LOWER BOUNDS FOR SOLUTIONS OF PARABOLIC DIFFERENTIAL INEQUALITIES

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1. Introduction. Let P be the parabolic differential operator

$$P = \partial/\partial t + E(x, t, \partial/\partial x),$$

where E is a linear elliptic operator of second order on $D \times [0, \infty)$, D being a bounded domain in R^n. The asymptotic behaviour of solutions $u(x, t)$ of differential inequalities of the form

$$(1) \qquad \int_D |Pu|^2\, dx \leqslant f(t) \int_D u^2\, dx + g(t) \int_D \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx$$

has been investigated by Protter **(4)**. He found conditions on the functions f and g under which solutions of (1), vanishing on the boundary of D and tending to zero with sufficient rapidity as $t \to \infty$, vanish identically for all $t \geqslant 0$. Similar results have been found by Lees **(1)** for parabolic differential inequalities in Hilbert space.

It is the purpose of this paper to establish for solutions of parabolic differential inequalities, lower bounds of the form

$$\int_D |u(x, t)|^2\, dx \geqslant e^{-K(t)} \int_D |u(x, 0)|^2\, dx$$

for all t sufficiently large, where K is a positive, non-decreasing function. Certain of the results of **(1)** and **(4)** follow from these lower bounds. The bounds can also be used to prove backward uniqueness theorems for the mixed problem for parabolic inequalities; cf. Lions and Malgrange **(2)**.

2. Terminology and assumptions. We shall use the abstract Hilbert space terminology of **(1)** and **(2)**. Let H and V be Hilbert spaces with V dense in H and suppose that the injection mapping of V into H is continuous. We shall denote the inner product in H by $\langle u, v\rangle$ and the corresponding norm by $|u|$. The norm in V is denoted by $||u||$.

For each $t \geqslant 0$, we are given a sesquilinear form

$$u, v \to a(t; u, v),$$

continuous on $V \times V$. We denote by $D[A(t)]$ the set of $u \in V$ such that the conjugate-linear form $v \to a(t; u, v)$ is continuous on V in the norm topology

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induced by H. Since V is dense in H, the form $v \to a(t; u, v)$ can be extended to a continuous conjugate-linear form on H. Thus there exists a linear operator $A(t)$ from $D[A(t)]$ into H such that

$$a(t; u, v) = \langle A(t)u, v\rangle$$

for all v in H.

We make the following assumptions:

(i) For all $u, v \in V$, the function $t \to a(t; u, v)$ is continuously differentiable on $[0, \infty)$.

(ii) For all $u, v \in V$ and $t \geqslant 0$,

$$a(t; u, v) = \overline{a(t; v, u)}.$$

(iii) There exists a positive function $m(t)$ such that

$$a(t; u, u) \geqslant m^2(t)||u||^2$$

for all $u \in V$ and $t \geqslant 0$.

(iv) There exists a non-negative function $\eta(t)$, with η/m in L_2 on every finite interval $[0, T]$, such that

$$d[a(t; u, u)]/dt = a'(t; u, u) \leqslant \eta^2(t)||u||^2$$

for all $u \in V$ and $t \geqslant 0$.

3. The principal result. Let $u(t)$ be a function from $[0, \infty)$ into $D[A(t)$ such that u is continuously differentiable in the norm topology of V. The functions $|u(t)|^2$ and $a(t; u(t), u(t))$ are continuously differentiable on $[0, \infty)$ and

$$d|u(t)|^2/dt = 2\,\mathrm{Re}\langle u(t), u'(t)\rangle, \tag{2}$$

$$d[a(t; u(t), u(t))]/dt = 2\,\mathrm{Re}\, a(t; u(t), u'(t)) + a'(t; u(t), u(t)). \tag{3}$$

Defining

$$Lu(t) = u'(t) + A(t)u(t), \tag{4}$$

we consider the differential inequality

$$|Lu(t)| \leqslant \phi(t)\,|u(t)| + \psi(t)\,||u(t)|| \tag{5}$$

for $t \geqslant 0$, where ϕ/m and ψ/m are functions in L_2 on each finite interval $[0, T]$. For convenience, we set

$$b(t) = \exp\left[-\int_0^t m^{-2}(\psi^2 + \eta^2)\,ds\right]. \tag{6}$$

THEOREM 1. *Suppose u is a solution of* (5) *with $|u(0)| \neq 0$. Then for t sufficiently large*

$$|u(t)| \geqslant |u(0)|e^{-K(t)}, \tag{7}$$

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where

$$K(t) = 3tb^{-1}(t)\left[\mu + \int_0^t b(s)\phi^2(s)\,ds\right] \tag{8}$$

and μ *is a positive constant depending on* u.

