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Nagoya Math. J.
Vol. 89 (1983), 77-87

ON p-ADIC ARTIN L-FUNCTIONS

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Introduction

In this paper we will discuss p-adic Artin L-functions. The existence of these functions is a simple consequence of a theorem of Deligne and Ribet [4]. One can formulate a "p-adic Artin conjecture" for these functions. Our primary purpose here is to relate this conjecture to the "main conjecture" discussed by Coates in [3]. We will describe the precise formulations of these conjectures that we will use later. Our main result will be that in fact the main conjecture implies the p-adic Artin conjecture.

Let k be a totally real number field. Let ψ be an Artin character for k. We denote by k_ψ the Galois extension of k attached to ψ. (Thus, ψ is the character of a faithful representation of $\mathrm{Gal}(k_\psi/k)$.) We will assume throughout this paper that k_ψ is also totally real. Otherwise the p-adic L-function for ψ will be identically zero. Let k_∞ denote the cyclotomic $\boldsymbol{Z}_p$-extension of k. We will say that ψ is of type S if $k_\psi \cap k_\infty = k$. We can associate to such ψ a purely algebraically defined polynomial $h_\psi(T)$. This polynomial will be constructed from a natural representation of $\mathrm{Gal}(k_\psi k_\infty/k)$ occurring in Iwasawa theory. Its formal properties are quite analogous to those of Artin L-functions. Assuming the main conjecture, we can relate this polynomial to the p-adic Artin L-function for ψ and thus prove the p-adic Artin conjecture in this case. One can extend this quite easily to the case when $\psi\rho$ is of type S, where ρ is a one-dimensional Artin character such that $k_\rho \subseteq k_\infty$. (Such Artin characters ρ will be said to be of type W.) However it is not difficult to find Artin characters ψ not satisfying the above conditions. We can still deduce the p-adic Artin conjecture for such ψ from the main conjecture, but the argument is somewhat more ad hoc.

Received May 6, 1981.
* Supported in part by a National Science Foundation grant.

https://doi.org/10.1017/S0027763000020250 Published online by Cambridge University Press

§1. Algebraically defined polynomials

Let ψ be a p-adic Artin character over k, that is, a character of a Galois representation on a vector space over Ω_p. Here Ω_p is an algebraic closure of the field $\boldsymbol{Q}_p$ of p-adic numbers. Let $G = \mathrm{Gal}\,(k_\psi k_\infty/k)$. The character ψ is of type S if and only if $G \cong \Delta \times \Gamma$, where $\Delta = \mathrm{Gal}\,(k_\psi/k)$ and $\Gamma = \mathrm{Gal}\,(k_\infty/k)$. The isomorphism arises from the restriction maps from G onto Δ and Γ. Let L denote the maximal abelian pro-p-extension of $k_\psi k_\infty$ in which only primes over p are ramified. Let $X = \mathrm{Gal}\,(L/k_\psi k_\infty)$. The group G acts on X by $x \to \bar{g}x\bar{g}^{-1}$ for $x \in X$ and $g \in G$, where $\bar{g}$ denotes any extension of g to an automorphism of L. Now since k_ψ is totally real, X is a torsion and noetherian Λ-module, where Λ is the completed group ring of Γ over $\boldsymbol{Z}_p$. (See [9].) This implies that $V = X \otimes_{\boldsymbol{Z}_p} \Omega_p$ is a finite dimensional vector space over Ω_p. Also G acts on V (through the first factor). Now assume that ψ is irreducible. Let e_ψ denote the idempotent element in the group ring $\Omega_p[\Delta]$ corresponding to ψ. Let $V_\psi = e_\psi V$. As a representation space for Δ, V_ψ is a direct sum of irreducible representations with character ψ. Now V_ψ is invariant under the action of Γ. If we let γ_0 be a fixed topological generator for Γ, then each root in Ω_p of the characteristic polynomial $c_\psi(T)$ of $\gamma_0 - \mathrm{id.}$ acting on V_ψ occurs with multiplicity divisible by $d_\psi = \psi(1)$, the degree of the irreducible character ψ. Define $h_\psi(T)$ to be the monic polynomial such that $h_\psi(T)^{d_\psi} = c_\psi(T)$.

