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MONOCHROMATIC SEQUENCES WHOSE GAPS BELONG TO $\{d, 2d, \ldots, md\}$

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For m and k positive integers, define a *k-term h_m-progression* to be a sequence of positive integers $\{x_1, \ldots, x_k\}$ such that for some positive integer d, $x_{i+1} - x_i \in \{d, 2d, \ldots, md\}$ for $i = 1, \ldots, k-1$. Let $h_m(k)$ denote the least positive integer n such that for every 2-colouring of $\{1, 2, \ldots, n\}$ there is a monochromatic h_m-progression of length k. Thus, $h_1(k) = w(k)$, the classical van der Waerden number. We show that, for $1 \leqslant r < m$, $h_m(m+r) \leqslant 2c(m+r-1)+1$, where $c = \lceil m/(m-r) \rceil$. We also give a lower bound for $h_m(k)$ that has order of magnitude $2k^2/m$. A precise formula for $h_m(k)$ is obtained for all m and k such that $k \leqslant 3m/2$.

1. INTRODUCTION

Van der Waerden's theorem on arithmetic progressions [**13**] says that for every positive integer k there is a smallest positive integer $w(k)$ such that for every 2-colouring of $\big[1, w(k)\big] = \big\{1, 2, \ldots, w(k)\big\}$ there is a monochromatic k-term arithmetic progression. The only known non-trivial values of the van der Waerden numbers $w(k)$ are $w(3) = 9$, $w(4) = 35$, and $w(5) = 178$. Furthermore, estimation of the function $w(k)$ remains a wide open problem (see [**4**] for a discussion of this).

If F is any family of sequences that includes the arithmetic progressions then by van der Waerden's theorem we may define, for each $k \in Z^+$, $F(k)$ to be the least positive integer such that for every 2-colouring of $\big[1, F(k)\big]$ there is a monochromatic k-term member of F. One such family of sequences, called the "quasi-progressions" was introduced by Brown, Erdős, and Freedman [**1**]. A *k-term quasi-progression of diameter n* is a sequence of positive integers $\{x_1, \ldots, x_k\}$ for which there is a positive integer d such that $x_{i+1} - x_i \in \{d, d+1, \ldots, d+n\}$ for $i = 1, \ldots, k-1$. If Q_n represents the family of all quasi-progressions of diameter n, it is clear that $w(k) = Q_0(k) \geqslant Q_1(k) \geqslant Q_2(k) \geqslant \cdots$. In [**9**] upper bounds were obtained for $Q_m(k)$ provided $k \leqslant 3m/2$. Examples of other families

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F containing the family of arithmetic progressions, and results on their associated van der Waerden-type functions $F(k)$, can be found in [**5, 6, 7, 8**].

In this paper we consider another family of sequences that includes the arithmetic progressions. Namely, for positive integers m and k, define a *k-term h_m-progression* to be a sequence of positive integers $\{x_1, x_2, \ldots, x_k\}$ such that, for some $d \in Z^+$, $x_{i+1} - x_i \in \{d, 2d, \ldots, md\}$ for $1 \leqslant i \leqslant k-1$. For positive integers m and k, let $h_m(k)$ denote the least positive integer n such that for every 2-colouring of $[1, n]$ there is a monochromatic k-term h_m-progression. By van der Waerden's theorem, $h_m(k)$ exists for all positive integers m and k. Furthermore, we see that

$$w(k) = h_1(k) \geqslant h_2(k) \geqslant \cdots .$$

In terms of gaining more information about $w(k)$, the functions $h_m(k)$ may be more useful than the functions $Q_m(k)$, for the following reason. Given positive integers m and k, define $g_m(k)$ to be the least positive integer s such that whenever $S = \{x_1, x_2, \ldots, x_s\}$ with $x_{i+1} - x_i \in \{1, 2, \ldots, m\}$ for $1 \leqslant i \leqslant s-1$, there is a k-term arithmetic progression in S (Rabung [**11**] showed that $g_m(k)$ exists for all m and k; further work on $g_m(k)$ can be found in [**2**] and [**10**]). Then for all k and m, $w(k) \leqslant h_m\big(g_m(k)\big)$. This is because any 2-colouring of $\big[1, h_m\big(g_m(k)\big)\big]$ will contain a monochromatic h_m-progression with $g_m(k)$ terms, and among these $g_m(k)$ terms must be a k-term arithmetic progression.

