X$, $M\colon D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$, and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, the following assertions are equivalent:*

(i) *for every $f \in B^s_{p,q}((0, 2\pi); X)$ there exists a unique strong $B^s_{p,q}$-solution of (3.1);*

(ii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and $\{\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier;*

(iii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} \|\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}\| < \infty$.*

Proof. (i) $\Rightarrow$ (iii) Suppose that for every $f \in B^s_{p,q}((0, 2\pi); X)$ there exists a unique strong $B^s_{p,q}$-solution of (3.1). Fix $x \in X$ and $k \in \mathbb{Z}$. Define $f(t) = \mathrm{e}^{\mathrm{i}tk}x$. Then $f \in B^s_{p,q}((0, 2\pi); X)$. By hypothesis there exists $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ with $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ such that $u(t) \in D(A)$ and

$$(Mu)'(t) = Au(t) + (a \dot{*} Au)(t) + f(t) \quad \text{a.e. } t \in (0, 2\pi).$$

By Lemma 2.7 we have $\mathrm{i}kM\hat{u}(k) = A\hat{u}(k) + a_kA\hat{u}(k) + x$. Following the same reasoning as in the proof of Theorem 3.4, we obtain that $\mathrm{i}k \in \rho_{M,\tilde{a}}(A)$ for all $k \in \mathbb{Z}$. Let $M_k := \mathrm{i}kM(\mathrm{i}kM - (1+a_k)A)^{-1}$. We shall see that $\{M_k\}_{k\in\mathbb{Z}}$ is bounded. Using the closed graph theorem, we have that there exists a constant C independent of f such that

$$\begin{aligned} \|Mu\|_{B^{s+1}_{p,q}((0,2\pi);X)} + \|Au\|_{B^s_{p,q}((0,2\pi);[D(A)])} + \|a \dot{*} Au\|_{B^s_{p,q}((0,2\pi);[D(A)])} \\ \leqslant C\|f\|_{B^s_{p,q}((0,2\pi);X)}. \end{aligned}$$

Note that for $f(t) = \mathrm{e}^{\mathrm{i}tk}x$ the solution u of (3.1) is given by

$$u(t) = (\mathrm{i}kM - (1 + a_k)A)^{-1}\mathrm{e}^{\mathrm{i}kt}x.$$

Hence,

$$\sup_{k\in\mathbb{Z}} \|\mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}x\| \leqslant C\|x\|.$$

(ii) ⇒ (i) Define $M_k = \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}$ and $N_k = (\mathrm{i}kM-(1+a_k)A)^{-1}$. Suppose that $\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. For $f \in B^s_{p,q}((0,2\pi);X)$ there exists $u \in B^s_{p,q}((0,2\pi);X)$ such that $\hat{u}(k) = \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. The identity $I = M_k - (1+a_k)AN_k$ implies that

$$\begin{aligned}\hat{u}(k) &= \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}\hat{f}(k)\\ &= (I+(1+a_k)AN_k)\hat{f}(k).\end{aligned}$$

So, we obtain $(\widehat{u-f})(k) = (1+a_k)AN_k\hat{f}(k)$. By Lemma 4.5, $\{I/(1+a_k)\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Thus, for $u-f \in B^s_{p,q}((0,2\pi);X)$ there exists $v \in B^s_{p,q}((0,2\pi);X)$ such that

$$\hat{v}(k) = \frac{1}{1+a_k}(\widehat{u-f})(k) = AN_k\hat{f}(k).$$

Since $0 \in \rho_{M,\tilde{a}}(A)$, we obtain that $A^{-1} \in \mathcal{B}(X)$, and therefore $w := A^{-1}v \in B^s_{p,q}((0,2\pi);X)$ and $\hat{w}(k) = N_k\hat{f}(k)$. Hence, $\mathrm{i}kM\hat{w}(k) - (1+a_kA)\hat{w}(k) = \hat{f}(k)$. Observe that, for all $k \in \mathbb{Z}$, we have

$$\hat{u}(k) = \mathrm{i}kM(\mathrm{i}kM-(1+a_k)A)^{-1}\hat{f}(k) = \mathrm{i}kM\hat{w}(k).$$

Thus, by uniqueness of Fourier coefficients, $u(t) = (Mw)'(t)$. Since $u \in B^s_{p,q}((0,2\pi);X)$, $(Mw)' \in B^s_{p,q}((0,2\pi);X)$, and therefore $Mw \in B^{s+1}_{p,q}((0,2\pi);X)$. Moreover, $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$.