4. Proof of Theorem 1. In **(3)**, lower bounds for solutions of the inequality

$$|u' + Au| \leqslant \phi\,|u|$$

were obtained in the case where A was a constant, symmetric operator. The present proof makes use of an identity which also played an important part in the derivation of the bounds of **(3)**. Namely, application of (2), (3), and (4) yields

$$\begin{aligned}\frac{d}{dt}\frac{a(t;u,u)}{|u|^2} &= \frac{d}{dt}\frac{\langle Au,u\rangle}{|u|^2}\\ &= \frac{1}{|u|^4}\{[2\,\mathrm{Re}\langle Au,u'\rangle + a'(t;u,u)]|u|^2 - 2\,\mathrm{Re}\langle u,u'\rangle\langle Au,u\rangle\}\\ &= -\frac{2}{|u|^4}\{|Au - \tfrac{1}{2}Lu|^2|u|^2 - [\mathrm{Re}\langle Au - \tfrac{1}{2}Lu,u\rangle]^2\\ &\qquad - \tfrac{1}{4}|Lu|^2\,|u|^2 + \tfrac{1}{4}[\mathrm{Re}\langle Lu,u\rangle]^2\} + \frac{a'(t;u,u)}{|u|^2}.\end{aligned}$$

Applying Schwarz's inequality to the identity, we obtain

$$\frac{d}{dt}\frac{a(t;u,u)}{|u|^2} \leqslant \frac{1}{2}\frac{|Lu|^2}{|u|^2} + \frac{a'(t;u,u)}{|u|^2}.$$

Making use of assumptions (iii) and (iv), and the inequality (5), we find that

$$\frac{d}{dt}\frac{a(t;u,u)}{|u|^2} \leqslant \phi^2 + m^{-2}(\psi^2+\eta^2)\frac{a(t;u,u)}{|u|^2}.$$

The solution of this ordinary differential inequality is easily found to be

$$\frac{a(t;u,u)}{|u|^2} \leqslant b^{-1}(t)\left(\lambda + \int_0^t b\phi^2\,ds\right), \tag{9}$$

where b is given by (6) and $\lambda = a(0;u(0),u(0))/|u(0)|^2$.

We next note that

$$\begin{aligned}\frac{d}{dt}\log|u| &= \frac{\mathrm{Re}\langle Lu - Au,\ u\rangle}{|u|^2}\\ &\geqslant -\phi - \psi\frac{||u||}{|u|} - \frac{a(t;u,u)}{|u|^2}.\end{aligned}$$

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Thus, we obtain the estimate

(10) $$\log \frac{|u(t)|}{|u(0)|} \geqslant -\int_0^t \phi\, ds - \int_0^t \psi \frac{\|u\|}{|u|}\, ds - \int_0^t \frac{a(s;u,u)}{|u|^2}\, ds$$
$$\geqslant -I_1(t) - I_2(t) - I_3(t).$$

We shall now find bounds for each of the integrals I_1, I_2, and I_3.

First, using Schwarz's inequality and the fact that b is positive and non-increasing, we see that

$$I_1(t) \leqslant \left(\int_0^t b^{-1}\, ds\right)^{\frac{1}{2}}\left(\int_0^t b\phi^2\, ds\right)^{\frac{1}{2}} \leqslant \left(tb^{-1}(t)\int_0^t b\phi^2\, ds\right)^{\frac{1}{2}}.$$