We first want to show that $h_\psi(T)$ is unchanged by "liftings" of ψ. Suppose k' is a finite Galois extension of k such that $k_\psi \subseteq k'$ and $k' \cap k_\infty = k$. Let $\Delta' = \mathrm{Gal}\,(k'/k)$ and let ψ' be the character of Δ' obtained from ψ by composition with the restriction map $\Delta' \to \Delta$. Let $V_{\psi'}$ and $h_{\psi'}(T)$ be defined just as above. Then we have the following proposition.

PROPOSITION 1. $h_{\psi'}(T) = h_\psi(T)$.

Proof. We will show that in fact $V_{\psi'} \cong V_\psi$ as representation spaces for Γ. Let L' be the maximal abelian p-ramified pro-p-extension of $k'k_\infty$ and let $X' = \mathrm{Gal}\,(L'/k'k_\infty)$. We have $F \subseteq F' \subseteq L'$ and $L \subseteq L'$, where $F = k_\psi k_\infty$, $F' = k'k_\infty$. Let $H = \mathrm{Gal}\,(F'/F)$. The image of X' under the natural homomorphism $X' \to X$ is of finite index in X. This natural homomorphism factors through $X'/I_H X'$, where I_H denotes the augmentation ideal in $\boldsymbol{Z}_p[H]$. Let a' and a denote the dimensions over $\boldsymbol{Q}_p$ of $(X'/I_H X') \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$ and $X \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$. Then $a' \geq a$. Also there is a subfield L'_0 of L' such that

$F' \subseteq L_0' \subseteq L'$, $X_0' = \mathrm{Gal}(L_0'/F') \cong Z_p^{a'}$, and L_0' is contained in the fixed field for $I_H X'$. Now $\mathrm{Gal}(L_0'/F)$ is a central extension of X_0' by H and the transfer homomorphism maps $\mathrm{Gal}(L_0'/F)$ onto a subgroup of X_0' containing $(X_0')^h$, where $h = |H|$. Thus $\mathrm{Gal}(L_0'/F)$ has a quotient group isomorphic to $Z_p^{a'}$. The field extension L_{00}'/F corresponding to this quotient group is unramified outside p (since the inertia groups for such primes have finite order dividing h). Thus $L_{00}' \subseteq L$ and hence $a' \leq a$. Letting $V' = X' \otimes_{Z_p} \Omega_p$, we have an isomorphism $V'/I_H V' \cong V$ induced from the map $X' \to X$. Since $e_{\psi'} V'$ is mapped to $e_\psi V$ and since $e_{\psi'} I_H V' = \{0\}$, we have $e_{\psi'} V' \cong e_\psi V$ (as Γ-spaces), proving Proposition 1.

There is now no need to assume that our characters are faithful. We also extend the definition of $h_\psi(T)$ to reducible characters by $h_{\theta+\eta}(T) = h_\theta(T) h_\eta(T)$. Let K be a finite totally real Galois extension of k such that $K \cap k_\infty = k$. Let $\Delta = \mathrm{Gal}(K/k)$ and let Δ_0 be a subgroup of Δ. Let ψ be a character of Δ_0 and let $\psi^* = \mathrm{Ind}_{\Delta_0}^{\Delta}(\psi)$.

Proposition 2. $h_{\psi^*}(T) = h_\psi(T)$.

Proof. Let L be the maximal p-ramified abelian pro-p-extension of Kk_∞ so that as before $V = \mathrm{Gal}(L/Kk_\infty) \otimes_{Z_p} \Omega_p$ is a representation space for $\mathrm{Gal}(Kk_\infty/k) \cong \Delta \times \Gamma$ (canonically). If χ is either a character of Δ or of Δ_0, we can of course define both V_χ and $h_\chi(T)$. Let $\alpha \in \Omega_p$. Consider $V_\alpha = \{v \in V \mid ((\gamma_0 - \mathrm{id.}) - \alpha)^t v = 0 \text{ for some } t\}$. V_α is invariant under the action of Δ. Let φ_α denote the character of the representation of Δ on V_α. If χ is an irreducible character of either Δ or Δ_0, then χ occurs in V_α with multiplicity $(\chi, \varphi_\alpha)_\Delta$ or $(\chi, \varphi_\alpha|_{\Delta_0})_{\Delta_0}$. Clearly this multiplicity will also be the multiplicity of α as a root of $h_\chi(T)$. If χ is reducible, the multiplicity of α as a root of $h_\chi(T)$ is also given by the above inner products. The proposition then follows immediately from the Frobenius Reciprocity Law $(\psi^*, \varphi_\alpha)_\Delta = (\psi, \varphi_\alpha|_{\Delta_0})_{\Delta_0}$.