Here we obtain an upper bound for $h_m(k)$ for all m and k such that $k < 2m$. We also give lower bounds for $h_m(k)$ that hold for all m and k. For $k \leqslant 3m/2$ we are able to give an exact formula for $h_m(k)$. One could say that we have had a bit more success in obtaining information about $h_m(k)$ than we had in [**9**] with regard to $Q_m(k)$, since the upper bound for Q_m is valid only for $k \leqslant 3m/2$ and since an exact formula for $Q_m(k)$ is known only for $k \leqslant m+2$. In Section 3 we include a table of exact values of $h_m(k)$ obtained by computer.

2. Results

It is easy to find a formula for $h_m(k)$ when $k \leqslant m$.

Proposition 1. *$h_m(k) = 2k-1$ if $k \leqslant m$.*

Proof: Any partition of $[1, 2k-2]$ into sets A and B, where $|A| = |B| = k-1$, avoids monochromatic k-element sets, so $h_m(k) \geqslant 2k-1$. On the other hand, every 2-colouring of $[1, 2k-1]$ contains a monochromatic sequence $x_1, \ldots, x_k$ with $x_{i+1} - x_i \in \{1, 2, \ldots, k\} \subseteq \{1, 2, \ldots, m\}$ for $1 \leqslant i \leqslant k-1$ (for otherwise $x_k - x_1 \geqslant k+1+(k-2)$). ∎

We are able to give an upper bound for $h_m(m+r)$ provided $r < m$. The smaller that r is, in proportion to m, the lower the upper bound we have. The result is described in the next theorem.

Theorem 2. *Let $m > r \geqslant 1$. Let $c = c(m,r) = \lceil m/(m-r) \rceil$. Then*

$$h_m(m+r) \leqslant 2c(m+r-1)+1.$$

Proof: Let $L = 2c(m+r-1)+1$ and let χ be any 2-colouring of $[1, L]$. We shall show that under χ there is a monochromatic $(m+r)$-term h_m-progression. It is clear that $[1, L]$ contains some monochromatic set of size at least $m+r$, all of whose elements are congruent to 1 modulo c. Let $X = \{x_1, x_2, \ldots, x_{m+r}\}$ consist of the least $m+r$ members of this monochromatic set, with $x_i < x_{i+1}$ for $i = 1, \ldots, m+r-1$, and assume $\chi(X) = 1$.

If $x_{i+1} - x_i \leqslant cm$ for each $i \in \{1, \ldots, m+r-1\}$, then X serves as the desired monochromatic h_m-progression, since $x_{i+1} - x_i \in \{c, 2c, \ldots, mc\}$ for all i. Thus, we assume that there is a $k \in \{1, \ldots, m+r-1\}$ such that $x_{k+1} - x_k = cs \geqslant c(m+1)$.

Let $A = \{x_k + ci : 1 \leqslant i \leqslant s-1\}$. Then $\chi(A) = 0$ and $|A| \geqslant m$. We consider two cases.

Case I. $c \geqslant 3$. Note that either $x_k > m - c$ or $x_{k+1} < L - (m-c) + 1$, since otherwise $\sum_{i \neq k} (x_{i+1} - x_i) \leqslant 2(m-c-1)$ which would contradict the fact that $\sum_{i \neq k} (x_{i+1} - x_i) \geqslant c(m+r-2)$. We shall assume $x_k > m - c$, as the case of $x_{k+1} < L - (m-c) + 1$ may be done by a symmetric argument.

