LETTER TO THE EDITOR

Dear Editor,

A note on distributions having the almost-lack-of-memory property

1. A case of the almost-lack-of-memory property

We consider the functional equation

(1) $$\bar{H}(c+z)=\bar{H}(c)\bar{H}(z) \qquad \text{for } z \geqq 0,$$

where H is a distribution on $[0, \infty)$, $\bar{H}=1-H$ and $c>0$ is a constant. It is seen that Equation (1) is equivalent to

(2) $$\bar{H}(nc+z)=\bar{H}(nc)\bar{H}(z) \qquad \text{for } n=1,2,\cdots \text{ and for } z \geqq 0,$$

because (1) implies that $\bar{H}(nc)=(\bar{H}(c))^n$ for $n=1,2,\cdots$ and is also equivalent to

$$\bar{H}(nc+z)=(\bar{H}(c))^n\bar{H}(z) \qquad \text{for } n=1,2,\cdots \text{ and for } z \geqq 0.$$

The property (2) is a particular case of what Chukova and Dimitrov (1992) define as the almost-lack-of-memory property. Chukova and Dimitrov (1992) give two examples of distribution H satisfying (2). To complement their results we solve the functional equation (1) (and hence (2)) as follows.

Theorem 1. Let Z be a non-negative random variable with distribution $H(z)=\Pr(Z<z)$ for $z \geqq 0$. Then H satisfies (1) if and only if

(3) $$H(z)=1-\alpha^{[z/c]}+\alpha^{[z/c]}H(z-[z/c]c) \qquad \text{for } z \geqq c>0,$$

where the constant $\alpha=\bar{H}(c)<1$ and $[z]$ is the least integer less than or equal to z.

Proof. Each of (1) and (3) implies that

$$\bar{H}(nc)=(\bar{H}(c))^n=\alpha^n \qquad \text{for } n=1,2,\cdots,$$

so that $\alpha<1$ due to the fact that $\bar{H}(z)\to 0$ as $z\to\infty$. Also, each of (1) and (3) is equivalent to

$$\bar{H}(nc+z)=\alpha^n\bar{H}(z) \qquad \text{for } n=1,2,\cdots \text{ and for } z\in[0,c).$$

This completes the proof.

Note that each distribution H with support contained in the interval $[0, c)$ is a solution to Equation (1). Also, each solution H to Equation (1) is uniquely determined by the function values $H(z)$, $z\in[0,c)$, for which no special requirement is needed.

2. A relationship to the single-server queueing system with unreliable server and service repetition

It is interesting to note that each distribution H of the form (3) is actually the distribution of blocking time (the total time taken by a customer) in a queueing system with instantaneous repairs after any failure of the constant-lifetime server, and vice versa. To see this, we adopt the notation of Chukova and Dimitrov (1992). Let $c > 0$ be a constant and let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. non-negative random variables with common distribution F. Then define $N = \min\{n : X_n < c\}$ and the blocking time $Z = \Sigma_{n=1}^{N} \min\{X_n, c\}$. Here, c can be considered as the lifetime of the server in the queueing system and X_n, $n \geqq 1$, are the times required by a call in its consecutive attempts for service.

Clearly, if $F(c) = 0$ then $N = \infty$ and hence $Z = \infty$ almost surely. Hereafter, we assume $F(c) > 0$. Then it is seen that $Z = (N - 1)c + X_N$ and

$$\Pr(X_N < x) = \Pr(X_1 < x \mid X_1 < c) = F(x)/F(c) \qquad \text{for } x \in [0, c].$$

As noted by Chukova and Dimitrov (1992), p. 694, the Laplace–Stieltjes transform of Z is

$$\phi_Z(s) = \boldsymbol{E} e^{-sZ} = \frac{(1-\alpha)\phi(s)}{1 - \alpha \exp(-sc)} \qquad \text{for } s \geqq 0, \tag{4}$$

where $\alpha = \bar{F}(c) < 1$ and $\phi(s) = \boldsymbol{E} \exp(-sX_N) = (1-\alpha)^{-1} \int_{[0,c]} \exp(-sx) dF(x)$ for $s \geqq 0$. Therefore, the distribution of blocking time Z is determined only by the values $F(x)$, $x \in [0, c)$. This means that there are infinitely many F (with $F(c) < 1$) resulting in the same distribution of Z. But if $F(c) = 1$, then $N = 1$ and hence $Z = X_1$ almost surely.

On the other hand, the Laplace–Stieltjes transform of H in (3) is exactly of the form (4) with $\alpha = \bar{H}(c)$ and $\phi(s) = (1-\alpha)^{-1} \int_{[0,c]} \exp(-sz) dH(z)$. Hence, each H in (3) is the distribution of blocking time Z in the above-mentioned queueing system with F satisfying $F(x) = H(x)$ for $x \in [0, c)$, and vice versa, so the distribution of blocking time Z can be written in the form (3).

Finally, we conclude the following theorem which shows that the result of Chukova and Dimitrov ((1992), Corollary 1, p. 695) is invertible.

Theorem 2. In the queueing system defined above, assume further that $F(c) > 0$. Then the blocking time Z is distributed as X_1 if and only if the distribution of X_1 is of the form (3).

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Reference

CHUKOVA, S. AND DIMITROV, B. (1992) On distributions having the almost-lack-of-memory property. *J. Appl. Prob.* **29**, 691–698.

Yours sincerely,

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