Now we consider two Artin characters ψ_1 and ψ_2 over k, both of type S, such that $\psi_1 = \psi_2 \rho$, where ρ is of type W. If $f(T)$ and $g(T)$ are two polynomials, we write $f(T) \sim g(T)$ when $f(T) = ug(T)$ with $u \in \Omega_p$ of absolute value 1. Then we have

Proposition 3. *Let* $\rho(\gamma_0) = \rho_0$. *Then* $h_{\psi_1}(T) \sim h_{\psi_2}(\rho_0(1 + T) - 1)$.

Proof. We can assume that ψ_1 and ψ_2 are irreducible. Let $K_i = k_{\psi_i}$ and $\Delta_i = \mathrm{Gal}(K_i/k)$ for $i = 1, 2$. Obviously, $K_1 k_\infty = K_2 k_\infty = F$, say. After identifying Δ_1 and Δ_2 with the corresponding subgroups of $\mathrm{Gal}(F/k)$, we actually have $\Delta_1 = \Delta_2 = \mathrm{Gal}(F/k_\infty)$. Regarding ψ_1 and ψ_2 as characters of $\mathrm{Gal}(F/k)$, their restrictions to $\mathrm{Gal}(F/k_\infty)$ coincide. Hence $V_{\psi_1} = V_{\psi_2}$. Let γ_1 and γ_2 be the topological generators of $\Gamma_1 = \mathrm{Gal}(F/K_1)$ and $\Gamma_2 = \mathrm{Gal}(F/K_2)$ such that $\gamma_1|_{k_\infty} = \gamma_0$. Clearly $\gamma_2 = \gamma_1\delta$ for some $\delta \in \mathrm{Gal}(F/k_\infty)$. Actually δ must be in the center of $\mathrm{Gal}(F/k_\infty)$. It is easy to determine δ since we must have $\gamma_2 \in \ker(\psi_2)$. Thus

$$d_{\psi_2} = \psi_2(\gamma_2) = \psi_1\rho^{-1}(\gamma_1\delta) = \psi_1(\gamma_1\delta)\rho^{-1}(\gamma_1\delta) = \psi_1(\delta)\rho^{-1}(\gamma_1) = \psi_1(\delta)\rho_0^{-1}$$

and therefore $\psi_1(\delta) = d_{\psi_2}\rho_0 = d_{\psi_1}\rho_0$, which determines δ since ψ_1 is faithful on $\mathrm{Gal}(F/k_\infty)$. Clearly δ acts on $V_{\psi_1} = V_{\psi_2}$ as multiplication by ρ_0. Proposition 3 follows immediately.

Now let ψ be an irreducible character of type S. If $\alpha \in \Omega_p$, let A denote $(\gamma_0 - \mathrm{id.}) - \alpha$ acting on V_ψ. Clearly, $\ker(A^t)/\ker(A^{t-1})$ has dimension over Ω_p divisible by d_ψ for all $t \geq 1$. Also, under certain assumptions on k_ψ, it is possible to show that V has a cyclic vector for the action of $\mathrm{Gal}(k_\psi k_\infty/k)$. For example, this would be true if $k = \boldsymbol{Q}$, the p-adic completions of k_ψ contained no primitive p-th (or 4th if $p = 2$) roots of unity, and Iwasawa's λ-invariant $\lambda(k_\psi)$ vanished. If V_ψ is cyclic for $\mathrm{Gal}(k_\psi k_\infty/k)$, then $\ker(A^t)/\ker(A^{t-1})$ would be cyclic for the action of Δ and would therefore have dimension over Ω_p at most d_ψ^2. Let $m_\psi(T)$ denote the minimal polynomial for $\gamma_0 - \mathrm{id.}$ acting on V_ψ. The above observations lead to the following result, which is closely related to Theorem 3 of [10].