Let

$$B = \{i \not\equiv 1 (\bmod\ c) : x_k - m + c \leqslant i < x_{k+1}\}.$$

We see that for each $b \in B$ there is some $a \in A$ such that $|a - b| \leqslant \max\{c, m\} = m$. Thus, if there is a set $B_0 \subseteq B$ with $|B_0| = r$ and $\chi(B_0) = 0$, then $A \cup B_0$ is a monochromatic h_m-progression with length at least $m + r$. We therefore assume that at most $r - 1$ members of B have colour 0. Let $Y = \{y_1, \ldots, y_t\}$ be those members of B having colour 1, listed in increasing order. To complete the proof of this case we show that Y gives us the monochromatic h_m-progression we seek by showing:

(1) $$y_{i+1} - y_i \leqslant m \text{ for } i = 1, \ldots, t-1$$

and

(2) $$t \geqslant m + r.$$

To establish (1), note that for any two elements b_j and $b_{j+\ell}$, of B,

$$b_{j+\ell} - b_j \leqslant \ell + \left\lceil \frac{\ell}{c-1} \right\rceil.$$

Hence, since $|B - Y| \leqslant r - 1$, for all i we have

(3) $$y_{i+1} - y_i \leqslant r + \left\lceil \frac{r}{c-1} \right\rceil = \left\lceil \frac{rc}{c-1} \right\rceil.$$

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Also, $c \geqslant m/(m-r)$ implies

$$\left\lceil \frac{rc}{c-1} \right\rceil \leqslant m. \tag{4}$$

By (3) and (4) we see that (1) is true.

To show (2), we first observe that $|B| \geqslant (c-1)(m + \lceil m/c \rceil)$. Since $|B - Y| \leqslant r$, (2) will follow if we can show

$$(c-2)m + (c-1)\lceil m/c \rceil + 1 \geqslant 2r. \tag{5}$$

Since $m > r$, (5) is obvious for $c \geqslant 4$. Inequality (5) also holds when $c = 3$, since in this case, using (4),

$$\begin{aligned} m + 2\left\lceil \frac{m}{3} \right\rceil + 1 &\geqslant \left\lceil \frac{3r}{2} \right\rceil + 2\left\lceil \frac{\lceil 3r/2 \rceil}{3} \right\rceil + 1 \\ &\geqslant \frac{3r}{2} + 2\left\lceil \frac{r}{2} \right\rceil + 1 \\ &\geqslant 2r. \end{aligned}$$

CASE II. $c = 2$. In this case we have $x_{m+r} - x_1 \leqslant 4(m + r - 1)$. Therefore, if $i \neq k$, then $x_{i+1} - x_i \leqslant m$, for otherwise we would have

$$\begin{aligned} x_{m+r} - x_1 &\geqslant x_{k+1} - x_k + \sum_{i \neq k} (x_{i+1} - x_i) \\ &\geqslant 2(m+1) + (m+1) + 2(m + r - 3) \\ &= 5m + 2r - 3 \\ &\geqslant 4m + 4r - 3. \end{aligned}$$

Let $B' = \{x_k + 2i + 1 : 0 \leqslant i \leqslant s - 1\}$. As in Case I, we may assume at most $r - 1$ members of B' have colour 0. Therefore, if $Y = \{y_1, \ldots, y_j\} = \{b \in B' : \chi(b) = 1\}$, then we have that $0 < y_{i+1} - y_i \leqslant 2r$ for $i = 1, \ldots, j-1$, and $x_{k+1} - y_j \leqslant 2r$ and $y_1 - x_k \leqslant 2r$. Since $2r \leqslant m$ (in this case), $X \cup Y$ forms a monochromatic h_m-progression with at least $m + r$ terms.

∎

REMARK. The proof of Theorem 2 actually shows a somewhat stronger result: that every 2-colouring of $\left[1, 2c(m + r - 1) + 1\right]$ has a monochromatic sequence $\{x_1, \ldots, x_k\}$ such that either $x_{i+1} - x_i \in \{1, 2, \ldots, m\}$ for all i, $1 \leqslant i \leqslant k - 1$, or $x_{i+1} - x_i \in \{c, 2c, \ldots, mc\}$ for all i, $1 \leqslant i \leqslant k - 1$.

For a fixed m, the largest value of k for which Theorem 2 gives an upper bound is $k = 2m - 1$. We single out this case as the next corollary.