Since A and M are closed operators and

$$\widehat{(Mw)'}(k) = \mathrm{i}kM\hat{w}(k) = (1+a_k)A\hat{w}(k) + \hat{f}(k) \quad \text{for all } k \in \mathbb{Z},$$

one has $(Mw)'(t) = Aw(t) + (a * Au)(t) + f(t)$ a.e. by Lemmas 2.6 and 2.7. We conclude that $w \in B^s_{p,q}((0,2\pi);X)$ is a strong $B^s_{p,q}$-solution to (3.1). Finally, the uniqueness follows the same way as in the proof of Theorem 3.4.

(iii) ⇔ (ii) This follows from Proposition 4.4. □

5. Maximal regularity on Triebel–Lizorkin spaces

In this section, we study the existence and uniqueness of solutions to (3.1) in the context of Triebel–Lizorkin spaces, $F^s_{p,q}((0,2\pi);X)$, where X is a Banach space, $1 \leqslant p,q \leqslant \infty$ and $s \in \mathbb{R}$. More details of these spaces can be found in [**8**] and the references therein.

The next definition and theorem are analogous to those in the previous sections.

Definition 5.1. Let $1 \leqslant p,q \leqslant \infty$ and $s \in \mathbb{R}$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is a $F^s_{p,q}$-multiplier if, for each $f \in F^s_{p,q}((0,2\pi);X)$, there exists a function $g \in F^s_{p,q}((0,2\pi);Y)$ such that

$$\hat{g}(k) = M_k\hat{f}(k), \quad k \in \mathbb{Z}.$$

https://doi.org/10.1017/S0013091513000606 Published online by Cambridge University Press

We recall the following result due to Bu and Kim [**8**].

Theorem 5.2 (Bu and Kim [8]). *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. Assume that*

$$\sup_{k\in\mathbb{Z}} \|M_k\| < \infty, \sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty, \tag{5.1}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+1} - 2M_k + M_{k-1})\| < \infty, \tag{5.2}$$

$$\sup_{k\in\mathbb{Z}} \|k^3(M_{k+2} - 3M_{k+1} + 3M_k - M_{k-1})\| < \infty, \tag{5.3}$$

where $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$. Then $\{M_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier.

Remark 5.3. If X, Y are UMD spaces in the above theorem, then the conditions (5.1) and (5.2) are sufficient for $\{M_k\}_{k\in\mathbb{Z}}$ to be an $F^s_{p,q}$-multiplier.

The definition of the solution of (3.1) in the Triebel–Lizorkin spaces is the same as that in the Besov case. The proof of following theorem is similar to Theorem 4.7. We omit the details.

Theorem 5.4. *Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. Let $A\colon D(A) \subseteq X \to X$, $M\colon D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3-regular. Then, the following assertions are equivalent:*

(i) *for every $f \in F^s_{p,q}((0, 2\pi); X)$ there exists a unique strong $F^s_{p,q}$-solution of (3.1);*

(ii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$, and $\{\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier;*

(iii) *$\{\mathrm{i}k\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} \|\mathrm{i}kM(\mathrm{i}kM - (1 + a_k)A)^{-1}\| < \infty$.*

6. Applications

We conclude the paper with some applications of the above results.

Example 6.1. Let us consider the boundary-value problem

$$\frac{\partial(m(x)u(t,x))}{\partial t} - \Delta u = \int_{-\infty}^{t} a(t-s)\Delta u(s,x)\,\mathrm{d}s + f(t,x) \quad \text{in } [0,2\pi]\times\Omega, \tag{6.1}$$

$$u = 0 \quad \text{in } [0,2\pi]\times\partial\Omega, \tag{6.2}$$

$$m(x)u(0,x) = m(x)u(2\pi,x) \quad \text{in } \Omega, \tag{6.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geqslant 0$ is a given measurable bounded function on Ω, and f is a function on $[0,2\pi]\times\Omega$.