PROPOSITION 4. *V_ψ is annihilated by $h_\psi(\gamma_0 - \mathrm{id.})$. Hence $m_\psi(T)$ divides $h_\psi(T)$. If V_ψ has a cyclic vector for the action of* $\mathrm{Gal}(k_\psi k_\infty/k)$, *then $h_\psi(T)$ divides $m_\psi(T)^{d_\psi}$.*

§ 2. *p*-adic Artin *L*-functions

First let ψ be a 1-dimensional Ω_p-valued Artin character over k. We still assume that k_ψ is totally real. The existence of the p-adic L-function $L_p(s, \psi)$ was first established by Deligne and Ribet. (See [4] and also [11].) Alternative approaches were given later by Barsky [1] and Cassou-Nogues [2]. Let $\sigma\colon \Omega_p \to \boldsymbol{C}$ be a fixed isomorphism. Then $\sigma \circ \psi$ is a complex-valued 1-dimensional Artin character. Let $L^*(z, \sigma \circ \psi)$ be the corresponding Artin L-function with all Euler factors for primes dividing p omitted. The values $L^*(1 - n, \sigma \circ \psi)$ are in $\boldsymbol{Q}$ (values of $\sigma \circ \psi$) and are nonzero for even n. Let $L^*(1 - n, \psi) = \sigma^{-1}(L^*(1 - n, \sigma \circ \psi))$, which is in fact independent of the choice of σ. The function $L_p(s, \psi)$ can be characterized as the unique, continuous, Ω_p-valued function on $\boldsymbol{Z}_p$ such that $L_p(1-n, \psi) = L^*(1-n, \psi)$ for all positive $n \equiv 0 \pmod{p-1}$ (or (mod 2) if $p = 2$). It is defined and analytic (except for at most a simple pole at $s = 1$ when ψ equals the principal character ψ^0) in a disc in Ω_p about zero of radius greater than 1. $L_p(s, \psi)$ is not identically zero.

Now assume ψ is an Ω_p-valued Artin character of arbitrary degree (with k_ψ still totally real). Let $\Delta = \mathrm{Gal}(K/k)$ where K is a finite, totally real Galois extension of k containing k_ψ. By Brauer's Induction Theorem,

$$\psi = \sum_{i=1}^{t} a_i \psi_i^* \tag{1}$$

where $a_i \in \boldsymbol{Z}$ and ψ_i is a 1-dimensional Artin character of a subgroup Δ_i of Δ for $i = 1, \cdots, t$. We define the p-adic Artin L-function for ψ by

$$L_p(s, \psi) = \sum_{i=1}^{t} L_p(s, \psi_i)^{a_i} . \tag{2}$$

It is clear that $L_p(1-n, \psi) = \sigma^{-1}(L^*(1-n, \sigma \circ \psi))$ for $n > 0$, $n \equiv 0$ (mod $p-1$ or 2), where $L^*(z, \sigma \circ \psi)$ is the complex Artin L-function with Euler factors for primes over p removed. $L_p(s, \psi)$ is therefore uniquely determined and is independent of (1). It of course has the basic properties: $L_p(s, \theta + \eta) = L_p(s, \theta)L_p(s, \eta)$, $L_p(s, \psi') = L_p(s, \psi)$, and $L_p(s, \psi^*) = L_p(s, \psi)$. Here the notation is as in the previous section. $L_p(s, \psi)$ will be meromorphic on some disc about zero in Ω_p.

Let $k' = k(\mu_p)$ (or $k(\mu_4)$ if $p = 2$), where μ_n denotes the n-th roots of unity in a fixed algebraic closure of k. Now k'_∞ contains all p-power roots of unity. The action of $\Gamma = \mathrm{Gal}(k_\infty/k) \cong \mathrm{Gal}(k'_\infty/k')$ on p-power roots of unity is given by a homomorphism $\kappa \colon \Gamma \to \boldsymbol{Z}_p^\times$. Let $\kappa_0 = \kappa(\gamma_0)$. Note that $\kappa_0 \equiv 1 \pmod{p^r}$ if $\mu_{p^r} \subseteq k'$.

If ψ is a 1-dimensional Artin character for k, then there exists a power series $G_\psi(T)$ with coefficients in $\mathcal{O}_\psi = \boldsymbol{Z}_p$ [values of ψ] such that

$$L_p(1-s, \psi) = \frac{G_\psi(\kappa_0^s - 1)}{H_\psi(\kappa_0^s - 1)} \tag{3}$$

for all $s \in \boldsymbol{Z}_p$ (except $s = 0$ if $\psi = \psi^0$). Here $H_\psi(T) = \xi(1+T) - 1$ if ψ is of type W, where $\xi = \psi(\gamma_0)$, and $H_\psi(T) = 1$ if ψ is not of type W. Also, if ρ is of type W, then we have

$$G_{\psi\rho}(T) = G_\psi(\rho_0(1+T) - 1) \quad \text{and} \quad H_{\psi\rho}(T) = H_\psi(\rho_0(1+T) - 1) \tag{4}$$

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where $\rho_0 = \rho(\gamma_0)$. (See [11] for an explanation.) As in chapter 6 of [8], one can use (3) to show that $L_p(s, \psi)$ is an analytic function in a certain disc about zero.