COROLLARY 3. *For all positive integers m, $h_m(2m - 1) \leqslant 4(m^2 - m) + 1$.*

PROOF: Letting $r = m-1$ in Theorem 2 yields $c = m$, and the corollary is immediate from Theorem 2. ∎

The following theorem gives a formula for $h_m(m+1)$. The formula depends on the parity of m. We note that it gives a generalisation of the trivial fact that $w(2) = h_1(2) = 3$.

THEOREM 4. *For all $m \geqslant 2$, $h_m(m+1) = 4m-1$ if m is odd, and $h_m(m+1) = 4m+1$ if m is even.*

PROOF: Note first that letting $r = 1$ in Theorem 2 shows that $h_m(m+1) \leqslant 4m+1$ for all $m \geqslant 2$. Thus to establish that the given expressions serve as upper bounds for their respective cases, we need only deal with the case of m odd.

Let m be odd and let χ be any 2-colouring of $[1, 4m-1]$. By Proposition 1 there is a monochromatic h_m-progression $X = \{x_1, \ldots, x_m\} \subseteq [m+1, 3m-1]$ with $x_{i+1} - x_i \in \{1, \ldots, m\}$ for $1 \leqslant i \leqslant m-1$. Without loss of generality, we shall assume $\chi(X) = 1$. We consider two subcases:

(i) $x_{i+1} - x_i > 1$ for some $i \in \{1, \ldots, m-1\}$;

(ii) $x_{i+1} - x_i = 1$ for all $i \in \{1, \ldots, m-1\}$.

First assume (i) holds. Let j be the least such i. If $\chi(x_j+1) = 1$, then $X \cup \{x_j+1\}$ is a monochromatic $(m+1)$-term h_m-progression. So we may assume $\chi(x_j+1) = 0$. Likewise, we can assume that each of $x_1 - 1, \ldots, x_1 - m$ have colour 0. Then $A = \big[x_1 - m, x_1 - 1\big] \cup \{x_j + 1\} \subseteq [1, 4m-1]$ is monochromatic, and since

$$(x_j + 1) - (x_1 - 1) = j + 1 \leqslant m,$$

A is an h_m-progression.

Now assume (ii) holds. Clearly, we may assume

$$\chi\Big([x_1 - m, x_1 - 1] \cup [x_m + 1, x_m + m]\Big) = 0.$$

Let

$$B = \left\{x_1 - (2i-1) : 1 \leqslant i \leqslant \frac{m+1}{2}\right\} \cup \left\{x_m + (2i-1) : 1 \leqslant i \leqslant \frac{m+1}{2}\right\}.$$

Then $B \subseteq [1, 4m-1]$ has size $m+1$, has colour 0, and each pair of consecutive elements of B has difference 2 or $m+1$. Hence, since $m+1$ is even and $m+1 \leqslant 2m$, B is a monochromatic h_m-progression.

To complete the proof we shall provide, for m odd, a 2-colouring of $[1, 4m-2]$ which avoids $(m+1)$-term monochromatic h_m-progressions; and, for m even, a 2-colouring of $[1, 4m]$ which avoids such progressions.

Let m be odd. Color $[1, 4m-2]$ with the colouring α defined as follows:

$$\underbrace{11\ldots1}_{m-1}\underbrace{00\ldots0}_{m}\underbrace{11\ldots1}_{m}\underbrace{00\ldots0}_{m-1}.$$

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Let $X = \{x_1, \dots, x_\ell\}$ be a maximum length monochromatic h_m-progression. By the symmetry of the colouring we may assume $\alpha(X) = 1$. Let $d = \min\{x_{i+1} - x_i : 1 \leqslant i \leqslant \ell - 1\}$. If $d = 1$, then $x_{i+1} - x_i \leqslant m$ for each i. It is then clear from the way α is defined that $\ell \leqslant m$. If $d \geqslant 2$, then each of the two blocks of 1's in the representation of α contains at most $m/2$ members of X, so again $\ell \leqslant m$. Hence, there is no monochromatic $(m+1)$-term h_m-progression under α.

Now assume m is even. Then the same explanation as that used in the odd case shows that the colouring of $[1, 4m]$ defined by the string

$$\underbrace{11\dots1}_{m}\underbrace{00\dots0}_{m}\underbrace{11\dots1}_{m}\underbrace{00\dots0}_{m}$$

has no monochromatic $(m+1)$-term h_m-progression. ∎

We now shift our attention to lower bounds for $h_m(k)$.