Let M be the multiplication operator induced by the function m. If we take $X = H^{-1}(\Omega)$, then by [**5**, p. 38] (see also the references therein) we have that there exists a constant $c > 0$ such that

$$\|M(zM - \Delta)^{-1}\| \leqslant \frac{c}{1+|z|},$$

whenever $\mathrm{Re}(z) \geqslant -c(1+|\mathrm{Im}(z)|)$. Thus, the inequality

$$\|\mathrm{i}kM(\mathrm{i}kM-(1+a_k)\Delta)^{-1}\| = \frac{|k|}{|1+a_k|}\left\|M\left(\frac{\mathrm{i}k}{1+a_k}M-\Delta\right)^{-1}\right\| \leqslant c$$

holds if

$$\mathrm{Re}\left(\frac{\mathrm{i}k}{1+a_k}\right) \geqslant -c\left(1+\left|\mathrm{Im}\left(\frac{\mathrm{i}k}{1+a_k}\right)\right|\right) \quad \text{for all } k\in\mathbb{Z},$$

that is, if

$$k\beta_k \geqslant -c((1+\alpha_k)^2+\beta_k^2+|k(1+\alpha_k)|) \tag{6.4}$$

is valid for all $k\in\mathbb{Z}$, where α_k and β_k denote the real and imaginary parts of a_k, respectively. In particular, if $a(t) := t^{b-1}/\Gamma(b)$, with b an even integer, then one can check that $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and $\beta_k = 0$ for all $k\in\mathbb{Z}$. Thus, the inequality (6.4) holds. Therefore, by Theorem 4.7 (or Corollary 3.5), we conclude that for all $f\in L^p_{2\pi}(\mathbb{R},H^{-1}(\Omega))$ there exists a unique solution for (6.1), (6.2).

Example 6.2. Consider, for $t\in[0,2\pi]$ and $x\in[0,\pi]$, the problem

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+1\right)u(t,x) &= -b\frac{\partial^2}{\partial x^2}u(t,x)-cu(t,x)\\ &\quad+\int_{-\infty}^{t}a(t-s)\left(b\frac{\partial^2}{\partial x^2}+c\right)u(s,x)\,\mathrm{d}s+f(t,x),\end{aligned} \tag{6.5}$$

$$u(t,0)=u(t,\pi)=\frac{\partial^2}{\partial x^2}u(t,0)=\frac{\partial^2}{\partial x^2}u(t,\pi)=0, \tag{6.6}$$

$$\left(\frac{\partial^2}{\partial x^2}+1\right)u(0,x)=\left(\frac{\partial^2}{\partial x^2}+1\right)u(2\pi,x), \tag{6.7}$$

where b is a positive constant and $-2b<c<4b$. We take $X=C_0([0,\pi])=\{u\in C([0,\pi])\colon u(0)=u(\pi)\}$ and K the realization of $\partial^2/\partial x^2$ with domain

$$D(K)=\left\{u\in C^2([0,\pi])\colon u(0)=u(\pi)=\frac{\partial^2}{\partial x^2}u(0)=\frac{\partial^2}{\partial x^2}u(\pi)=0\right\}.$$

Define $M=K+I$ and $A=bM+(c-b)I$. By [**5**, Example 1.2, p. 39] we have that

$$\|M(zM-A)^{-1}\| \leqslant \frac{d}{1+|z|}$$

for all $\mathrm{Re}(z)\geqslant -d(1+|\mathrm{Im}(z)|)$, d being a suitable positive constant. Therefore, as in Example 6.2, if for all $k\in\mathbb{Z}$ the inequality

$$k\beta_k \geqslant -d((\alpha_k-1)^2+\beta_k^2+|k(\alpha_k-1)|) \tag{6.8}$$

is valid, then for all $f\in B^s_{p,q}((0,2\pi),C_0([0,\pi]))$, $s>0$, $1\leqslant p,q\leqslant\infty$, by Theorem 4.7, we conclude that the problem (6.5)–(6.7) has a unique strong solution u with regularity

$$\frac{\partial^2 u}{\partial x^2}\in B^s_{p,q}((0,2\pi),C_0([0,\pi])).$$

In particular, if $a(t) := e^{\gamma t}$, where $\gamma \in \mathbb{R}$, we can check that $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and the inequality (6.8) holds with $d = 1$.