We can derive a similar expression for $L_p(s, \psi)$ where ψ is of arbitrary degree by using (1), (2), and (3). However note that if f is an extension of k (corresponding to a subgroup of Δ), then $\mathrm{Gal}(f_\infty/f)$ will be identified with a subgroup of Γ of finite index. A topological generator will be a power of γ_0. The p-adic L-functions for 1-dimensional Artin characters over f will have expressions similar to (3), but with possibly different arguments. We will however always write the power series and polynomials involved in terms of $T = \kappa_0^s - 1$. (This is clearly possible.) Define $H_\psi(T) = \prod_{i=1}^t H_{\psi_i}(T)^{a_i}$, where the ψ_i's and a_i's are as in (1). It is not a difficult exercise to show that $H_\psi(T)$ is well-defined. In fact, if ψ is 1-dimensional, $H_\psi(T)$ will be just as defined above. If ψ is irreducible of degree greater than 1, then $H_\psi(T) = 1$. We also define $G_\psi(T) = \prod_{i=1}^t G_{\psi_i}(T)^{a_i}$, a quotient of power series with coefficients in the integers of a finite extension of $\boldsymbol{Q}_p$. Then (3) will hold for all s in some disc containing $\boldsymbol{Z}_p$ except possibly for finitely many values of s. Consequently, $G_\psi(T)$ depends only on ψ (and on the choice of topological generator γ_0 of Γ). We can now formulate one of the two conjectures mentioned in the introduction.

p-adic Artin conjecture: $G_\psi(T) \in \boldsymbol{Z}_p[[T]] \otimes_{\boldsymbol{Z}_p} \Omega_p$.

Some remarks are in order. It is easy to see that $G_\psi(T)$ is in the quotient field of $\mathcal{O}_\psi[[T]]$. Thus the above conjecture amounts to the statement that $cG_\psi(T) \in \mathcal{O}_\psi[[T]]$ for some nonzero $c \in \mathcal{O}_\psi$. A stronger form of the above conjecture would be that $G_\psi(T)$ itself is in $\mathcal{O}_\psi[[T]]$. We will discuss this question later. Also the p-adic Artin conjecture implies that if ψ does not have ψ^0 as a constituent, then $L_p(s, \psi)$ is in fact defined and holomorphic in the disc $|s| < r$ where $r = |\kappa_0 - 1|^{-1}|p|^{1/(p-1)}$. (See the end of chapter 6 of [8] for an explanation of this point.) However this last statement is not equivalent to the conjecture.

By the Weierstrass Preparation Theorem, we can uniquely write $G_\psi(T)$ in the form $G_\psi(T) = \pi^m u(T) g_\psi(T)$, where π is a uniformizing parameter, $m \in \boldsymbol{Z}$, $u(T)$ is an invertible element of $\mathcal{O}_\psi[[T]]$, and $g_\psi(T) = g'(T)/g''(T)$, where g' and g'' are monic polynomials in $\mathcal{O}_\psi[T]$ whose nonleading coefficients are divisible by π. We define $m_\psi = m/e$ and $\ell_\psi = \deg(g') - \deg(g'')$. Here e denotes the ramification index for the extension $\boldsymbol{Q}_p(\psi)/\boldsymbol{Q}_p$.

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We close this section by simply listing several easily proved properties.

(i) Let θ and η be Artin characters over k.
$G_{\theta+\eta}(T)=G_\theta(T)G_\eta(T)$, $H_{\theta+\eta}(T)=H_\theta(T)H_\eta(T)$, $g_{\theta+\eta}(T)=g_\theta(T)g_\eta(T)$,
$m_{\theta+\eta}=m_\theta+m_\eta$, $\ell_{\theta+\eta}=\ell_\theta+\ell_\eta$

(ii) Let $\psi_1=\psi_2\rho$ where ρ is of type W. Let $\rho_0=\rho(\gamma_0)$.
Then $G_{\psi_1}(T)=G_{\psi_2}(\rho_0(1+T)-1)$, $H_{\psi_1}(T)=H_{\psi_2}(\rho_0(1+T)-1)$,
$g_{\psi_1}(T)\sim g_{\psi_2}(\rho_0(1+T)-1)$, $m_{\psi_1}=m_{\psi_2}$, $\ell_{\psi_1}=\ell_{\psi_2}$.