The next theorem gives a lower bound for $h_m(k)$ for all m and k.

Theorem 5. *Let m be a fixed positive integer and let $\lambda(k,m) = \left\lceil (k-1)/\lceil k/m \rceil \right\rceil$. Then*

$$h_m(k) \geqslant 2(k-1)\left(\left\lceil \frac{k}{\lambda(k,m)} \right\rceil - 1\right) + 1 = \frac{2}{m}k^2\big(1 + o(1)\big).$$

Proof: Let k and m be given. Let $\lambda = \lambda(k,m)$ and let $M = 2(k-1)\big(\lceil k/\lambda \rceil - 1\big)$. Color $[1, M]$ with the colouring $ABAB\dots AB$ where A and B each appear $\lceil k/\lambda \rceil - 1$ times, and where A represents a block of $k-1$ ones and B represents a block of $k-1$ zeros. To prove the theorem we show that under this colouring there is no k-term monochromatic h_m-progression.

Assume $X = \{x_1, \dots, x_k\}$ is a k-term h_m-progression. Let d be a positive integer such that $x_{i+1} - x_i \in \{d, 2d, \dots, md\}$ for all i, $1 \leqslant i \leqslant k-1$. Since each block has size $k-1$, there is some i such that $x_{i+1} - x_i \geqslant k$. Therefore $d \geqslant \lceil k/m \rceil$. Hence, each block contains at most $\lceil (k-1)/d \rceil \leqslant \lambda$ members of X. It follows that at most $\lambda\big(\lceil k/\lambda \rceil - 1\big)$ members of X can be of the same colour. Since (for any positive integers k and λ) $\lambda\big(\lceil k/\lambda \rceil - 1\big) \leqslant k - 1$, the proof is complete. ∎

Remark. For a fixed integer i, $0 \leqslant i \leqslant k-2$, define $\lambda(k,m,i)$ to be $\left\lceil (k-i-1)/\lceil (k-i)/m \rceil \right\rceil$. Then a slight modification of the proof of Theorem 5 gives us the more general inequality

$$h_m(k) \geqslant 2(k-i-1)\left(\left\lceil \frac{k}{\lambda(k,m,i)} \right\rceil - 1\right) + 1. \tag{6}$$

That is, we change the colouring so that there are $\left\lceil k/\lambda(k,m,i) \right\rceil - 1$ blocks of each colour, and each block has length $k-i-1$. The lower bounds of (6) do not improve

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the asymptotic lower bound of Theorem 5, but do improve the lower bound for some particular values of k and m. For example, by Theorem 5, $h_4(11) \geqslant 41$. Taking $i = 1$ in (6) gives $h_4(11) \geqslant 55$.

In some instances the following lower bound is better than that provided by (6). For example, according to Theorem 6, $h_4(8) \geqslant 29$, while the best lower bound by means of inequality (6) is $h_4(8) \geqslant 25$.

THEOREM 6. *If $k > m + 1$, then $h_m(k) \geqslant 4(k-1) + 1$.*

PROOF: We consider two cases.

CASE I. k is odd. Using the notation of Theorem 5, since $\lceil k/m \rceil \geqslant 2$, $\lambda(k,m) \leqslant (k-1)/2$. Hence, by Theorem 5, $h_m(k) \geqslant 2(k-1)(3-1) + 1$.

CASE II. k is even. Color $\left[1, 4(k-1)\right]$ as follows:

$$11\ \underbrace{00\ldots0}_{k-2}\underbrace{11\ldots1}_{k-2}\underbrace{00\ldots0}_{k-2}\underbrace{11\ldots1}_{k-2}\ 00.$$

Let $X = \{x_1, \ldots, x_M\}$ be a maximal length monochromatic h_m-progression. By the symmetry of the colouring we may assume X has colour 1. We know there is a positive integer d such that $x_{i+1} - x_i \in \{d, 2d, \ldots, md\}$ for all i, $1 \leqslant i \leqslant M-1$. If $d = 1$ then, since $k - 1 \geqslant m + 1$, all members of X must belong to the same block of 1's, so that $M \leqslant k - 2$. If $d \geqslant 2$, then at most $(k-2)/2$ members of X belong to each of the long blocks of 1's, so $M \leqslant k - 1$. ∎

We see that the lower bound given by Theorem 6 coincides with the upper bound of Theorem 2 whenever $c = 2$. This gives us a precise formula for $h_m(m + r)$ whenever $m \geqslant 2r$, which we state in the following corollary.