To estimate $I_2(t)$ we first observe that, since b is non-increasing and $b(0) = 1$,

$$-\log b(t) = -\int_0^t b^{-1}b'\, ds \leqslant b^{-1}(t).$$

Consequently,

$$\int_0^t m^{-2}\psi^2\, ds \leqslant b^{-1}(t).$$

Making use of assumption (iii), we then find that

$$I_2(t) \leqslant \int_0^t m^{-1}\psi\left[\frac{a(s;u,u)}{|u|^2}\right]^{\frac{1}{2}} ds$$
$$\leqslant \left(\int_0^t m^{-2}\psi^2\, ds\right)^{\frac{1}{2}}\left(\int_0^t \frac{a(s;u,u)}{|u|^2}\, ds\right)^{\frac{1}{2}}$$
$$\leqslant (b^{-1}(t) I_3(t))^{\frac{1}{2}}.$$

Finally, the estimate (9) implies that

$$I_3(t) \leqslant \int_0^t b^{-1}(s)\left[\lambda + \int_0^s b(y)\phi^2(y)\, dy\right] ds$$
$$\leqslant tb^{-1}(t)\left[\lambda + \int_0^t b(s)\phi^2(s)\, ds\right].$$

It follows that if μ is a positive constant with $\mu \geqslant \lambda$, say $\mu = \max(\lambda, 1)$, then for t sufficiently large,

$$I_i(t) \leqslant tb^{-1}(t)\left[\mu + \int_0^t b(s)\phi^2(s)\, ds\right], \qquad i = 1, 2, 3.$$

Application of these bounds to (10) then yields the desired estimate (7) for sufficiently large t.

Throughout the proof, we have tacitly assumed that $|u(t)| \neq 0$ for $t \geqslant 0$. This is an easy consequence of the assumption $|u(0)| \neq 0$ and the lower bound (7).

5. Applications. Theorem 1 can be applied to parabolic differential inequalities of the type studied by Lees **(1)**. Let

$$Pu = u' + A_0(t)u + B(t)u,$$

where $a_0(t; u, v) = \langle A_0(t)u, v\rangle$ is a sesquilinear form satisfying conditions (i), (ii), (iv), and

(iii′) there exist positive constants m and r such that

$$a_0(t; u, u) \geqslant m^2 \|u\|^2 - r\,|u|^2 \qquad \text{for all } u \in V \text{ and } t \geqslant 0.$$

The operator $B(t)$ is assumed to satisfy

$$|B(t)u| \leqslant \beta(t)\|u\|$$

for all $u \in V$ and $t \geqslant 0$, where β is in L_2 on each finite interval $[0, T]$. Suppose u satisfies the differential inequality

$$|Pu| \leqslant f\,|u| + g\,\|u\|. \tag{11}$$

Then letting $Lu = u' + Au$, where $A = A_0 + r$, we have

$$|Lu| \leqslant (f + r)\,|u| + (g + \beta)\,\|u\|.$$

Consequently, solutions of (11) satisfy the inequality (7) with $\phi = f + r$ and $\psi = g + \beta$.

If $t_0 \geqslant 0$, then it is easily seen that the proof of Theorem 1 also establishes the lower bound

$$|u(t)| \geqslant |u(t_0)|e^{-K(t)}$$

for t sufficiently large. Thus, the following asymptotic results are immediate consequences of Theorem 1.

COROLLARY 1. *If u is a solution of* (5) *satisfying*

$$\lim_{t\to\infty} e^{K(t)}|u(t)| = 0,$$

then $u \equiv 0$.

COROLLARY 2. *Let ψ/m and η/m be in $L_2(0, \infty)$. Suppose ϕ is bounded on each finite interval* $[0, T]$ *and satisfies*

$$\phi^2(t) = O(t^{\delta-2}) \qquad as \quad t \to \infty,$$

for some $\delta > 1$. If u is a solution of (5) *satisfying*

$$\lim_{t\to\infty} e^{\beta t^{\delta}}|u(t)| = 0$$

for all $\beta > 0$, then $u \equiv 0$.

We conclude with a backward uniqueness theorem for solutions of the inequality (5) on a finite interval.

THEOREM 2. *Let* ϕ, ψ/m, *and* η/m *be in* L_2 *on the finite interval* $[0, T]$. *If* u *is a solution of* (5) *satisfying* $u(T) = 0$, *then* $u \equiv 0$ *on* $[0, T]$.

Proof. Indeed, the estimates obtained in the proof of Theorem 1 imply that there is a positive constant C such that

$$|u(t)| \leqslant C\,|u(T)|$$

for $0 \leqslant t \leqslant T$.

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