(As before, $\sim$ means the two rational functions differ by a factor in Ω_p of absolute value 1.)

§ 3. The Main Conjecture

We now state the main conjecture for the p-adic L-functions constructed by Deligne and Ribet.

MAIN CONJECTURE. *Let ψ be a* 1-*dimensional Artin character of type* S. *Then* $h_\psi(T)=g_\psi(T)$.

Note that this is slightly different than the usual version of the main conjecture, which is in terms of the (-1)-eigenspace for complex conjugation in a vector space constructed from an inverse limit of ideal class groups. The above statement is in fact equivalent. The equivalence is discussed in detail in [6] for $k=\boldsymbol{Q}$, but the arguments there extend without change to the more general case. Note also that if ρ is a character of type W and if $\psi\rho$ is also of type S, then the main conjecture for ψ is equivalent to the main conjecture for $\psi\rho$ by Proposition 3.

Now let K be a finite, totally real Galois extension of k such that $K\cap k_\infty=k$, and let $\Delta=\mathrm{Gal}(K/k)$. Let θ be a 1-dimensional character of a subgroup Δ_0 of Δ. If k_0 is the fixed field for Δ_0, then clearly, the choice of γ_0 and κ_0 for the new base field k_0 can be the same as for k. We will then have $g_\theta(T)=g_{\theta^*}(T)$, where, as before, $\theta^*=\mathrm{Ind}_{\Delta_0}^{\Delta}(\theta)$. By proposition 2, we also have $h_\theta(T)=h_{\theta^*}(T)$. If we assume that the main conjecture is true for all such θ, then, by Brauer's Induction Theorem, we get the following consequence: If ψ is a character of Δ of arbitrary degree, then $h_\psi(T)=g_\psi(T)$ and hence $g_\psi(T)$ is in fact a polynomial. Therefore, the main conjecture implies the p-adic Artin conjecture for Artin characters ψ of type S. Indeed, a "main conjecture for characters of type S" would be true. The general case is dealt with in the next proposition.

PROPOSITION 5. *The main conjecture implies the p-adic Artin conjecture.*

Proof. Let ψ be an Artin character of arbitrary degree for k. The base fields for which we will assume the main conjecture will be subfields of k_ψ.

First assume that $\psi\rho$ is of type S for some ρ of type W. Until now, the polynomial $h_\psi(T)$ has been defined only when ψ is of type S. If $\psi\rho$ is of type S, we define $h_\psi(T)$ to be the monic polynomial satisfying $h_\psi(T) \sim h_{\psi_\rho}(\rho_0^{-1}(1+T)-1)$, where $\rho_0 = \rho(\gamma_0)$. By Proposition 3, $h_\psi(T)$ is well defined. Now, by the discussion preceding Proposition 5 and by (ii) of the previous section, it is clear that $g_\psi(T) = h_\psi(T)$ and hence $g_\psi(T)$ is a polynomial in this case too.

Let ψ be arbitrary. Let $K = k_\psi$ and let $\Delta = \mathrm{Gal}(K/k)$. If f is an extension of k in K and if θ is a 1-dimensional Artin character over f, then obviously $\theta\rho$ is of type S for some ρ of type W over f. Thus $h_\theta(T_e)$ is defined. Here $T_e = (1+T)^{p^e} - 1$, where $p^e = [f \cap k_\infty : k]$. (The role of κ_0 and γ_0 is played by $\kappa_0^{p^e}$ and $\gamma_0^{p^e}$ over f.) We define $h_{\theta^*}(T) = h_\theta(T_e)$, where $\theta^* = \mathrm{Ind}^{\Delta}_{\mathrm{Gal}(K/f)}(\theta)$. If $\psi = \sum_{i=1}^t a_i\psi_i^*$ as in (1), we then define a rational function $h_\psi(T)$ by $h_\psi(T) = \prod_{i=1}^t h_{\psi_i^*}(T)^{a_i}$. We will show (unconditionally) that $h_\psi(T)$ is well-defined and is in fact a polynomial. The proposition follows from this because the main conjecture implies that $g_\psi(T) = h_\psi(T)$.