COROLLARY 7. *Let $r \geqslant 2$ and $m \geqslant 2r$. Then $h_m(m + r) = 4(m + r - 1) + 1$.*

3. COMPUTATIONS AND FINAL REMARKS

We have run a computer program to calculate $h_m(k)$. The program also gives the 2-colourings of $\left[1, h_m(k) - 1\right]$ that contain no monochromatic k-term h_m-progressions. Table I below shows the known values of $h_m(k)$ for $m \leqslant 8$ and $k \leqslant 13$. Of course, by Proposition 1, Theorem 4, and Corollary 7, we know the value of $h_m(k)$ for all m and k such that $k \leqslant 3m/2$. The first row of the table contains the known values of $w(k)$ (these may be found in [**3**] and [**12**]). The symbol $\geqslant$ appearing before a number means that computer time became excessive and that at the point at which the program was halted, we were able to infer from the output of the program that the value of $h_m(k)$ was not less than this number.

We notice from the table that $h_m(k) = 6(k-1) + 1$ for each of the following pairs (m, k): (3,6), (5,9), (6,10), (6,11), (6,12), (7,12), (7,13). Only in the case of (6,10) does this value agree with the upper bound of Theorem 2. We wonder if there is some natural extension of Corollary 7 for which $6(k-1)+1$ is the value of $h_m(k)$. It is reasonable to think that for those cases in which $h_m(k) = 6(k-1)+1$ there must be some simple colouring of $\big[1, 6(k-1)\big]$ that is similar to those used in the proofs of Theorem 5 or 6. However for none of the pairs (m, k) listed above are there any obvious patterns in the colourings of $\big[1, 6(k-1)\big]$ that avoid monochromatic k-term h_m-progressions. This is the case even though, for example, there are thirty six different colourings of $[1, 48]$ that avoid 9-term h_5-progressions; and there are twenty six different colourings of $[1, 54]$ that avoid 10-term monochromatic h_6-progressions.

We also notice that for each $m \in \{3, 4, 5, 6\}$, $h_m(2m) = 6(2m-1)+\varepsilon$ where $\varepsilon = 1$ or 2. Perhaps $h_m(2m) = 12m\big(1+o(1)\big)$. The only values of k and m in the table for which $h_m(k) = 8(k-1)+1$ are $(m, k) = (2, 5)$ and $(m, k) = (4, 9)$. We wonder if it holds for $(m, k) = (6, 13)$ or more generally, for $(m, k) = (2j, 4j+1)$.

A study analogous to that of this paper can be made where instead of using 2-colourings we use r-colourings. If one denotes the corresponding function by $h_m(k, r)$, then the van der Waerden numbers for r colours satisfy the inequality $w(k, r) \leqslant h_m(g_m(k), r)$, where g is the function defined in Section 1. We have no idea of the rate of growth of $h_m(k, r)$.

Table I. Values of $h_m(k)$

m / k	2	3	4	5	6	7	8	9	10	11	12	13
1	3	9	35	178	?	?	?	?	?	?	?	?
2	3	9	17	33	55	$\geqslant 87$	$\geqslant 125$	$\geqslant 177$	?	?	?	?
3	3	5	11	19	31	71	97	$\geqslant 117$	?	?	?	?
4	3	5	7	17	21	35	44	65	$\geqslant 75$	$\geqslant 84$	?	?
5	3	5	7	9	19	25	33	49	56	$\geqslant 69$	$\geqslant 76$	?
6	3	5	7	9	11	25	29	33	55	61	67	$\geqslant 73$
7	3	5	7	9	11	13	27	33	37	47	67	73
8	3	5	7	9	11	13	15	33	37	41	45	71

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