Acknowledgements. C.L. is partly supported by FONDECYT Grant 1100485.

References

1. H. Amann, Operator-valued Fourier multipliers, vector-valued Besov spaces and applications, *Math. Nachr.* **186** (1997), 5–56.
2. H. Amann, On the strong solvability of the Navier–Stokes equations, *J. Math. Fluid. Mech.* **2** (2000), 16–98.
3. W. Arendt and S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, *Math. Z.* **240** (2002), 311–343.
4. W. Arendt and S. Bu, Operator-valued Fourier multiplier on periodic Besov spaces and applications, *Proc. Edinb. Math. Soc.* **47**(2) (2004), 15–33.
5. V. Barbu and A. Favini, Periodic problems for degenerate differential equations, *Rend. Instit. Mat. Univ. Trieste* (1997) **28**(Supplement), 29–57.
6. J. Bourgain, Some remarks on Banach spaces in which martingale differences sequences are unconditional, *Ark. Mat.* **21** (1983), 163–168.
7. S. Bu and Y. Fang, Maximal regularity for integro-differential equation on periodic Triebel–Lizorkin spaces, *Taiwan. J. Math.* **12**(2) (2008), 281–292.
8. S. Bu and J. M. Kim, Operator-valued Fourier multipliers on periodic Triebel spaces, *Acta Math. Sinica (Engl. Ser.)* **21** (2005), 1049–1056.
9. D. L. Burkholder, A geometrical condition that implies the existence of certain singular integrals on Banach-space-valued functions, in *Proc. Conf. on Harmonic Analysis in Honor of Antoni Zygmund, Chicago, IL, 1981*, pp. 270–286 (Wadsworth, Belmont, CA, 1983).
10. D. L. Burkholder, Martingales and singular integrals in Banach spaces, in *Handbook of the geometry of Banach spaces*, Volume 1, pp. 233–269 (North-Holland, Amsterdam, 2001).
11. T. A. Burton and B. Zhang, Periodic solutions of abstract differential equations with infinite delay, *J. Diff. Eqns* **90** (1991), 357–396.
12. P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Die Grundlehren der Mathematische Wissenschaften, Volume 145 (Springer, 1967).
13. R. Denk, M. Hieber and J. Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Memoirs of the American Mathematical Society, Volume 166 (American Mathematical Society, Providence, RI, 2003).
14. A. Favaron and A. Favini, Maximal time regularity for degenerate evolution integro-differential equations, *J. Evol. Eqns* **10**(2) (2010), 377–412.
15. A. Favini and A. Yagi, *Degenerate differential equations in Banach spaces*, Pure and Applied Mathematics, Volume 215 (Dekker, New York, 1999).
16. M. Girardi and L. Weis, *Criteria for R-boundedness of operator families*, Lecture Notes in Pure and Applied Mathematics, Volume 234, pp. 203–221 (Dekker, New York, 2003).
17. M. Girardi and L. Weis, Operator-valued Fourier multiplier theorems on Besov spaces, *Math. Nachr.* **251** (2003), 34–51.
18. V. Keyantuo and C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, *J. Lond. Math. Soc.* **69**(3) (2004), 737–750.
19. V. Keyantuo and C. Lizama, Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces, *Studia Math.* **168**(1) (2005), 25–50.
20. V. Keyantuo, C. Lizama and V. Poblete, Periodic solutions of integro-differential equations in vector-valued function spaces, *J. Diff. Eqns* **246**(3) (2009), 1007–1037.

21. J. Lagnese, *Boundary stabilization of thin plates* (SIAM, Philadelphia, PA, 1989).
22. C. Lizama and R. Ponce, Periodic solutions of degenerate differential equations in vector-valued function spaces, *Studia Math.* **202** (2011), 49–63.
23. J. Prüss, *Evolutionary integral equations and applications*, Monographs in Mathematics, Volume 87 (Birkhäuser, Basel, 1993).
24. H. J. Schmeisser and H. Triebel, *Topics in Fourier analysis and function spaces* (Wiley, 1987).
25. L. Weis, *A new approach to maximal L_p-regularity*, Lecture Notes in Pure and Applied Mathematics, Volume 215, pp. 195–214 (Marcel Dekker, New York, 2001).

https://doi.org/10.1017/S0013091513000606 Published online by Cambridge University Press