Now let $\alpha \in \Omega_p$. We will prove that the multiplicity of α in $h_\psi(T)$ depends only on ψ and α and is nonnegative. Put $F = Kk_\infty$ and $L =$ the maximal abelian pro-p-ramified extension of F. If $G = \mathrm{Gal}(F/k)$, then $\mathrm{Gal}(L/F) \otimes_{Z_p} \Omega_p$ is, as before, a finite dimensional space for G over Ω_p. We let V be the semisimplification of the above representation of G. Letting $N = \mathrm{Gal}(F/k_\infty)$, we have an exact sequence $1 \to N \to G \to \Gamma \to 1$ which is in fact split. We can choose $g_0 \in G$ such that $g_0|_{k_\infty} = \gamma_0$ that such and g_0 is a topological generator of a complement for N in G. We let

$$V_\alpha = \{v \in V \mid (g_0^{p^r} - (1+\alpha)^{p^r})v = 0 \text{ for some } r\}\,.$$

V_α is a G-invariant subspace which does not depend on our choice of g_0. Now we have a homomorphism $\chi_\alpha\colon G \to \Omega_p^\times$ which factors through Γ and sends g_0 to $1+\alpha$. We let $W_\alpha = V_\alpha \otimes \chi_\alpha^{-1}$ so that G acts on W through a finite quotient group $\bar{G} = G/H$. We may assume that K is contained in the fixed field for H. Then Δ is a homomorphic image of $\bar{G}$ and all the characters we are considering can be viewed as characters of $\bar{G}$ or subgroups of $\bar{G}$. Let φ_α be the character of the representation of $\bar{G}$ on W_α.

Consider an arbitrary field f such that $k \subseteq f \subseteq K$. Let $\Delta_f = \mathrm{Gal}(K/f)$ and let $\bar{G}_f$ be the inverse image of Δ_f in $\bar{G}$. Let θ be a 1-dimensional character of Δ_f. Assume $[f \cap k_\infty : k] = p^e$ and let $T_e = (1+T)^{p^e} - 1$ as before. For some character ρ of type W for f, $\eta = \theta\rho$ will be of type S and so $\mathrm{Gal}(f_\theta f_\infty/f) = \mathrm{Gal}(f_\eta f_\infty/f)$ will be canonically isomorphic to $\mathrm{Gal}(f_\eta/f) \times \Gamma_f$, where $\Gamma_f = \mathrm{Gal}(f_\infty/f)$. Let $\gamma_f \in \Gamma_f$ be such that $\gamma_f|_{k_\infty} = \gamma_0^{p^e}$. It is easy to see that the multiplicity of $\alpha_e = (1+\alpha)^{p^e} - 1$ as a root of $h_\eta(T_e)$ is given by $(\varphi_\alpha|_{\bar{G}_f}, \eta)_{\bar{G}_f}$. (We can assume that H is chosen so that η factors through $\bar{G}_f$. Also, see the argument used to prove Proposition 1.) Let $\xi = \rho(\gamma_f)^{-1}$. Choose $\beta \in \Omega_p$ so that $(1+\beta)^{p^e} = \xi(1+\alpha)^{p^e}$. Clearly $V_\beta = V_\alpha$ and $\varphi_\beta|_{\bar{G}_f} = (\varphi_\alpha|_{\bar{G}_f})\cdot\rho$. The multiplicity of $\beta_e = (1+\beta)^{p^e} - 1$ as a root of $h_\eta(T_e)$ equals $(\varphi_\beta|_{\bar{G}_f}, \eta)_{\bar{G}_f} = (\varphi_\alpha|_{\bar{G}_f}, \eta\rho^{-1})_{\bar{G}_f} = (\varphi_\alpha|_{\bar{G}_f}, \theta)_{\bar{G}_f}$. The multiplicity of β_e as a root of $h_\eta(T_e)$ is also the multiplicity of α_e as a root of $h_\theta(T_e)$. This in turn equals the multiplicity of α as a root of $h_{\theta^*}(T)$. To summarize, α has multiplicity $(\varphi_\alpha|_{\bar{G}_f}, \theta)_{\bar{G}_f} = (\varphi_\alpha, \theta^*)_{\bar{G}}$ in the polynomial $h_{\theta^*}(T)$. Applying this to $\theta = \psi_i$ for $i = 1, \cdots, t$, we see that α has (non-negative) multiplicity $(\varphi_\alpha, \psi)_{\bar{G}}$ in $h_\psi(T)$, proving that $h_\psi(T)$ is indeed a polynomial.

Q.E.D.

§4. Some questions

The p-adic Artin conjecture would seem to be more approachable than the complex Artin conjecture in some ways. For a given ψ and p, it might quite conceivably be possible to verify the p-adic Artin conjecture by a purely computational approach. For example, one might be able to eliminate possible poles by using different representations of $L_p(s, \psi)$ in the form of (2). On the other hand, there is a rich circle of connections between the complex Artin conjecture and the theory of modular forms and representation theory. These connections have led to proofs of the complex Artin conjecture for some non-monomial characters. It would be interesting to find similar connections in the p-adic case. Can the p-adic Artin conjecture be related directly to the complex conjecture? Of course, both are valid for monomial characters.

Behavior at $s = 1$. For 1-dimensional Artin characters ψ, it should be true that $L_p(s, \psi)$ has a pole (which must be simple) at $s = 1$ when $\psi = \psi^0$ and is nonzero at $s = 1$ when $\psi \neq \psi^0$. The same result would then follow for irreducible Artin characters ψ of arbitrary degree. It would be interesting to know more about the values at $s=1$. There is a p-adic Stark conjecture (formulated by Serre). Now our proof of Proposition 5 is based on the conjectured properties of the representation space V. If we could somehow construct a similar representation space (perhaps by proving a weak form of a p-adic Stark conjecture for Deligne-Ribet p-adic L-functions), it might be possible to prove the p-adic Artin conjecture unconditionally. It would be interesting to examine exactly what is involved in constructing such a representation space.

Behavior at $s=0$. This topic has been discussed in some detail by Gross in [7]. Let ω be the Teichmuller character over k (i.e. the 1-dimensional Ω_p-valued Artin character over k giving the Galois action on p-th or 4th roots of unity). If ψ is an arbitrary Artin character, we define $\psi_n = \psi\omega^{-n}$ for every integer n. For 1-dimensional even Ω_p-valued characters ψ, we have for $n \geq 1$

$$L_p(1-n, \psi) = \sigma^{-1}(L^*(1-n, \sigma\circ\psi_n)) \tag{5}$$

and also these values are non-zero for $n \geq 2$. Using (2), one can easily extend (5) to the case when ψ is of arbitrary degree provided $n \geq 2$. It would be interesting to prove that $L_p(s, \psi)$ is in fact analytic at $s=0$ and has the correct value. The order of vanishing at zero of $L_p(s, \psi)$ and $L^*(z, \sigma\circ\psi_1)$ should be the same too. If ψ is 1-dimensional, let a and b denote the order of vanishing of $L_p(s, \psi)$ at $s=0$ and $L^*(z, \sigma\circ\psi_1)$ at $z=0$, respectively. Also, let c be the order of vanishing of $h_\psi(T)$ at $T=\kappa_0-1$. The only relationships between a, b, and c that seem to be known at present are the following: (i) If one is nonzero, they all are and (ii) $c \geq b$. (Proofs omitted.) Note that b is easily described. It is simply the number of prime ideals $\mathfrak{p}$ of k dividing p such that $\psi_1(\mathfrak{p}) = 1$.

Let $\theta = \psi\rho$, where ψ is an Artin character of arbitrary degree and ρ is of type W. If ρ is of sufficiently large order, then it is easy to see that in fact

$$L_p(0, \theta) = \sigma^{-1}(L(0, \sigma\circ\theta_1)) . \tag{6}$$

(The Euler factors for primes dividing p will be trivial.) Let λ_ψ denote the degree of $h_\psi(T)$. One can use (6), the arguments in section 5 of [5], and the arguments in section 2 of [6] to prove the following result: If ψ is such that ψ_1 is rational valued, then $\ell_\psi = \lambda_\psi$. One could also relate

m_ψ to Iwasawa's μ-invariant. If the well known conjecture that μ vanishes for the cyclotomic Z_p-extension of an arbitrary base field is true, then it would follow that $m_\psi = 0$ if p is odd and that $m_\psi = \psi(1)\ [k: Q]$ if $p = 2$. One would first prove this for all ψ such that ψ_1 is rational-valued, then for 1-dimensional ψ, and finally for ψ of arbitrary degree. Thus, the vanishing of Iwasawa's μ-invariant would imply that the p-adic Artin conjecture is equivalent to the stronger conjecture that $G_\psi(T) \in \mathcal{O}_\psi[[T]]$.

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